INCOMPLETE DISPLACEMENT FUNCTION FOR DERIVING THE STIFFNESS MATRIX

By

T. NAGY

Department of Civil Engineering Mechanics, Technical University, Budapest

(Received December 1, 1973)

Presented by Prof. Dr. S. KALISZKY

Introduction

An essential problem of the finite elements displacement method applied in structural analysis is the selection of the stiffness matrix. Calculation accuracy may be improved theoretically at will by using finer mesh, but practically, the abrupt growth of storage and running time needs (in general, however, according to the third power of nodes) limits the mesh fineness. The good selection of the elementary stiffness matrix, (with other words, of the displacement function) is of utmost importance, likely to much improve the accuracy for the same number of nodes. Methods are known for determining the elementary stiffness matrix in case of a displacement function containing as many free parameters as the degrees of freedom of the element. PIAN has described the method of "condensing" the stiffness matrix [6] where the number of free parameters in the displacement function is greater than the degrees of freedom of the element ($f > s$), a method contained already in recent manuals [7]. In what follows, a method will be presented for establishing the stiffness matrix in case of incomplete displacement functions i.e. containing less free parameters than the degrees of freedom of the element. This method will be applied for rectangular plane stress elements, and the obtained result will be compared to those from calculations applying other displacement functions. (Definitions will be given at the end of this paper.)

1. Methods of establishing the elementary stiffness matrix

Referring to the literature [7, 9, 10], a concise description will be given of the establishment of the elementary stiffness matrix in each of the three possible cases:

A) the displacement function has less free parameters than the degrees of freedom of the element:

$$f < s,$$

B) the displacement function has as many free parameters as the degrees of freedom of the element:

$$f = s,$$

C) the displacement function has more free parameters than the degrees of freedom of the element:

$$f > s.$$

To simplify the presentation, only elastic deformations and nodal loads will be taken into consideration.

First, let us assume the displacement function in the form

$$\underset{(m,1)}{\mathbf{u}} = \underset{(m,f)}{\mathbf{N}} \; \underset{(f,1)}{\alpha} \tag{1}$$

where

u is the displacement vector of an arbitrary point of the element;

m displacement degree of freedom of a point of the element;

Nα matrix form of the displacement function, where **N** is the matrix containing the combinations of the node co-ordinates, α being the vector of unknown coefficients.

The displacement function has to be established in dependence of nodal displacements. Let matrix $\underset{(s,f)}{\mathbf{C}}$ be defined as:

$$\underset{(s,f)}{e} = \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{e}_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{e}_n \end{bmatrix} = \underset{(s,f)}{\mathbf{C}} \; \underset{(f,1)}{\alpha} = \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{c}_i \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{c}_n \end{bmatrix} \alpha \tag{2}$$

C is seen to be a hypervector of as many blocks $\mathbf{c}_i$ as there are nodes in the element, all blocks $\mathbf{c}_i$ correspond to matrix $\underset{(m,f)}{\mathbf{N}}$, replaced by actual co-ordinates of the i-th node. Vector **e** contains nodal displacements of the element.

A) For $f < s$, matrix equation $\mathbf{e} = \mathbf{C}\alpha$ contains more scalar equations than unknowns. In this case the general inverse (corresponding to the least squares method) will be applied:

$$\underset{(f,1)}{\alpha} = \underset{(f,s)}{\bar{\mathbf{C}}^{-1}} \; \underset{(s,1)}{\mathbf{e}} \tag{3}$$

where

$$\underset{(f,s)}{\bar{\mathbf{C}}^{-1}} = (\underset{(f,s)}{\mathbf{C}^*} \underset{(s,f)}{\mathbf{C}})^{-1} \underset{(f\cdot s)}{\mathbf{C}^*} \tag{4}$$

Expressing deformations by the proper derivatives of the displacement function:

$$\epsilon = \mathbf{B}\alpha = \mathbf{B}\bar{\mathbf{C}}^{-1}\mathbf{e}\,. \tag{5}$$

The stress field of the element can be determined from the physical equations of elasticity:

$$\sigma = \mathbf{D}\epsilon = \mathbf{D}\mathbf{B}\bar{\mathbf{C}}^{-1}\mathbf{e}\,, \tag{6}$$

where $\mathbf{D}$ is the matrix of material properties.

Potential energy of the element:

$$\Pi = 1/2 \int_{(V)} \epsilon^* \sigma \, dV - \mathbf{e}^* \mathbf{q}\,, \tag{7}$$

the first term being the potential energy of internal forces. Substituting:

$$\Pi = 1/2\ \mathbf{e}^* \bar{\mathbf{C}}^{-1*} \int_{(V)} \mathbf{B}^* \mathbf{D}\mathbf{B}\, dV\, \bar{\mathbf{C}}^{-1}\mathbf{e} - \mathbf{e}^* \mathbf{q}\,. \tag{8}$$

To find the minimum potential vs. nodal displacements, the minimum condition is:

$$\frac{\partial \Pi}{\partial \mathbf{e}_i} = 0\,. \tag{9}$$

Derivation leads finally to:

$$\mathbf{K}\mathbf{e} + \mathbf{q} = 0 \tag{10}$$

with the elementary stiffness matrix

$$\underset{(s,s)}{\mathbf{K}} = \bar{\mathbf{C}}^{-1*} \int_{(V)} \mathbf{B}^* \mathbf{D}\mathbf{B}\, dV\, \bar{\mathbf{C}}^{-1}\,. \tag{11}$$

B) For $f = s$, i.e. there are as many free parameters as the degrees of freedom of the element, and $\mathbf{C}$ is a non-singular matrix:

$$\underset{(s,1)}{\alpha} = \underset{(s,s)}{\mathbf{C}^{-1}} \underset{(s,1)}{\mathbf{e}}\,. \tag{12}$$

Producing the element potential as usual:

$$\Pi = 1/2\ \mathbf{e}^* \mathbf{C}^{-1*} \int_{(V)} \mathbf{B}^* \mathbf{D}\mathbf{B}\, dV\, \mathbf{C}^{-1}\mathbf{e} - \mathbf{e}^* \mathbf{q}\,. \tag{13}$$

To find the minimum potential energy vs. nodal displacements:

$$\frac{\partial \Pi}{\partial \mathbf{e}_i} = 0,$$

leading to the stiffness matrix:

$$\mathbf{K} = \mathbf{C}^{-1*} \int \mathbf{B}^* \mathbf{D} \mathbf{B} \, dV \, \mathbf{C}^{-1} . \tag{14}$$

C) For $f > s$, i.e. there are more free parameters than the degrees of freedom of the element, matrices $\mathbf{C}$ and α will be partitioned:

$$\mathbf{e} = [\mathbf{C}_a, \mathbf{C}_b] \begin{bmatrix} \alpha_a \\ \alpha_b \end{bmatrix} \tag{15}$$

expressing α_a:

$$\mathbf{e} = \mathbf{C}_a \alpha_a + \mathbf{C}_b \alpha_b,$$

and assuming C_a to be non-singular

$$\alpha_a = \mathbf{C}_a^{-1} (\mathbf{e} - \mathbf{C}_b \alpha_b) \tag{16}$$

yields for α_a:

$$\alpha = \begin{bmatrix} \alpha_a \\ \alpha_b \end{bmatrix} = \begin{bmatrix} \mathbf{C}_a^{-1} & \mathbf{C}_a^{-1} \ \mathbf{C}_b \\ 0 & \mathbf{E} \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \alpha_b \end{bmatrix} = \mathbf{W} \bar{\mathbf{e}} \tag{17}$$

Accordingly, the displacement function will be of the form:

$$\mathbf{u} = \mathbf{N} \mathbf{W} \bar{\mathbf{e}} . \tag{18}$$

Potential energy of the element:

$$\Pi = 1/2 \bar{\mathbf{e}}^* \mathbf{W} \int_{(V)} \mathbf{B}^* \mathbf{B} \mathbf{D} \, dV \, \mathbf{W} \bar{\mathbf{e}} - \bar{\mathbf{e}}^* \bar{\mathbf{q}} , \tag{19}$$

with: $\mathbf{q} = \begin{bmatrix} \mathbf{q} \\ 0 \end{bmatrix}$.

Now, the potential energy minimum has to be sought for with respect to all elements of vector $\bar{\mathbf{e}}$, hence:

$$\left. \begin{aligned} \frac{\partial \Pi}{\partial \mathbf{e}_i} &= 0 \qquad & i &= 1, 2, \ldots, s \\ \frac{\partial \Pi}{\partial \alpha_j} &= 0 \qquad & j &= s + 1, \ldots, f \end{aligned} \right\} \tag{20}$$

The elementary stiffness matrix is of the form:

$$\underset{(f,f)}{\bar{\mathbf{K}}} = \underset{(f,f)}{\mathbf{W}} \underset{(f,f)}{\mathbf{B}^*} \int \underset{(k,k)}{\mathbf{D}} \underset{(k,f)}{\mathbf{B}}\, dV \underset{(f,f)}{\mathbf{W}} \tag{21}$$

k being the number of deformation components involved within the element.

Partitioning the equation $\underset{(f,f)}{\bar{\mathbf{K}}} \underset{(f,1)}{\bar{\mathbf{e}}} + \underset{(f,1)}{\bar{\mathbf{q}}} = 0$:

$$\begin{bmatrix} \underset{(s,s)}{\mathbf{K}_{aa}} & \underset{(s,f-s)}{\mathbf{K}_{ab}} \\ \underset{(f-s,s)}{\mathbf{K}_{ba}} & \underset{(f-s,f-s)}{\mathbf{K}_{bb}} \end{bmatrix} \begin{bmatrix} \underset{(s,1)}{\mathbf{e}} \\ \underset{(f-s,1)}{\alpha_b} \end{bmatrix} + \begin{bmatrix} \underset{(s,1)}{\mathbf{q}} \\ \underset{(f-s,1)}{\mathbf{0}} \end{bmatrix} = 0 . \tag{22}$$

Expanding:

$$\mathbf{K}_{aa}\,\mathbf{e} + \mathbf{K}_{ab}\alpha_b + \mathbf{q} = 0 ;$$

$$\mathbf{K}_{ba}\,\mathbf{e} + \mathbf{K}_{bb}\,\alpha_b = 0 . \tag{23}$$

Expressing α_b from the latter equation and resubstituting:

$$(\mathbf{K}_{aa} - \mathbf{K}_{ab}\,\mathbf{K}_{bb}^{-1}\,\mathbf{K}_{ba})\,\mathbf{e} + \mathbf{q} = 0 \tag{24}$$

leading to the modified elementary stiffness matrix:

$$\underset{(s,s)}{\mathbf{K}_m} = (\underset{(s,s)}{\mathbf{K}_{aa}} - \underset{(s,f-s)}{\mathbf{K}_{ab}} \underset{(f-s,f-s)}{\mathbf{K}_{bb}^{-1}} \underset{(f-s,s)}{\mathbf{K}_{ba}}) . \tag{25}$$

2. Comparison of stiffness matrices of a rectangular plane stress element

In 1973, Miss Gy. Horváth, graduating student in civil engineering, presented in her diploma work actual computations to establish various elementary stiffness matrices for a rectangular plane stress element [3]. Tested displacement functions:

I $\quad u_x = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$

$\quad\; u_y = \alpha_5 + \alpha_6 x + \alpha_7 y + \alpha_8 xy$

II $\quad u_x = \alpha_1 + \alpha_2\, x + \alpha_3\, y + \alpha_4\, xy + \alpha_9 \sin \frac{x\pi}{a} \sin \frac{y\pi}{b}$

$\quad\; u_y = \alpha_5 + \alpha_6\, x + \alpha_7\, y + \alpha_8\, xy + \alpha_{10} \sin \frac{x\pi}{a} \sin \frac{y\pi}{b}$

$$\text{III}\quad u_x = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy + \alpha_9 \left(\frac{x}{a} - \frac{x^2}{a^2}\right) + \alpha_{10} \left(\frac{y}{b} - \frac{y^2}{b^2}\right)$$

$$u_y = \alpha_5 + \alpha_6 x + \alpha_7 y + \alpha_8 xy + \alpha_{11} \left(\frac{x}{a} - \frac{x^2}{a^2}\right) + \alpha_{12} \left(\frac{y}{b} - \frac{y^2}{b^2}\right)$$

$$\text{IV}\quad u_x = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy + \alpha_9 x^2 + \alpha_{10} y^2$$

$$u_y = \alpha_5 + \alpha_6 x + \alpha_7 y + \alpha_8 xy + \alpha_{11} x^2 + \alpha_{12} y^2$$

$$\text{V}\quad u_x = \alpha_1 + \alpha_2 x + \alpha_3 y$$

$$u_y = \alpha_4 + \alpha_5 x + \alpha_6 y\,.$$

I is the simplest displacement function corresponding to the linear displacement field. In the cases of II, III and IV, $f > s$: II and III meet boundary conditions (the excess term being a so-called disturbing function), while displacement function IV is a complete second-order polynomial. III is known (7) to be a displacement function corresponding to linear stress field. In case of displacement function V, $f < s$.

Computations showed the elementary stiffness matrix with either of the four displacement functions (II, III, IV, V) deviating from the fundamental case, to differ from the elementary stiffness matrix by a matrix of identical structure:

$$\mathbf{K}_m = \mathbf{K}_a + \mathbf{M} \tag{26}$$

where:

$$\mathbf{M} = z \begin{bmatrix} -x & & +x & & -x & & +x & \\ & -y & & +y & & -y & & +y \\ +x & & -x & & +x & & -x & \\ & +y & & -y & & +y & & -y \\ -x & & +x & & -x & & +x & \\ & -y & & +y & & -y & & +y \\ +x & & -x & & +x & & -x & \\ & +y & & -y & & +y & & -y \end{bmatrix} \tag{27}$$

x, y and z values for each displacement function are tabulated as:

Type	x	y	z
II	$\frac{+1}{2\beta^{-1}+\beta(1-\nu)}$	$\frac{+1}{2\beta+\beta^{-1}(1-\nu)}$	$\frac{32E(1+\nu)\delta}{\pi^6(1-\nu)}$
III	$+\nu^2\beta+\frac{1-\nu}{2}\beta^{-1}$	$+\nu^2\beta^{-1}+\frac{1-\nu}{2}\beta$	$\frac{E\delta}{12(1-\nu^2)}$
IV	$+\nu^2\beta+\frac{(1-\nu)^2}{4}\beta^{-1}$	$+\nu^2\beta^{-1}+\frac{(1-\nu)^2}{4}$	$\frac{E\delta}{12(1-\nu^2)}$
V	$+\beta+\frac{1-\nu}{2}\beta^{-1}$	$+\beta^{-1}+\frac{1-\nu}{2}$	$\frac{E\delta}{12(1-\nu^2)}$

δ being the element thickness, and $\beta = b/a$ the side ratio.

The identical structure of the excess term of stiffness matrices related to different shape functions can also theoretically be verified. Namely, it can be proved that for any rectangular plane element, all elements of the stiffness matrix can be produced as a function of the upper left block elements, and in case of plane stress state, as a function of elements k_{11} and k_{22} [8].

Although a general method is known for comparing different displacement functions [4], it seemed simpler to compare actual problem outcomes. The selected structure is shown in Fig. 1. The vertical displacement of point C, taking shear effect into consideration, amounts to $e = 0.0512$ m.

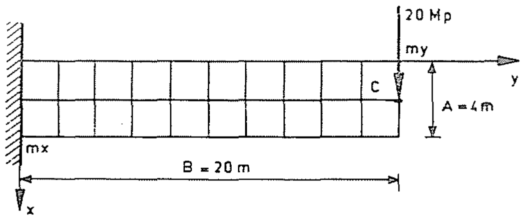

Fig. 1. Thickness $\delta = 0.1$ [m]; cross-section area $F = 0.4$ [m^2]; moment of inertia $I_x = 0.1 \cdot 4^3$ 12 [m^4]; shape factor $\varrho = 1.2$; Poisson's ratio $\nu = 0$; Young's modulus $E = 10^6$ [Mp/m^2]; shear modulus $G = 5 \cdot 10^5$ [Mp/m^2]

Outcomes with different mesh finenesses are shown in Fig. 2.

The results have led to the following conclusions:

— in the tested problem, the result obtained with the "incomplete" displacement function is converging to the theoretical value, although "from above";

— in plane stress element problems, the best approximation is obtained with the displacement function assuming linear stress distribution, in spite of the lack of continuity along the edges of such an element;

— selection of the displacement function fundamentally affects the accuracy, hence it merits to be considered.

The incomplete displacement function meets but partly the usual convergence requirements, among them the conditions for rigid-body motions and for the elementary stress fields (it corresponds to a constant stress field) but fails in nodal boundary conditions. Remind, however, that the quoted convergence conditions have been established empirically rather than theoretically, and they are only known to generally bring about convergence in case of fine mesh. Among recent research reports [1] some present displacement functions not perfectly meeting boundary conditions for shells.

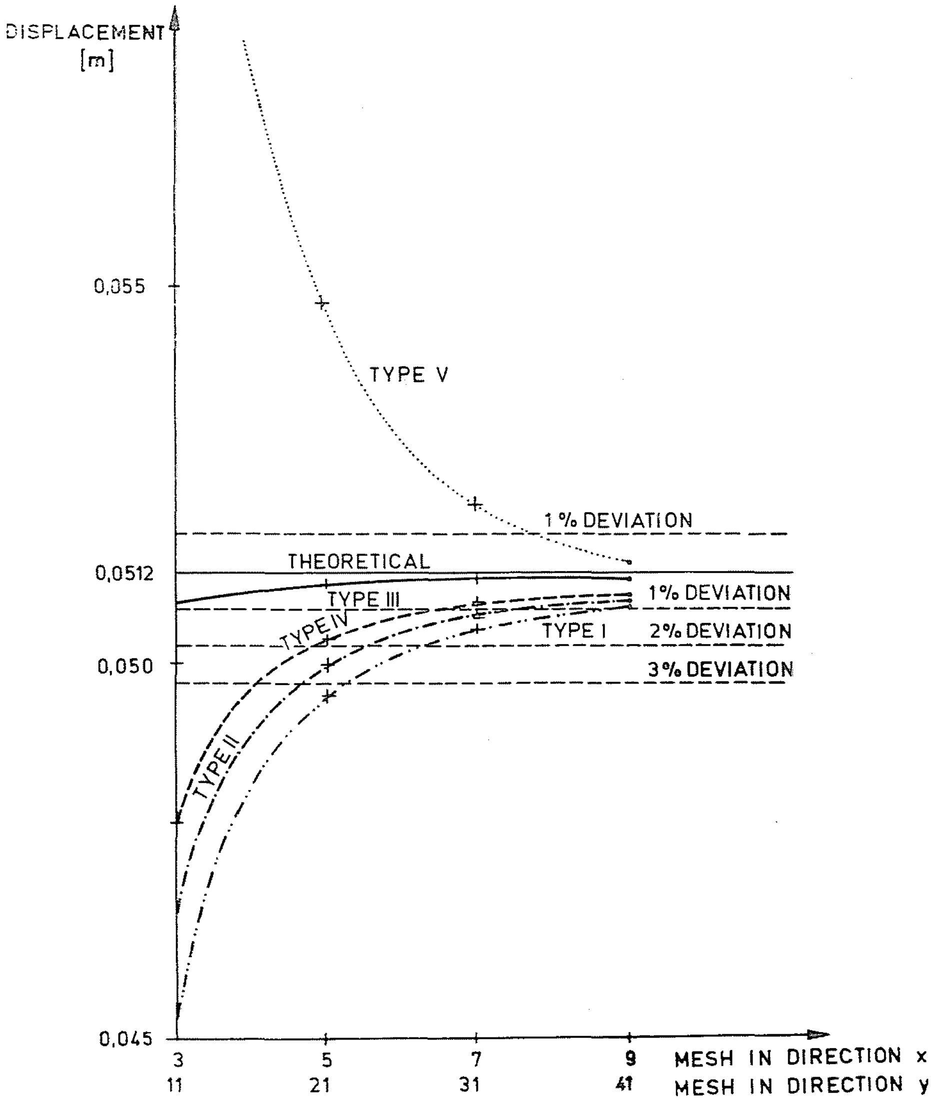

Fig. 2

It is felt advisable to continue the research and to examine the suitability of elementary stiffness matrices derived from incomplete displacement functions for other element types.

Summary

Establishment of the elementary stiffness matrix in case of displacement functions containing as many free parameters as, less than, and more than, the element's degrees of freedom, respectively. Stiffness matrices of a rectangular plane stress element, established with five different displacement functions are compared, together with the resulting convergence. Calculations showed the incomplete displacement function to yield results also converging to the theoretical value, although "from above".

References

1. Brebbia, C. A.: A Comparison of Recent Shallow Shell Finite Elements (Int. J. of Mech. Sci. Vol. 12, 1972.)
2. Hooley, R. F.—Hibbert, P. D.: Bounding Plane Stress Solutions by Finite Elements. (Journ. of the Struct. Div. -Proc. ASCE, 1966.)
3. Horváth, Gy.: Plane Stress Element Analysis by the Displacement Method of Finite Elements.* Diploma work, Budapest, 1973. Manuscript.
4. Khanna, J.: Criterion for Selecting Stiffness Matrix (AIAA J ournal, Vol. 3. No. 10. Oct. 1965.)
5. Nagy, T.—Havas, I.: On a Suitable Elementary Stiffness Matrix of Rectangular Plane Stress Element.* (Építés— Építészettudomány, Vol. IV. No 3—4. 1972.)
6. Pian, T. H. H.: Derivation of Element Stiffness Matrix. (AIAA Journal, Vol. 2. No. 3. March. 1964.)
7. Przemieniecki, J. S.: Theory of Matrix Structural Analysis. McGraw-Hill, New York etc. 1968.
8. Shaposhnikov, N. N.: Some Properties of the Stiffness Matrix of Rectangular Elements and their Application in the Finite Elements Method. (In Russian). MIIT Reports No. 422, Moskow, 1972.
9. Szabó, J.—Szilágyi, Gy.: Structural Analysis IV.* University notebook. Budapest, 1971.
10. Zienkiewicz, O. C.: The Finite Element Method in Structural and Continuum Mechanics. McGraw-Hill, London 1967.

Senior Assist. Dr. Tamás Nagy, H-1521. Budapest

* In Hungarian