

ANALYSIS OF FLEXURAL STIFFNESS OF UNIFORMLY LOADED STEEL MEMBERS ON THE YIELD PLATEAU

by

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1. Stability analysis in the plastic range

There exist two different theories for determining the critical load of a column buckling in the plastic range: the Engesser – Kármán and the Engesser – Shanley theories. Both express the flexural stiffness, the moment-curvature relationship similarly to that for the elastic range, taking the variation of the stress-strain diagram $\sigma-\varepsilon$ in the range of proportional limit σ_a to static yield point σ_F to obtain the buckling load.

Figs 1a and b show the $\sigma-\varepsilon$ and $\sigma-\lambda$ diagrams of aluminium alloy, a material having no sharply defined yield point.

For structural steels, having a sharply defined yield point, Figs 1c and d show the $\sigma-\varepsilon$ diagram and the $\sigma_{cr}-\lambda$ relationship, respectively [1]; at the static yield stress σ_F , the tangent modulus E_t is reduced steadily to become zero at the yield point, i. e. where the $\sigma_{cr}-\lambda$ diagram intersects yield strength at M . Along the $\sigma-\varepsilon$ diagram for σ within the strain-hardening range, critical stresses beyond the yield point may be obtained to result in the branch $M-P-N$ of the column curve.

Fig. 1d shows results of continuously loaded specimens of structural steel (e. g. KÁRMÁN [2]). For short, stocky columns ($\lambda = l/i \sim 30$) the buckling strength σ_{cr} is far above the yield strength σ_F and test curves σ_a-M-N are of the form according to the dashed line, rather than of the form $M-P-N$.

This deviation between test results and theory does not appear in practice, since columns of slenderness $\lambda = 30$ are exceptional.

This deviation becomes a problem when examining the lateral torsional plastic buckling of beams under uniform moment.

Taking the lateral torsional buckling of beams under uniform moment as the buckling of compression flange (the web does not restrain the flange of the beam section from buckling) after the complete yield of the compression flanges according to the previously mentioned theories the beam will buckle laterally — irrespective of length —, thus, in fact, beams under uniform moment will buckle with the development of plastic moment M_t , independent of the beam length.

This condition has not been verified experimentally (Fig. 2); but experience has shown that for "long" beams the range of plastic rotation where the plastic moment M_t [3] is supported is less than for "short" beams.

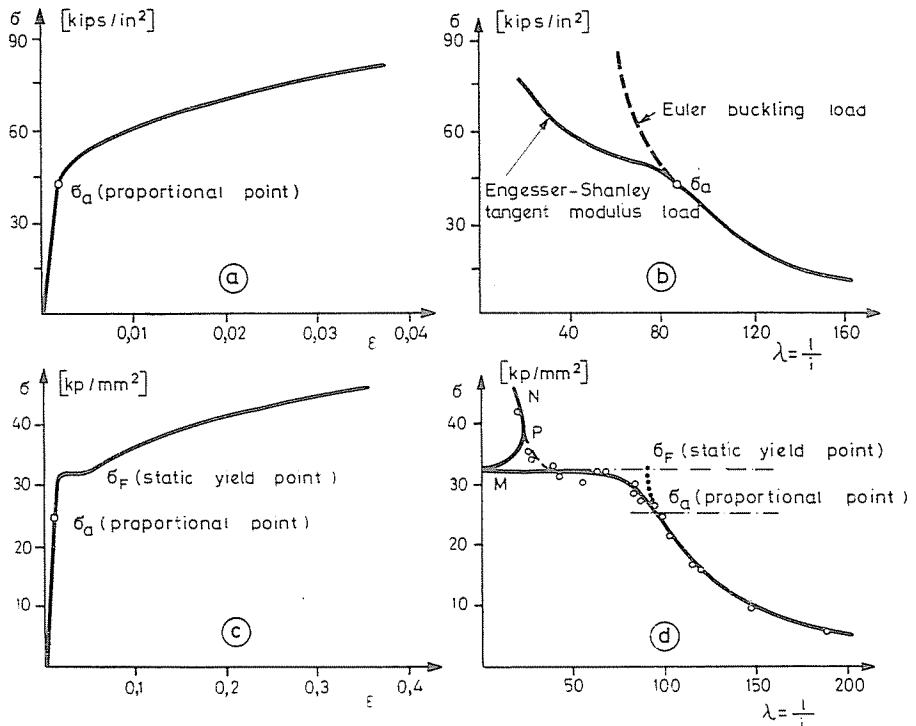


Fig. 1

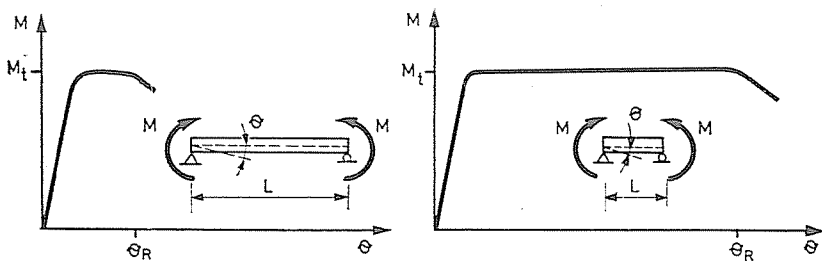


Fig. 2

Thus a physical model was required likely to give the relationship $\sigma_{cr} - \lambda$ corresponding to the dashed line in Fig. 1d. The model has been established according to the statement of F. BLEICH (p. 22) [1]: "The explanation may be that, owing to variations in the homogeneity of the material, yielding does not occur simultaneously over the entire cross section and that, in one region

strain hardening already has taken place while in other regions yielding begins. In this way it is possible that bending, which should start at the yield point, is more or less delayed, and buckling takes place at average stresses above the yield point stress."

2. Yielded steel behaviour

At yield point, characteristic, regular line systems appear on the surface of steel specimens indicating that plastic strain is not uniform at yield, but is concentrated in thin bands, their traces on the surface being referred to as "Lüders—Hartmann lines". These bands appear abruptly at the yield point, they densify and broaden with increasing load.

In his famous book [4] NÁDAI summarized assumptions and test results for the yield of steel.

ENDRE REUSS has also dealt with the Lüders—Hartmann lines developed in a twisted, round bar.

The $L-H$ lines on the surface of steel specimens indicate that on the yield plateau the material is inhomogeneous, having discontinuous characteristics.

There is a finite jump in strain from the yield (or slip) strain to the strain-hardening (or resistance) strain.

BEEDLE, L. S. and TALL, L. [6] examined the problem whether the stress to create a slip plane is higher than the stress to maintain it. At a strain rate near zero ($1.0 \mu \cdot \text{sec}^{-1}$) the ratio of "dynamic" to "static" yield stress was found to be close to $i = 1.05$.

3. Discontinuous stress-strain laws

The idealized elasto-plastic stress-strain diagram (Fig. 3) of steel is well known. (The continuous line indicates the post-elastic strains occurring above the proportional limit σ_a , whilst the dashed line shows the idealized elasto-plastic stress-strain condition.)

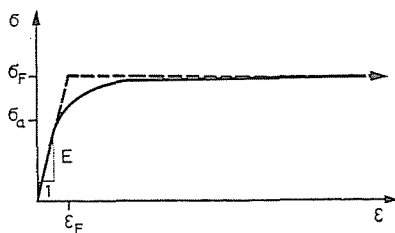


Fig. 3

The discontinuous stress-strain laws — after Nádai — are shown in Fig. 4. The stress-strain diagram allows for the discontinuity of yield by averaging strains ε in the range ε_F to $\bar{\varepsilon} = s \cdot \varepsilon_F$ rather than taking actual ε values.

In case of a single yield line, strain ε of column of length L is shown in Fig. 5. Be the length in yield state ΦL , that in elastic condition $(1-\Phi)L$, then

$$L - \delta L = L - (1 - \Phi)L \cdot \varepsilon_F - \Phi L \cdot \bar{\varepsilon} \quad (1)$$

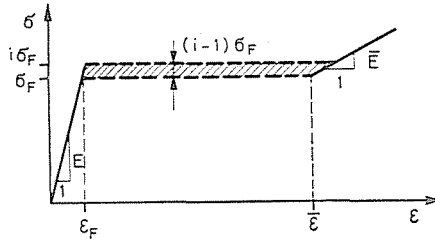


Fig. 4

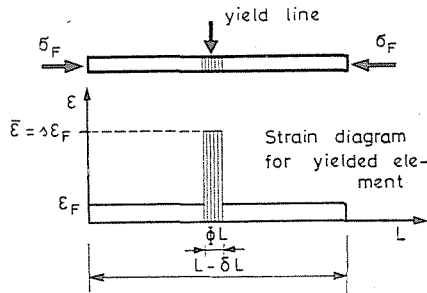


Fig. 5

The over-all strain:

$$\varepsilon = \frac{\delta L}{L} = (1 - \Phi) \cdot \varepsilon_F + \Phi s \varepsilon_F \quad (2)$$

hence:

$$\Phi = \frac{\varepsilon / \varepsilon_F - 1}{s - 1} \quad (3)$$

If the load is increased above $i\sigma_F$, the relationship between deformations and stress increase is expressed by the strain-hardening modulus \bar{E} .

4. Flexural stiffness of members in discontinuous yield state for the cases of bending during and after axial deformation

For the subsequent theoretical and experimental studies, only the flexural stiffness affected by axial load and moment will be considered. As mentioned before, to the lateral torsional buckling of beam in the plastic range, upon the

average strain $\varepsilon (\varepsilon > \varepsilon_F)$ in the compression flange a strain due to the virtual lateral disturbance moment is superposed.

Now, an "over-all modulus" will be sought for, describing the flexural stiffness of idealized discontinuous elasto-strain-hardening materials undergoing strain $\varepsilon > \varepsilon_F$ due to combined axial load and disturbance moment, considering that the disturbance moment may increase the effect of the axial load.

To this aim, response of the member in compression or tension on the yield plateau to small disturbance moments will be studied.

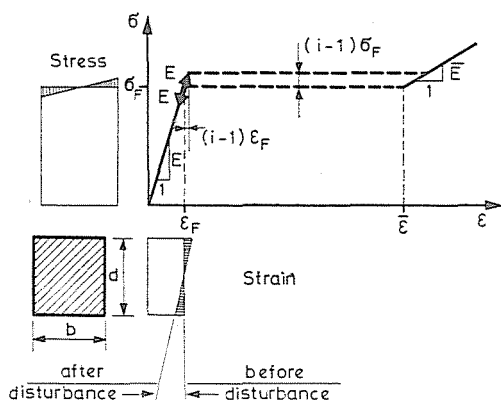


Fig. 6

The "over-all modulus" will be defined from the differential behaviour of the two parts of steel after yield, namely those characterized by strain at onset of strain-hardening and yield strain, respectively. Again, it will be examined, how "blending", i. e. distribution along the length of specimen affects flexural stiffness.

Consider an axially loaded column of rectangular cross section. The average strain ε of the column is that understood between the ends of the member.

a) The behaviour of the parts, in which the yield strain is developed (Fig. 6):

If no stresses $> i\sigma_F$ are produced by disturbance moment anywhere then no new yield lines appear and the parts will be in the elastic range, thus, for these parts the moment-curvature is:

$$M = E \cdot J_x \cdot \kappa \tag{4}$$

The maximum disturbance moment satisfying this condition:

$$M \leq (i-1)M_F \tag{5}$$

b) The behaviour of strain-hardening parts (yield lines) is developed in which strain-hardening strain $\bar{\varepsilon} = s\varepsilon_F$.

The axial load and the disturbance moment may be taken about the $N-N$ axis (Fig. 7). Assuming the disturbance moment not to increase the axial load, a relationship similar to the moment-curvature valid in the elastic range is obtained, the elastic modulus E being replaced by the "reduced elastic modulus" T :

$$M = T \cdot J_x \cdot \kappa. \quad (6)$$

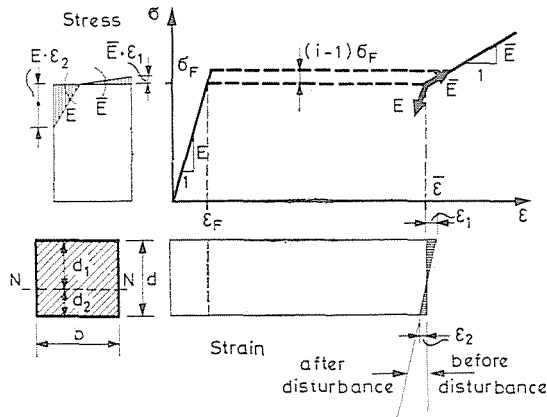


Fig. 7

The above considerations seem to be similar to the Engesser—Kármán theory, the latter, however, assumes the material of the column to be homogeneous, though behaving differently during loading than unloading; the present theorem is valid only to the yield lines indicating strain-hardening.

The ratio of reduced to elastic modulus for a rectangle cross section:

$$m_K = \frac{T}{E} = \frac{4E \cdot \bar{\varepsilon}}{(\sqrt{E} + \sqrt{\bar{E}})^2} = \left[\frac{2}{1 + \sqrt{h}} \right]^2. \quad (7)$$

In view of the discontinuous stress-strain laws, at first it seems as if only the previous solution could be used for determining the moment-curvature of the yield lines, since it is preassumed of the physical model that in the yield lines a strain hardening strain $\bar{\varepsilon}$ has developed, without any further axial strain in their plane until the entire region under yield stress σ_F has reached the strain-hardening strain $\bar{\varepsilon}$.

Nevertheless stresses $(i-1)\sigma_F$ inserted between the preexisting yield lines (planes) leading to ever more yield lines produce additional strains:

$$\Delta\varepsilon = \frac{(i-1)\sigma_F}{\bar{E}} = (i-1)h \cdot \varepsilon_F. \tag{8}$$

These strain excesses enable the disturbance moment to permit no unloading in the cross section belonging to the yield line, thus, all the yield planes in their cross section undergo an increase of stress (Fig. 8). Thus, the moment-curvature relationship:

$$M = \bar{E} \cdot J_x \cdot \kappa. \tag{9}$$

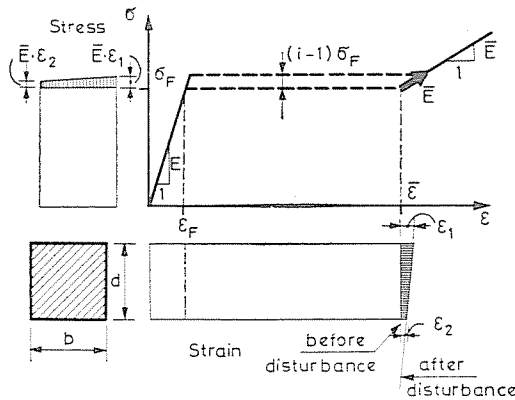


Fig. 8

This consideration, allowing for the possibility of axial load increase in the yield line cross section, is similar to the Engesser—Shanley theory, it should be emphasized, however, that this kind of flexural stiffness holds only for the yield lines. This is the so-called “tangent modulus” solution.

Here

$$m_T = \frac{\bar{E}}{E} = \frac{1}{h} \tag{10}$$

is the ratio of tangent (strain-hardening) to elastic modulus.

Let us see now how the distribution, the “blending” of elastic parts and yield lines along the length of the column affects the moment-curvature relationship valid throughout the column length.

From experiments it has long been known that yield lines begin to develop in tensile specimens at the ends, whilst for compression specimens the first yield lines develop at mid-length. Thus, the moment-curvature differs be-

tween compression and tensile members. Nevertheless the assumption usual for steel materials, namely that steel behaves alike in compression and in tension is reasonable also to apply now, possible by assuming the distribution of the yield lines along the length of the column.

Thus, the curvature of the entire column, similarly to the average strain ε :

$$\alpha_{\text{avg}} = (1 - \Phi) \cdot \alpha_{\text{el}} + \Phi \cdot \alpha_{\text{pl}}. \quad (11)$$

Two solutions have been developed for strain-hardening parts (yield lines):

1. Applying the "reduced modulus"

$$\frac{M}{E_{0K} J_x} = (1 - \Phi) \frac{M}{E J_x} + \Phi \frac{M}{E_K \cdot J_x}. \quad (12)$$

Be

$$m_{0K} = \frac{E_{0K}}{E} \quad (13)$$

$$m_{0K} = \frac{1}{1 + \Phi \left[\left(\frac{1 + \sqrt{h}}{2} \right)^2 - 1 \right]}. \quad (14)$$

2. Applying the "tangent modulus"

$$\frac{M}{E_{0T} J_x} = (1 - \Phi) \frac{M}{E J_x} + \Phi \frac{M}{E_T \cdot J_x}. \quad (15)$$

Be

$$m_{0T} = \frac{E_{0T}}{E} \quad (16)$$

$$m_{0T} = \frac{1}{1 + \Phi(h - 1)}. \quad (17)$$

For $\Phi = 0$ (the steel material is elastic) $m_{0K} = m_{0T} = 1$; for $\Phi = 1$ (the steel material is in full yield) $m_{0K} = m_K$ and $m_{0T} = m_T$. In the range $0 < \Phi < 1$, m_0 is zero, if $m_K = m_T = 0$, consequently $h = \infty$, i. e. the strain-hardening modulus is zero (idealized elasto-plastic material).

Making use of experimental results ($h = 32.2$; $s = 12.7$; $\varepsilon_F = 0.00115$) Fig. 9 shows the form of Eq. (17).

Applying Eqs. (14) or (17) a relationship $\sigma_{cr} - \lambda$ can be written for the compression column, which describes the condition marked with dashed line in Fig. 1d.

Consequently, a yield condition does not *a priori* imply that a buckling condition also exists.

5. Tests to determine the flexural stiffness of strain-hardening steel members under axial load and disturbance moment

In order to verify the relationships in item 4, tensile tests were made applying axial loads to prevent stability problems from even indirectly arising.

The axial tension caused in the column an average strain $\varepsilon > \varepsilon_F$. Subsequently, moments were introduced at both ends applying a disturbance and the effect of bending moment permitted to conclude on the flexural stiffness of the member in yield.

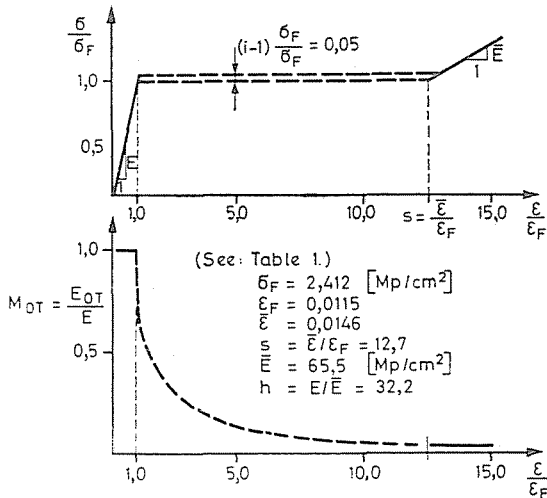


Fig. 9

5.1. The testing device

To this aim a tensile testing machine has been constructed, for applying bending moments on the ends of columns in various strain conditions.

Both the tensile load and the bending moment were applied by taking the elastic response of the load system into consideration and thus, after 15 mins of rest left for the static yield stress σ_F to develop, the disturbance bending moment could be introduced at a predefined strain.

The specimens cut out from structural steel have been taken from nearly the same place to provide as homogeneous material properties as possible.

The ends of the specimens were supplied with wedges of steel K 1 to transmit the tension (phase I) as shown in Fig. 10; load-transmitting wedges are on the lower shafts of two rigid *T* members; by rotating the horizontal shafts of *T* around point *C*, the tension was applied. At the predetermined

plastic strain value, bending moment was applied at both ends of the column (phase II; Fig. 11).

The loading apparatus is shown in Figs 12a and b.

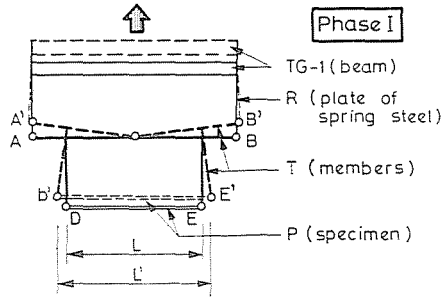


Fig. 10

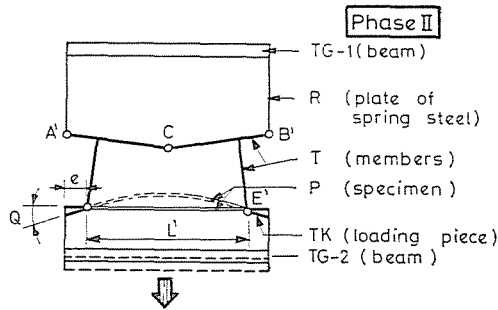


Fig. 11

5.2. Test method

The tension H has been determined with electric resistance strain-gauges type Huggenberger BP 2/120 p. mounted on spring steel plates R (Fig. 10).

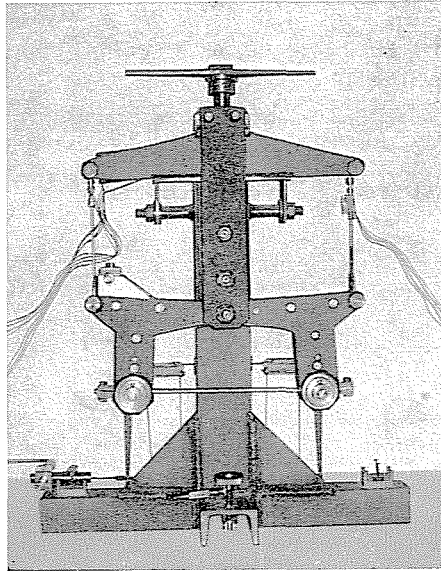
The extension $\Delta L = L' - L$ in the specimen due to tension H has been determined from the horizontal displacement of the lower shafts of T at points D and E using two inductive transmitters W 10.

The value of the bending moment has been obtained by means of electric resistance strain gauges type Huggenberger BP 2/120 p.

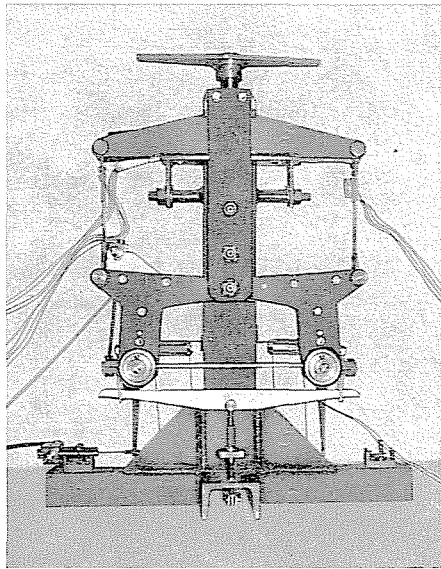
The rotation θ has been determined from the vertical displacement of lever arm e_1 and tensile force P using an inductive transmitter W 10.

The extension ΔL of the specimen versus tensile load H was recorded by a XY-recorder type EFKI.

The rotation of end cross section θ versus bending moment $M = P \cdot e$ was recorded by a Honeywell XYY' recorder.



a



b

Figs. 12a and 12b

5.3. Experimental

After having precisely measured the specimen, it was placed into the testing machine and after having carefully balanced the instruments, the tension H was applied by lifting the beam $TG-1$ by means of a spindle screw.

At the predesigned strain ($\varepsilon > \varepsilon_F$) a rest of 15 mins has been left for strain recovery and then the bending moment was applied in small increments, always with an interval of 15 mins between.

The applied maximum bending moment, still not producing a new yield-line:

$$M \leq (i-1) M_F = (i-1) \frac{2EJ_x}{d} \varepsilon_F. \quad (18)$$

The test results on specimens marked B7 and B19 are shown in Fig. 13.

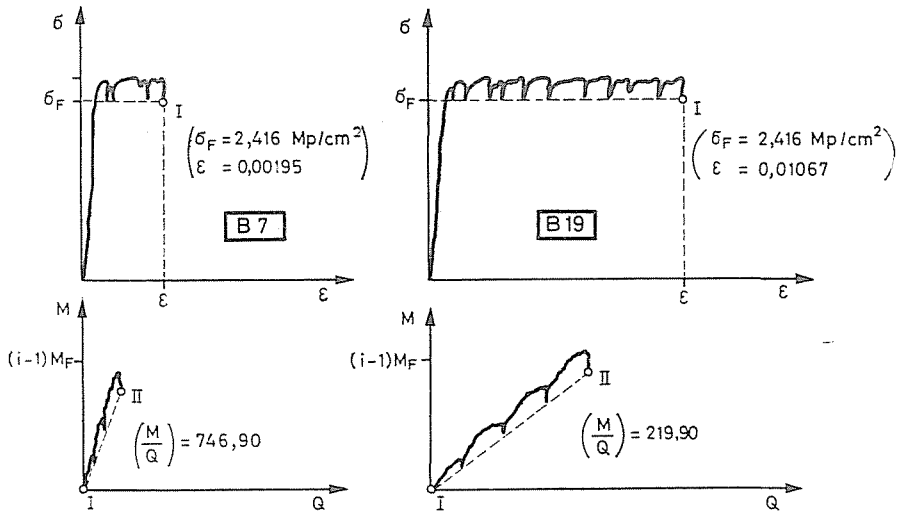


Fig. 13

5.4. Discussion of test results

The rotations due to the disturbance moment M/Θ in end cross sections of specimens subject to a given tension and axial extension are shown in Fig. 14 as a function of strain ε .

The stresses and displacements of the specimens are given in Fig. 15. The bending moment and the rotation of the end cross section are related by the so-called "stiffness stability functions" [7] taking the axial load into consideration; in case of tension these functions contain hyperbolic functions since the applied moments at points D and E are equal, thus

$$\frac{M_D}{\Theta_D} = \frac{M_E}{\Theta_E} = \frac{E^* J_x}{l} S(1+C) = \frac{E^* J_x}{l} S', \quad (19)$$

equation used to evaluate the test results, where $E^* = m_p \cdot E$ is an "over-all modulus" belonging to a given strain ε ; S' has been determined as a function of ϱ

$$\varrho = \frac{P}{P_E} \tag{20}$$

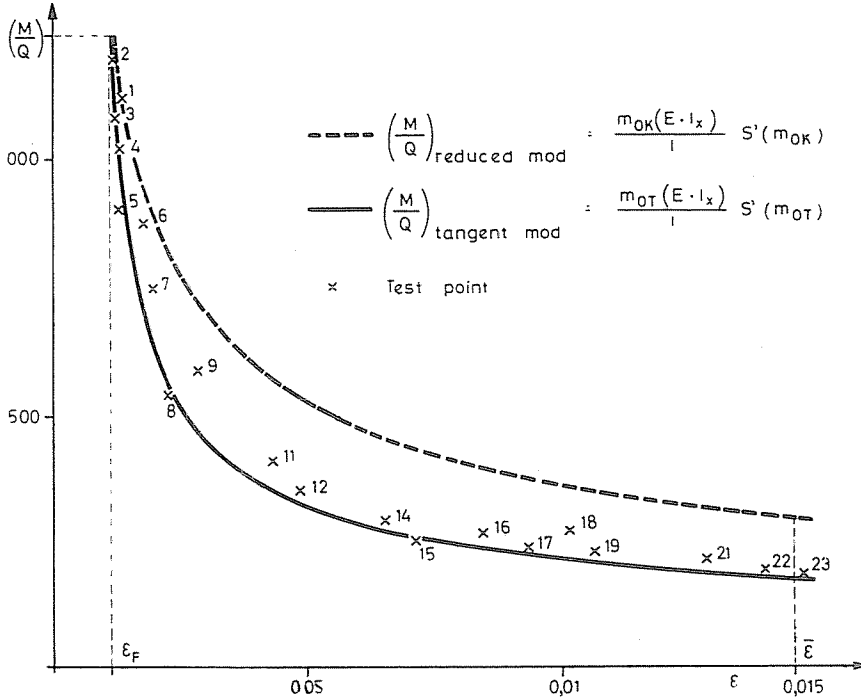


Fig. 14

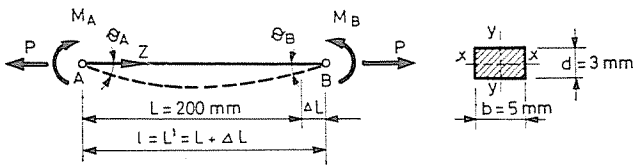


Fig. 15

The M/θ ratio obtained on the specimen in strain condition ε was intended to determine the effective value of m_p ; since, however, m_p occurs also in the stability function S' , it is difficult to directly determine, so the test results have been processed by simple iteration in a desk-computer. The obtained m_p values are shown in Fig. 16 as a function of $\varepsilon/\varepsilon_F$. Test data and test results are compiled in Tables 1 and 2. (Specimens A and B were used for determining the material properties, and the bending moment, respectively.)

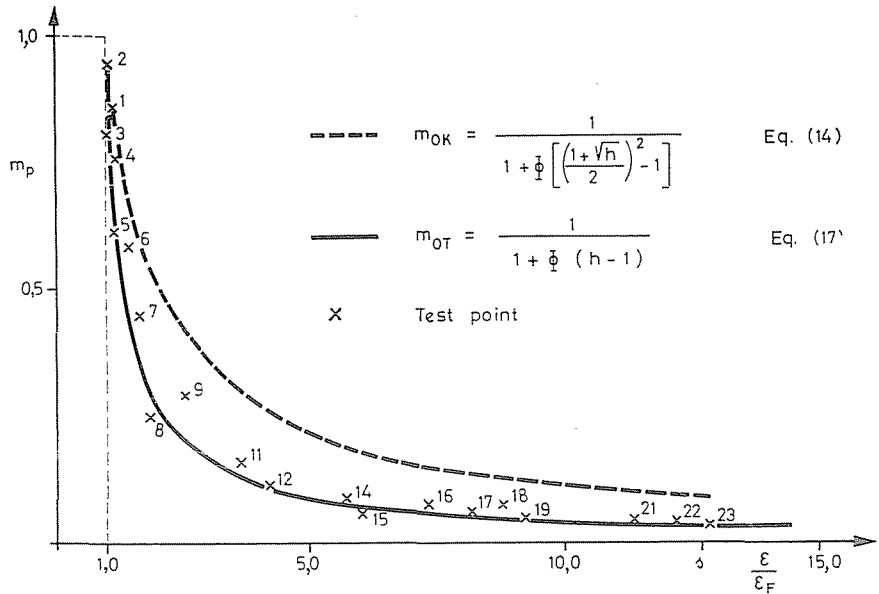


Fig. 16

6. Conclusions

From the observation of Figs 14 and 16 and Tables 1 and 2 it can be stated that the experiments verified the assumption of uniformly yielding steel members for the case of tension. The discontinuous yield concept is applied for inelastic stability problems [8] concerning the lateral buckling of beams under uniform moment and under moment gradient as well as the inelastic local buckling.

Table I

Test No.	L	b	d	σ_F	$\epsilon_F = \frac{\sigma_F}{E}$	$\bar{\epsilon}$	$s = \frac{\bar{\epsilon}}{\epsilon_F}$	\bar{E}	$h = \frac{E}{\bar{E}}$
	mm	mm	mm	Mp/cm ²	—	—	—	Mp/cm ²	—
A. 1	200.05	5.04	3.03	2.461	0.00117	0.0151	12.9	68.4	30.7
A. 2	200.02	5.03	2.98	2.290	0.00109	0.0138	12.7	63.2	33.2
A. 3	199.98	4.98	3.01	2.397	0.00114	0.0144	12.6	70.1	30.0
A. 4	200.01	5.02	3.04	2.385	0.00113	0.0140	12.3	67.3	31.2
A. 5	200.03	4.99	2.97	2.490	0.00119	0.0152	12.8	59.2	35.5
A. 6	199.97	5.05	3.05	2.450	0.00116	0.0150	13.0	64.9	32.4
				2.412	0.00115	0.0146	12.7	65.5	32.2

Table II

Test No.	L	b	d	σ_F	$\varepsilon_F = \frac{\sigma_F}{E}$	ε	$\frac{\varepsilon}{\varepsilon_F}$	$\left(\frac{M}{\Theta}\right)$	m_p
	mm	mm	mm	Mp/cm ²	—	—	—	—	—
B. 1	200.02	5.05	3.02	2.437	0.00116	0.00133	1.146	1124.30	0.860
B. 2	200.05	5.02	3.05	2.416	0.00115	0.00116	1.010	1197.00	0.940
B. 3	199.97	5.04	3.01	2.461	0.00117	0.00119	1.018	1070.05	0.801
B. 4	200.01	5.01	2.98	2.391	0.00114	0.00131	1.148	1036.40	0.758
B. 5	200.05	4.99	3.03	2.473	0.00118	0.00124	1.050	907.20	0.613
B. 6	200.02	5.03	3.01	2.424	0.00116	0.00172	1.483	878.00	0.584
B. 7	200.04	5.01	3.00	2.416	0.00115	0.00195	1.695	746.90	0.450
B. 8	200.05	5.02	2.99	2.454	0.00117	0.00221	1.890	530.10	0.250
B. 9	199.96	5.03	2.98	2.380	0.00113	0.00288	2.550	583.38	0.295
B. 10	199.98	5.04	3.02						
B. 11	200.07	5.01	3.02	2.509	0.00119	0.00432	3.630	403.20	0.154
B. 12	200.04	4.99	3.01	2.422	0.00115	0.00489	4.250	349.30	0.119
B. 13	200.01	4.98	3.00						
B. 14	199.99	4.97	3.03	2.394	0.00114	0.00656	5.760	291.60	0.085
B. 15	200.02	5.02	3.03	2.513	0.00119	0.00715	6.010	250.60	0.064
B. 16	199.98	5.01	3.01	2.433	0.00116	0.00849	7.320	268.00	0.073
B. 17	200.01	5.03	3.01	2.408	0.00115	0.00938	8.160	236.60	0.058
B. 18	200.02	5.02	2.98	2.436	0.00116	0.01020	8.780	270.50	0.074
B. 19	199.97	5.02	3.04	2.417	0.00115	0.01067	9.270	219.90	0.051
B. 20	200.02	5.04	2.99						
B. 21	200.01	5.02	3.01	2.384	0.00113	0.01289	11.400	211.70	0.047
B. 22	200.01	5.01	3.01	2.415	0.00115	0.01400	12.170	188.90	0.038
B. 23	200.02	4.99	3.02	2.416	0.00115	0.01480	12.890	174.70	0.032

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Notation

The following symbols have been adopted for use in this paper:

- d — width of an element;
- E — Young's modulus;
- \bar{E} — strain-hardening modulus;

E_{oK}	— “over-all” modulus (reduced modulus);
E_{oT}	— “over-all” modulus (tangent modulus);
E^*	— “over-all” modulus for a given strain ε ;
h	— ratio of elastic to strain-hardening modulus;
J_x	— moment of inertia;
i	— ratio of stress to create a slip plane to σ_F ;
L	— specimen length;
l	$= L + \Delta L = L(1 + \varepsilon)$;
M	— moment;
M_t	— plastic moment;
m	— ratio of flexural stiffness at yield to elastic flexural stiffness;
m_K	— m corresponding to “reduced modulus” conditions;
m_P	— expected m ;
m_T	— m corresponding to “tangent modulus” conditions;
S'	$= S(1 + C)$ — stiffness stability function [7];
s	— ratio of strain-hardening strain to yield strain;
T	— reduced modulus;
ε	— strain;
ε_F	— yield strain;
ε	$= s \cdot \varepsilon_F$ — strain at onset of strain-hardening;
\varkappa	— curvature;
\varkappa_{el}	— elastic curvature;
\varkappa_{pl}	— yield curvature;
\varkappa_{avg}	— average curvature;
Θ	— rotation of the specimen;
λ	— slenderness ratio;
σ	— stress;
σ_a	— proportional limit;
σ_F	— static yield stress.

Summary

The effect of strain-hardening of structural steel has been studied both experimentally and theoretically. Comparison between test results and theoretical studies on the effect of strain-hardening proved a good agreement.

References

1. BLEICH, F.: Buckling Strength of Metal Structures. McGraw-Hill Book C., Inc. New York, 1952
2. KORÁNYI, I.: Stability Problems in Engineering Practice.* Akadémiai Kiadó, Budapest, 1965
3. LAY, M. G.—GALAMBOS, T. V.: The Inelastic Behavior of Closely Braced Steel Beams under Uniform Moment. Lehigh University Inst. of Research Fritz Engrg Lab., Report 297, 9, 1964
4. NÁDAI, A.: Theory of Flow and Fracture of Solids. McGraw-Hill Book C. New York, 1950 (Sec. ed.)
5. LAY, M. G.: Yielding of Uniformly Loaded Steel Members. Journal of the Structural Division ASCE, Vol. 91, No. ST 6, December 1965
6. BEEDLE, L. S.—TALL, L.: Basic Column Strength. Journal of the Structural Division, ASCE, Vol. 86., No. ST 7, July 1960
7. HORNE, M. R.—MERCHANT, W.: The Stability of Frames. Pergamon Press, Oxford, 1965
8. IVÁNYI, M.: Thesis.* Manuscript, Budapest, 1972

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