GENERALIZATION OF THE STABILITY ANALYSIS OF ELASTIC SYSTEMS

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1. Introduction

In this country, research has been dealt since years with matrix equations of bar structures, to apply them for the case of great displacements and for the stability analysis of the complete structure [1, 2, 3, 4]. Based on, and joining previous results, in this paper the generalized eigenvalue problem of the stability analysis of systems with finite numbers of freedom will be discussed.

The structure may consist either of bars or other models of finite elements. Its material is supposed to be ideally elastic. Displacements are generally large, that is, not infinitesimal. Responses are due to loads, displacements (initial strains e.g. change of temperature) or their combination.

The processus needs iterated application of equations of the second-order theory, hence the second-order theory should be discussed first.

2. The second-order theory; equations and solutions

References discuss the second-order theory in detail, so it is merely outlined for sake of understanding.

Some modifications have been made in the discussion method of our references, as necessitated by the present problem.

A structure previously subjected to stresses has been investigated. Their vector is $\mathbf{s}$.

If the load on the structure changes by vector $\Delta\mathbf{q}$ and its thermal strain system by vector $\Delta\mathbf{t}$, the change of state is expressed by the equilibrium equation

$$\mathbf{D}\Delta\mathbf{u} + \mathbf{G}^* \Delta\mathbf{s} + \Delta\mathbf{q} = 0 \tag{1}$$

and the compatibility equation

$$\mathbf{G}\Delta\mathbf{u} + \mathbf{F}\Delta\mathbf{s} + \Delta\mathbf{t} = 0 \tag{2}$$

where $\Delta\mathbf{u}$ and $\Delta\mathbf{s}$ are changes of the nodal displacement vector and of the characteristic stress vector, respectively.

In the first equation $\mathbf{G}^*\Delta\mathbf{s}$ is the equilibrium force produced by the change of internal forces with respect to the original network, $\mathbf{D}\Delta\mathbf{u}$ being due to the stresses already developed, because of the change of network.

Measure of this latter effect is the second-order stiffness matrix

$$\mathbf{D} = [D_{ij}] ;$$

or

$$D_{ij} = \sum_k L_{ijk}\, s_k$$

in terms of the three-index tensor

$$\mathfrak{L} = [L_{ijk}]$$

its symbol

$$\mathbf{D} = \mathfrak{L}\,\mathbf{s} . \tag{3}$$

The elements of $\mathfrak{L}$ can be approximated by taking into account the square expressions of elongation, based on energy aspects, according to the principles of the finite element method [5].

The changes of state of the structure characterized by vectors $\Delta\mathbf{u}$ and $\Delta\mathbf{s}$ can be calculated either by the force method or by the displacement method using Eqs (1) and (2).

Superposition being justified, the vectors take the form:

$$\begin{aligned} \Delta\mathbf{u} &= \Delta\mathbf{u}_q + \Delta\mathbf{u}_t \\ \Delta\mathbf{s} &= \Delta\mathbf{s}_q + \Delta\mathbf{s}_t . \end{aligned} \tag{4}$$

3. The two-parameter problem of the stability analysis

In case of great displacements the stability analysis can be carried out by iteration. Namely the problem is linearized assuming that the structure attained the nearly critical state, therefore the change resulting in the critical state can be determined by the second-order (although linear) theory. Correctness of this supposition can be checked ulteriorly. In the negative case the computation has to be continued.

The effect on the structure is of the form

$$\mathbf{q}' = \lambda\mathbf{q}$$

$$\mathbf{t}' = \mu\mathbf{t}$$

where $\mathbf{q}$ is the basic load, and $\mathbf{t}$ the basic thermal strains. These are completely fixed, scalar parameters being λ and μ, respectively. Parameter value changing from one instant to the other determines the actual load. In the former, pre-critical state of the structure let $\lambda_e\mathbf{q}$, $\mu_e\mathbf{t}$, $\mathbf{u}_e$ and $\mathbf{s}_e$ be vectors of loads, initial strains, nodal displacements and stresses, respectively, all being known to us.

Critical state may occur by a small step, the four quantities changing by

$$\Delta\lambda\mathbf{q}\,,\quad \Delta\mu\mathbf{t}\,,\quad \Delta\mathbf{u}\,,\quad \Delta\mathbf{s}$$

respectively. In this state, however, the structure has further — infinitesimally near — equilibrium forms, described, with respect to the former ones, by

load and initial strain differential	0
displacement differential	$d\mathbf{u}$, and
stress differential	$d\mathbf{s}$ (Fig. 1).

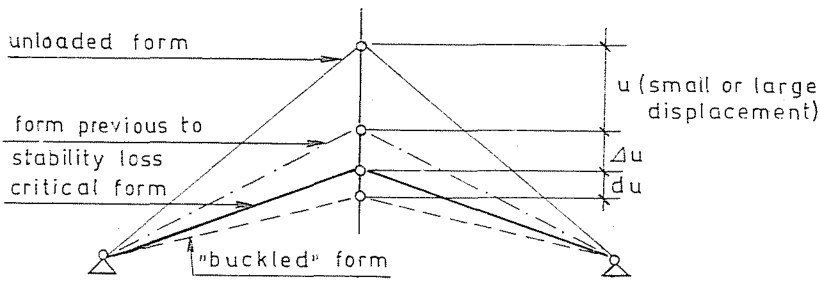


Fig. 1

Thus, the infinitesimal change in the critical state is characterized by the fact that stress changes entraining differential displacements constitute a self-equilibrating force system.

The internal forces $\mathbf{s}$ constitute a balancing force $\mathbf{G}^*\mathbf{s}$ on the network of geometry matrix $\mathbf{G}$, therefore in critical state

$$d(\mathbf{G}^*\,\mathbf{s})_{kr} = \mathbf{G}^*_{kr}\,d\mathbf{s} + d\mathbf{G}^*_{kr}\,\mathbf{s}_{kr} = 0\,. \tag{5}$$

However

$$\mathbf{G}^*_{kr}\,d\mathbf{s} = \mathbf{G}^*_e\,d\mathbf{s} + \Delta\mathbf{G}^*\,d\mathbf{s}\,,$$

therefore

$$\mathbf{G}^*_e\,d\mathbf{s} + \Delta\mathbf{G}^*\,d\mathbf{s} + d\mathbf{G}^*_{kr}\,\mathbf{s}_{kr} = 0\,. \tag{6}$$

The first term stands for those balancing effects which could be developed by the differentials of internal forces in the pre-critical form. These are expressed by the stiffness matrix assigned to the appropriate form:

$$\mathbf{G}^*_e\,d\mathbf{s} = -\mathbf{K}_e\,d\mathbf{u}\,. \tag{7}$$

The second two terms represent the forces due to network changes, so both can be written by means of the tensor of the second-order stiffness matrix:

$$\Delta \mathbf{G}^* \, d\mathbf{s} = \mathfrak{L}_e \, d\mathbf{s} \cdot \Delta \mathbf{u} = \mathfrak{L}_e \, \Delta \mathbf{u} \cdot d\mathbf{s} \, ; \tag{8}$$

similarly

$$d\mathbf{G}^*_{kr} \, \mathbf{s}_{kr} = \mathfrak{L}_{kr} \, \mathbf{s}_{kr} \cdot d\mathbf{u} \cong \mathfrak{L}_e \, \mathbf{s}_{kr} \cdot d\mathbf{u} \, . \tag{9}$$

This approximation is necessary to keep the problem linear. Replacing terms (7), (8) and (9) into Eq. (6):

$$-\mathbf{K}_e \, d\mathbf{u} + \mathfrak{L}_e \, \Delta \mathbf{u} \cdot d\mathbf{s} + \mathfrak{L}_e \, \mathbf{s}_{kr} \cdot d\mathbf{u} = 0 \, . \tag{10}$$

To write the eigenvalue problem in the usual form, the second two terms of the equation are also written in form corresponding to $\mathbf{K} \, d\mathbf{u}$.

Taking into account that

$$\mathbf{s}_{kr} = \mathbf{s}_e + \Delta \mathbf{s} \, , \tag{11}$$

further, according to the compatibility equation of the second-order theory

$$d\mathbf{s} = -\mathbf{F}^{-1} \, \mathbf{G}^* \, d\mathbf{u} \, . \tag{12}$$

Besides, according to relationships (4), in case of effects characterized by scalar parameters

$$\begin{aligned} \Delta \mathbf{u} &= \Delta\lambda \mathbf{u}_q + \Delta\mu \mathbf{u}_t \\ \Delta \mathbf{s} &= \Delta\lambda \mathbf{s}_q + \Delta\mu \mathbf{s}_t \end{aligned} \tag{13}$$

where

$$\Delta\lambda = \lambda_{kr} - \lambda_e \qquad \Delta\mu = \mu_{kr} - \mu_e$$

$\mathbf{u}_q$ and $\mathbf{s}_q$; $\mathbf{u}_t$ and $\mathbf{s}_t$ are displacement and internal force vectors calculable from the basic load and from the basic initial strain, respectively, delivered either by the force method, by the displacement method or by their combination, while the network corresponds to the pre-critical state of index e, rather than to the unloaded condtion.

Substituting (11), (12) and (1i3) into (10), and arranging, the following two-parameter eigenvalue problem results:

$$(\mathbf{A} - \lambda_{kr} \, \mathbf{B} - \mu_{kr} \, \mathbf{C}) d\mathbf{u} = 0$$

here

$$\begin{aligned} \mathbf{B} &= \mathfrak{L}_e \, \mathbf{u}_q \, \mathbf{F}^{-1} \, \mathbf{G}^* - \mathfrak{L}_e \, \mathbf{s}_q \\ \mathbf{C} &= \mathfrak{L}_e \, \mathbf{u}_t \, \mathbf{F}^{-1} \, \mathbf{G}^* - \mathfrak{L}_e \, \mathbf{s}_t \\ \mathbf{A} &= -\mathbf{K}_e + \mathfrak{L}_e \, \mathbf{s}_e + \lambda_e \, \mathbf{B} + \mu_e \mathbf{C} \, . \end{aligned} \tag{14}$$

In practice, the problem can only be solved by fixing first one, and then the other parameter. Solutions of the one-parameter problems yield sets of eigenvalues and eigenvectors.

4. One- and multi-parameter problems

If the structure is subjected to loads alone, then the eigenvalue problem is

$$(\mathbf{A} - \lambda_{kr}\mathbf{B})\, d\mathbf{u} = 0 \quad \text{where} \quad \mathbf{A} = -\mathbf{K}_e + \mathfrak{L}_e \mathbf{s}_e + \lambda_e \mathbf{B}\,. \tag{15}$$

If only initial strain load occurs:

$$(\mathbf{A} - \mu_{kr}\,\mathbf{C})\, d\mathbf{u} = 0 \quad \text{where} \quad \mathbf{A} = -\mathbf{K}_e + \mathfrak{L}_e \mathbf{s}_e + \mu_e\,\mathbf{C}\,. \tag{16}$$

Applying *the standard displacement method* to compute $\mathbf{u}_q$, $\mathbf{s}_q$, $\mathbf{u}_t$ and $\mathbf{s}_t$ and indicating the solution of the equation by the inverse of the coefficient matrix, the response due to loads in view of the stability:

$$\Delta\mathbf{u} = \Delta\lambda(\mathbf{K} - \mathbf{D})_e^{-1}\,\mathbf{q} \qquad \Delta\mathbf{s} = -\mathbf{K}_{s,e}\mathbf{G}_e\,\Delta\mathbf{u}$$

($\mathbf{K}_{s,e}$ and $\mathbf{K}_e$ being the hypermatrix assembled of the stiffness matrices of the structural elements, and the stiffness matrix of the complete structure in the pre-critical state, resp.) thus:

$$\mathbf{B} = \mathfrak{L}_e(\mathbf{K} - \mathbf{D})_e^{-1}\,\mathbf{q}\cdot\mathbf{K}_{s,e}\,\mathbf{G}_e + \mathfrak{L}_e\,\mathbf{K}_{s,e}\,\mathbf{G}_e(\mathbf{K} - \mathbf{D})_e^{-1}\,\mathbf{q}\,. \tag{17}$$

The introductory references deduce the eigenvalue problem by means of the above computing method. In fact, (15) and (17) occur in them. Response due to initial strain in view of stability loss:

$$\Delta\mathbf{u} = -\Delta\mu(\mathbf{K} - \mathbf{D})_e^{-1}\,\mathbf{G}_e^*\,\mathbf{K}_{s,e}\,\mathbf{t} \qquad \Delta\mathbf{s} = -\mathbf{K}_{s,e}\,\mathbf{G}_e\,\Delta\mathbf{u} - \Delta\mu\mathbf{K}_{s,e}\,\mathbf{t}$$

and

$$\mathbf{B} = -\mathfrak{L}_e\,\mathbf{K}_{s,e}\,\mathbf{G}_e(\mathbf{K} - \mathbf{D})_e^{-1}\,\mathbf{q}_t - \mathfrak{L}_e\,\mathbf{K}_{s,e}\,\mathbf{t} - \mathfrak{L}_e(\mathbf{K} - \mathbf{D})_e^{-1}\,\mathbf{q}_t\cdot\mathbf{K}_{s,e}\,\mathbf{G} \tag{18}$$

where

$$\mathbf{q}_t = -\mathbf{G}_e^*\,\mathbf{K}_{s,e}\,\mathbf{t}\,,$$

the nodal force system due to initial strain appearing in the equilibrium equation, called *thermal load*.

If there is stability loss even in case of *small* displacements, the formulae feature

$$\lambda_e = 0 \quad \text{and} \quad \mu_e = 0 \quad \text{and} \quad \mathbf{s}_e = 0\,.$$

The critical values λ_k and μ_k representing the solution of the one-parameter problem depend on the parameter of the selected pre-critical state:

$$\lambda_{kr} = f(\lambda_e) \quad \text{and} \quad \mu_{kr} = g(\mu_e) \, .$$

For a correct solution, the difference between the two states is negligible,

$$\lambda_{kr} = \lambda_e \quad \text{and} \quad \mu_{kr} = \mu_e \, .$$

The graph of the resulting non-linear equation is shown in Fig. 2. For a correct solution the computation has to be repeated at least twice, except for small displacements where an immediate result is obtained, because $\lambda_e = 0$.

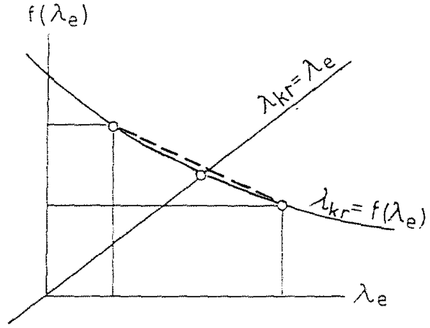


Fig. 2

The eigenvalue problem can easily be generalized for the case of an arbitrary number of effect parameters, i.e. for loads consisting of the complex of a multi-parameter load system and a multi-parameter initial strain system, characterized by vectors $\sum_i \lambda_i \mathbf{q}_i$ and $\sum_j \mu_j \mathbf{t}_j$, respectively. The one- or multi-parameter character of the load occurred in the deduction only when determining vectors $\Delta \mathbf{u}$ and $\Delta \mathbf{s}$, these being, however, given by the *linear* theory. Therefore the principle of superposition is valid, and so the eigenvalue problem for the general case can be written directly, without repeating the deduction:

$$(\mathbf{A} - \sum_i \lambda_{i,kr} \mathbf{B}_i - \sum_j \mu_{j,kr} \mathbf{C}_j) d\mathbf{u} = 0$$

where

$$\mathbf{A} = -\mathbf{K}_e + \mathfrak{S}_e \mathbf{s}_e + \sum_i \lambda_{i,e} \mathbf{B}_i + \sum_j \mu_{j,e} \mathbf{C}_j \tag{19}$$

formulae for $\mathbf{B}_i$ and $\mathbf{C}_j$ being interpreted according to item 3.

Of course, this case can also be analyzed by a set of one-parameter problems.

5. Outlines of computerization

Based on this theory, algorithm and program of the general stability analysis of bar structures have been developed at the Department of Civil Engineering Mechanics of the Technical University, Budapest, for an ODRA-1204 computer.

In its actual form, the program suits for planar bar structures with a general lay-out consisting of straight bars of constant cross-section. Nodes may be either rigid or hinged. They may be subject to arbitrary forces or couples of any direction. The bars may be subject to four different effects:

1. concentrated force perpendicular to the bar axis;
2. linearly varying distributed load perpendicular to the bar axis;
3. even temperature change;
4. uneven temperature change.

Structure of the program is seen on the flow chart of the main stages of computation.

6. Examples

The framework seen in Fig. 3 is subject to two concentrated forces. Stiffness data of the bars are identical ($EA = 3.6 \cdot 10^5$ Mp, $EJ = 4.8 \cdot 10^3$ m^{2}Mp).

Determining matrices in (15) at parameter $\lambda_e = 0$ and solving the eigenvalue problem, we obtain $\lambda_{kr} = 116.98$. The eigenvector yielding the affin presentation of the buckling mode is:

$$\mathbf{du} = \begin{bmatrix} du_{1x} \\ du_{1y} \\ d\varphi_1 \\ du_{2x} \\ du_{2y} \\ d\varphi_2 \end{bmatrix} = \begin{bmatrix} 0.68776 \\ 0.00143 \\ 0.02893 \\ 0.68903 \\ 0.06307 \\ 0.21773 \end{bmatrix}.$$

Temperature of the unloaded structure in Fig. 4 is 20°C. Let us test the structure at an inside temperature decreased to 5°C.

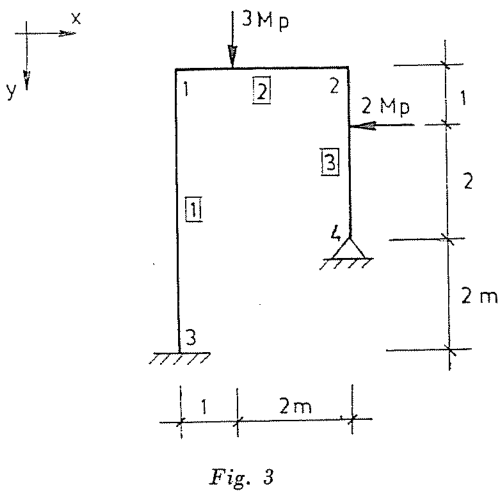


Fig. 3

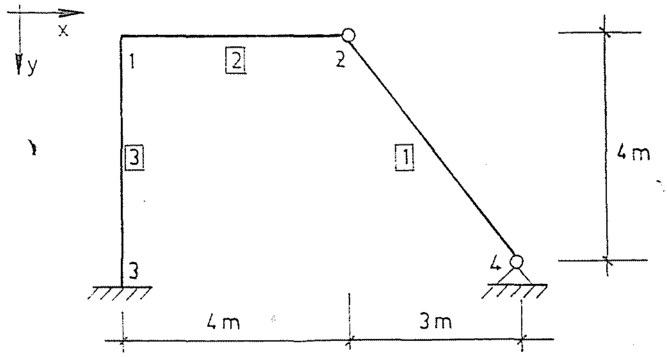


Fig. 4

The bars are of T cross section ($EA = 1.8 \cdot 10^4$ Mp, $EJ = 1.2 \cdot 10^3$ m²Mp). The effect of the change of temperature is composed of two parts. The structure is cooled to the temperature of the neutral axis, on the other hand it is subjected to a temperature gradient of 15°C (Fig. 5).

Solving the eigenvalue problem starting from $\mu_e = 0$ a value $\mu_{kr} = 119.09$ is obtained, the eigenvector being:

$$d\mathbf{u} = \begin{bmatrix} 0.40452 \\ 0.01263 \\ 0.22569 \\ 0.44088 \\ -0.42780 \\ -0.63866 \end{bmatrix}.$$

In the presented examples actually no expressed critical load occurs, the structure fails by gradual but immense deformation. In such a case the function $f(\lambda_e)$ in Fig. 2 does not intersect the straight $\lambda_{kr} = \lambda_e$, but approaches it asymptotically. Thus, the obtained critical parameters are only informative. The obtained buckling mode shows where it is most economical to strengthen he structure if necessary.

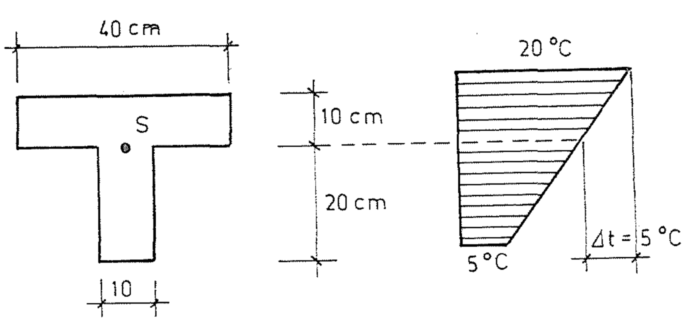
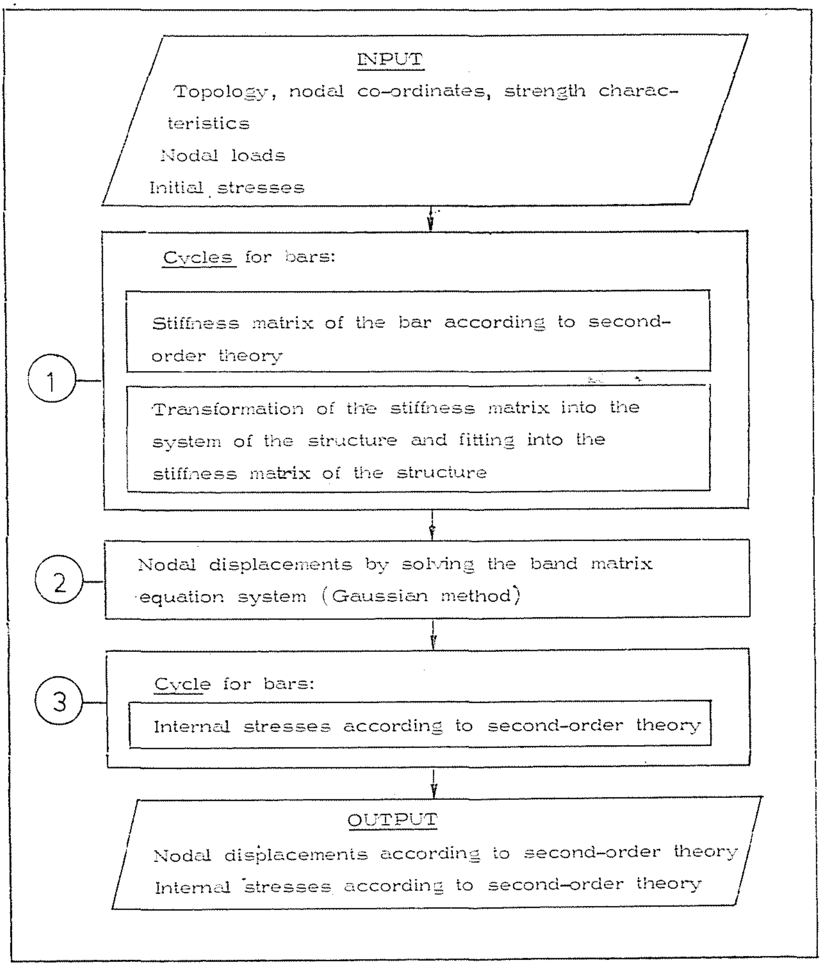


Fig. 5

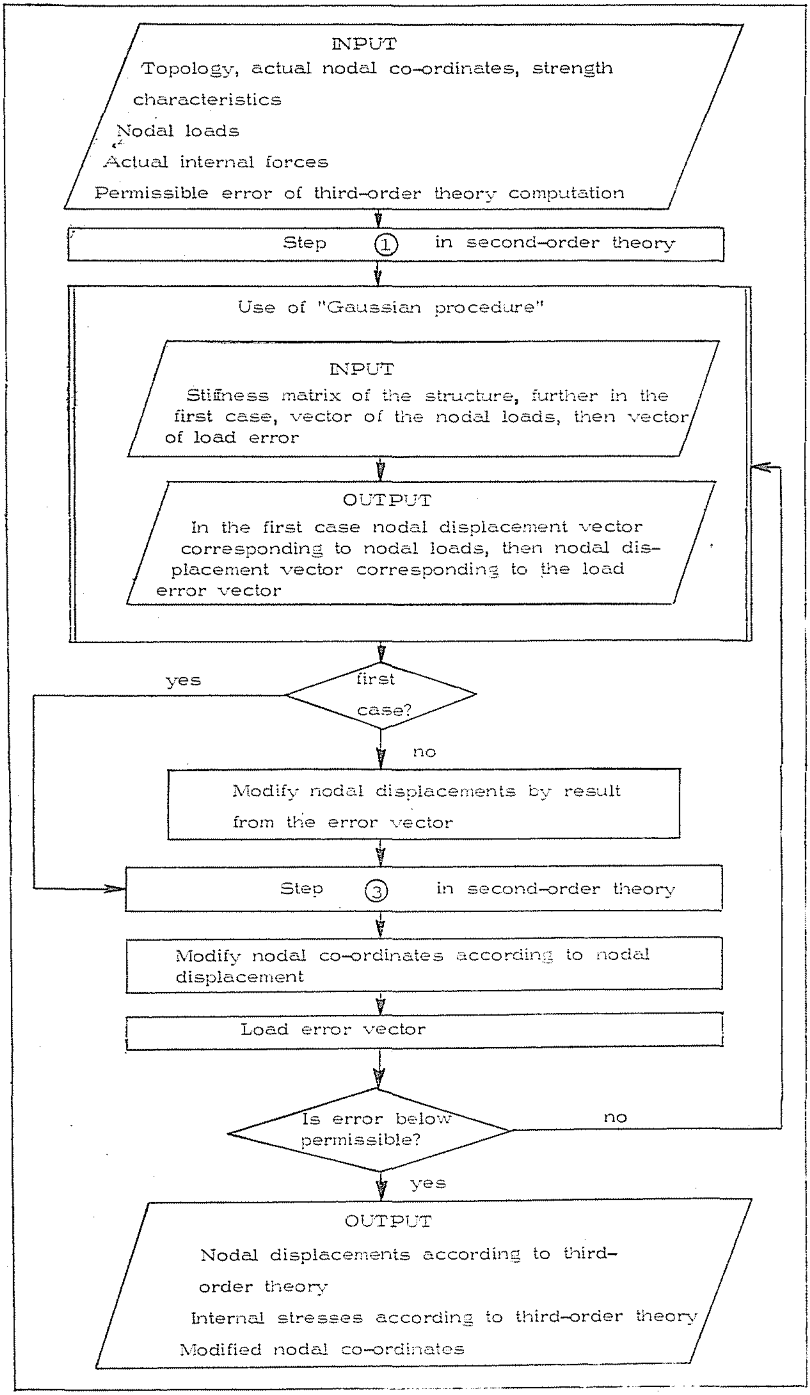
GENERAL FLOW CHART ACCORDING TO THE SECOND-ORDER THEORY



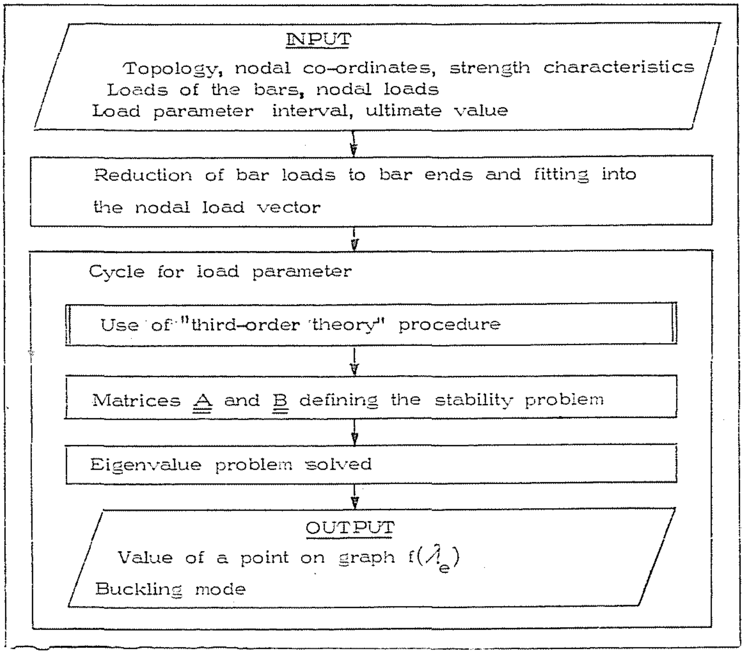
Summary

Generalized eigenvalue problem of the stability analysis of systems with several degrees of freedom is described. The system is assumed to be perfectly elastic and capable of either small or great displacements. Deformations are small. Responses may be due either to loads or to thermal stresses. This theory has been applied to develop a program written in ALGOL for an ODRA—1204 computer. Flow chart of the program and short numerical examples are presented.

GENERAL FLOW CHART ACCORDING TO THE THIRD-ORDER THEORY



GENERAL FLOW CHART OF STABILITY ANALYSIS



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