RANDOM FLIGHT ON SPATIAL LATTICE POINTS

by

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There are several papers concerned with random walk on lattice points of an n-dimensional space (in particular the cases where $n = 1, 2, 3$), few results are known, however, of cases where the moving particle can do more in one step than to pass to lattice points neighbouring its momentary place, namely, to jump.

The studied case will be that of a particle moving at random in an n-dimensional space so that it can pass in one step from a given point not only to any neighbouring point with a given probability but to any point of the defined finite neighbourhood of its momentary place. Let this random walk be called a random flight.

For the sake of understanding, this method will first be illustrated by the case of random flight on the lattice points of the real line with integer coordinates, then extended to multi-dimensional cases. Notice that random flight, just as the usual random walk, forms a homogeneous Markov chain, and our primary object is to determine the N-step transition probabilities, fundamental for a comprehensive analysis of the Markov chain.

The one-dimensional random flight is derived from the random flight on a polygon. The one step transition probability matrix of the random flight on a polygon is known (e. g. [1]) to be a cyclic matrix, and N-step transition probabilities are easy to calculate by the familiar spectral decomposition of the cyclic matrix.

1. One-dimensional random flight

Let us consider the case first where the particle can fly at random to any of the n vertices of a regular polygon, and it can make jumps of $0, 1, 2, \ldots, l$ units ($l < n/2$) from its momentary place in both directions meeting at the polygon vertices with probabilities

$$p_{-l}, \ldots, p_{-1}, p_0, p_1, \ldots p_l. \quad \text{Of course} \quad \sum_{i=-l}^{l} p_i = 1 .$$

In this case the matrix of one-step transition probabilities will be the cyclic matrix

$$\mathbf{P} = \begin{bmatrix} p_0 & p_1 & p_2 \cdots & p_{-2} & p_{-1} \\ p_{-1} & p_0 & p_1 \cdots & p_{-3} & p_{-2} \\ p_{-2} & p_1- & p_0 \cdots & & p_{-3} \\ & & & & \\ p_1 & p_2 & p_3 & p_{-1} & p_0 \end{bmatrix} \tag{1}$$

that can be written as the polynomial of

$$\Omega = \begin{bmatrix} 0 & 1 & 0 & 0 \dots\dots & 0 \\ 0 & 0 & 1 & 0 \dots\dots & \\ 0 & 0 & 0 & 1 \dots\dots & 0 \\ & & & & \\ 0 & 0 & 0 & 0 \dots\dots & 1 \\ 1 & 0 & 0 & 0 \dots\dots & 0 \end{bmatrix} \tag{2}$$

the so-called elementary cyclic matrix, hence:

$$\mathbf{P} = \sum_{j=-l}^{l} p_j \, \Omega^j \,. \tag{3}$$

N-step transition probabilities are given by elements of matrix $\mathbf{P}^N$, determined by the spectral decomposition of matrix $\mathbf{P}$. Namely, it is known that eigenvalues of the elementary cyclic and unitary matrix Ω are the n-th unit roots, i. e. let eigenvalues be denoted by $\omega_k(k = \overline{0, n-1})$ then

$$\omega_k = e^{i\frac{2k\pi}{n}} \qquad k = 0, 1, \dots, n-1 \,. \tag{4}$$

Elements of eigenvectors $\mathbf{u}_k$ belonging to eigenvalues (4):

$$u_{jk} = \frac{1}{\sqrt{n}} \, \omega_k^{j-1} \qquad \begin{matrix} (j = 1, 2, \dots, n; \\ k = 0, 1, \dots, n-1). \end{matrix} \tag{5}$$

Accordingly, spectral decomposition of the requested matrix $\mathbf{P}^N$ can be written as:

$$\mathbf{P}^N = \sum_{k=0}^{n-1} \left(\sum_{j=-l}^{l} p_j \, \omega_k^j \right)^N \mathbf{u}_k \, \overline{\mathbf{u}}_k^* . \tag{6}$$

The probability for the particle to pass by this random flight from the r-th to the s-th vertex in N steps, is given by the s-th element of the r-th row of matrix $\mathbf{P}^N$:

$$\mathbf{P}^{N}{}_{(r,s)} = \frac{1}{n} \sum_{k=0}^{n-1} \left(\sum_{j=-l}^{l} p_j \, e^{i \frac{2k\pi}{n} j} \right)^{N} e^{i(r-s)\frac{2k\pi}{n}} \,. \tag{7}$$

This formula can be considered as a Riemann integral sum. Introducing notation $s-r = \nu$, $\frac{k}{n} = x$, $\frac{1}{n} = \Delta x$, for $n \to \infty$ we obtain:

$$\mathbf{P}_{\nu}^{N} = \int_{0}^{1} \left(\sum_{j=-l}^{l} p_j \, e^{i2\pi jx} \right)^{N} e^{i\nu 2\pi x} \, dx \,. \tag{8}$$

Transformation $e^{i2\pi x} = z$, $dx = \frac{1}{2\pi i} \frac{dz}{z}$ leads from (8) to the integral

$$\mathbf{P}_{\nu}^{N} = \frac{1}{2\pi i} \oint_{|z|=1} \left(\sum_{j=-l}^{l} p_j \, z \right)^{N} z^{\nu-1} \, dz \tag{9}$$

solved e. g. by the residue theorem:

$$\mathbf{P}_{\nu}^{N} = \sum_{|z|=1} \mathrm{Res} \left(\sum_{j=-l}^{l} p_j z^{j} \right)^{N} z^{\nu-1} \,. \tag{10}$$

As a special case, this formula delivers the known result for the usual random walk where the particle cannot pass from any point but to either of the two neighbouring lattice points with probabilities of 1/2 each. Now, the transition matrix will be:

$$\mathbf{P} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & \dots & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \dots & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \dots & 0 \\ & & & & & \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

Hence

$$\lambda_k = \frac{1}{2} (\omega_k + \bar{\omega}_k) = \frac{1}{2} \left(e^{i\frac{2k\pi}{n}} + e^{-i\frac{2k\pi}{n}} \right) = \frac{1}{2} \left(z + \frac{1}{z} \right).$$

Thus, according to (9):

$$\mathbf{P}_\nu^N = \frac{1}{2\pi i} \oint\limits_{|z|=1} \frac{1}{2^N} \left(z + \frac{1}{z}\right)^N z^{\nu-1}\, dz = \tag{11}$$

$$= \frac{1}{2\pi i} \frac{1}{2^N} \oint\limits_{|z|=1} \left[\sum_{k=0}^{N} \binom{N}{k} z^{k+\nu-1} \left(\frac{1}{z}\right)^{N-k}\right] dz.$$

The residue theorem leads to the well-known result:

$$\mathbf{P}_\nu^N = \frac{1}{2^N}\binom{N}{\frac{N-\nu}{2}}$$ (N and ν being at the same time even or odd)

Provided

$$p_i = \frac{1}{2l+1}\,(j = -l, \ldots, 0, \ldots l)$$

i. e. the particle may jump to (or remain at) any point accessible by a single step with the same probability, and making use of the substitution $2\pi x = t$ and of the fact that for $z = e^{it}$,

$$\sum_{k=-l}^{l} z^k = \frac{z^{l+1/2} - z^{-l-1/2}}{z^{1/2} - z^{-1/2}} = \frac{\sin\left(l + \frac{1}{2}\right) t}{\sin \frac{1}{2} t}$$

holds, (8) becomes:

$$\mathbf{P}_\nu^N = \frac{1}{(2l+1)^N} \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\sin\left(l + \frac{1}{2}\right) t}{\sin \frac{1}{2} t}\right)^N e^{i\nu t}\, dt. \tag{12}$$

In this case the probability for the particle to return by N steps to its initial position is:

$$(1)\, \mathbf{P}_0^N = \frac{1}{(2l+1)^N} \frac{2^{N-1}}{\pi} \int_0^{2\pi} \left[\frac{\sin\left(l + \frac{1}{2}\right) t}{2 \sin \frac{1}{2} t}\right]^N dt. \tag{13}$$

2. Two-dimensional random flight

Let us consider now the case of a particle moving on plane lattice points, able to jump from any point (x_1, x_2) to any point $(x_1 + i_1,\ x_2 + i_2)$ of the rectangle $(x_1 - l_1, x_1 + l_1)$ long by $(x_2 - l_2, x_2 + l_2)$ wide with probabilities $p_{i_1 i_2}$; $(i_1 = -\overline{l_1, l_1},\ i_2 = -\overline{l_2, l_2})$

where

$$\sum_{i_1=-l_1}^{l_1} \sum_{i_2=-l_2}^{l_2} p_{i_1 i_2} = 1 .$$

This random flight is derived from the random flight on a torus.

Assume the parallel and meridian circles of the torus to be of n_1 and n_2 unit lengths of perimeter, respectively. Again, assume $l_1 < \frac{n_1}{2}$, $l_2 < \frac{n_2}{2}$. Now, the matrix of the one-step transition probabilities is the following hypermatrix consisting of cyclic blocks:

$$\mathbf{P} = \left[\begin{array}{ccccc|ccc|c|cccc}
p_{00} & p_{01} & p_{02} & \cdots & p_{0-1} & p_{10} & p_{11} & p_{1-1} & \cdots & p_{-10} & p_{-11} & p_{-12} & p_{-1-1} \\
p_{0-1} & p_{00} & p_{01} & \cdots & p_{0-2} & p_{1-1} & & & & p_{-1-1} & p_{-10} & p_{-11} & \cdots \\
\vdots & & & & & & & & & & & & \\
p_{01} & p_{02} & p_{03} & \cdots & p_{00} & & & & & p_{-11} & p_{-12} & \cdots & p_{-10} \\
\hline
p_{-10} & p_{-11} & p_{-12} & & & & & & & & & & \\
& & & & & & & & & & & & \\
& & & & p_{-10} & & & & \cdots & & & & \\
\hline
& & & & & & & & & & & & \\
& & & & & & & & & & & & \\
& & & & & & & & & & & &
\end{array}\right]$$

Making use of the elementary cyclic matrices $\mathbf{\Omega}_{n_1}$ and $\mathbf{\Omega}_{n_2}$ of n_1 and n_2 order, respectively, matrix $\mathbf{P}$ can be written as the direct polynomial

$$\mathbf{P} = \sum_{i_1=-l_1}^{l_1} \left(\sum_{i_2=-l_2}^{l_2} p_{i_1 i_2} \mathbf{\Omega}_{n_2}^{i_2} \cdot\times \mathbf{\Omega}_{n_1}^{i_1} \right).$$

Taking the spectral decomposition of the elementary cyclic matrices, as well as the Egerváry—Stéphanos theorem for the spectral decomposition of direct polynomials into consideration, spectral decomposition of matrix $\mathbf{P}^N$ to yield transition probabilities of N steps can be written as:

$$\mathbf{P}^N = \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} \left(\sum_{i_1} \sum_{i_2} p_{i_1 i_2} \omega_{k_1}^{(n_1) i_1} \omega_{k_2}^{(n_2) i_2} \right)^N (\mathbf{u}_{k_1}^{(n_1)} \cdot \times \mathbf{u}_{k_2}^{(n_2)})(\overline{\mathbf{u}}_{k_1}^{(n_1)} \cdot \times \overline{\mathbf{u}}_{k_2}^{(n_2)})^*$$

where

$$\omega_{k_1}^{(n_1)} = e^{i \frac{2k_1 \pi}{n_1}}, \quad \omega_{k_2}^{(n_2)} = e^{i \frac{2k_2 \pi}{n_2}}$$

$$u_{i_1 k_1}^{(n_1)} = \frac{1}{\sqrt{n_1}} \omega_{k_1}^{(n_1)(i_1-1)}$$

and

$$u_{i_2 k_2}^{(n_2)} = \frac{1}{\sqrt{n_2}}\, \omega_{k_2}^{(n_2)(i_2-1)}.$$

The probability for the particle to get during its random flight on the torus from point $(x_1 x_2)$ to $(x_1 + r_1, x_2 + r_2)$ by N steps is delivered by element $(x_1, x_1 + r_1)$ of the block $(x_2, x_2 + r_2)$ of the matrix $\mathbf{P}^N$, i. e.:

$$\mathbf{P}^N_{(x_1,x_2)\to(x_1+r_1,x_2+r_2)} = \frac{1}{n_1 n_2} \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} \left[\sum_{i_1} \sum_{i_2} p_{i_1 i_2}\, e^{2\pi i\left(\frac{k_1}{n_1} i_1 + \frac{k_2}{n_2} i_2\right)}\, e^{2\pi i\left(\frac{k_1}{n_1} r_1 + \frac{k_2}{n_2} r_2\right)}\right]. \qquad (14)$$

By analogy to the one-dimensional case, random walk on the torus yields random flight on plane lattice points by limit procedure.

Introducing denotations

$$\frac{k_1}{n_1} = x_1, \frac{k_2}{n_2} = x_2, \quad \frac{1}{n_1} = \Delta x_1, \quad \frac{1}{n_2} = \Delta x_2$$

for $n_1, n_2 \to +\infty$

$$\mathbf{P}^N_{r_1,r_2} = \int_0^1 \int_0^1 \left(\sum_{i_1=-l_1}^{l_1} \sum_{i_2=-l_2}^{l_2} p_{i_1 i_2}\, e^{2\pi i(i_1 x_1 + i_2 x_2)}\right)^N e^{2\pi i(r_1 x_1 + r_2 x_2)}\, dx_1\, dx_2 \qquad (15)$$

Using transformation $e^{2\pi i x_1} = z_1$, $e^{2\pi i x_2} = z_2$

$$\mathbf{P}^N_{r_1 r_2} = \frac{1}{(2\pi i)^2} \oint\limits_{|z_1|=1} \oint\limits_{|z_2|=1} \left(\sum_{i_1} \sum_{i_2} p_{i_1 i_2} z_1^{i_1} z_2^{i_2}\right)^N z_1^{r_1-1} z_2^{r_2-1}\, dz_1\, dz_2. \qquad (16)$$

By applying a perfectly analogous consideration, making use of the theorem on the spectral decomposition of direct products of n factors [1], the suitable formula for the N-step transition probabilities of the particle flying at random on lattice points of an n-dimensional space is obtained.

If the particle can pass by one step from point $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ to any point $\mathbf{x} + \mathbf{i} = (x_1 + i_1, x_2 + i_2, \ldots, x_n + i_n)$ of the neighbourhood of the n-dimensional rectangular polyhedron with edge lengths $(-l_1, l_1)$, $(-l_2, l_2), \ldots (-l_n, l_n)$ with a probability $p_{i_1 i_2 \ldots i_n}$

$$\left[-l_k \le i_k \le l_k, (k = 1,2, \ldots, n), \sum_{i_1, i_2, \ldots, i_n} p_{i_1 i_2 \ldots i_n} = 1\right]$$

then

$$\mathbf{P}^N_{\mathbf{x}\to\mathbf{x}+\mathbf{r}} = \int_0^1 \underset{(n)}{\cdots} \int_0^1 \left(\sum_{-l_1}^{l_1} \cdots \sum_{-l_n}^{l_n} p_{i_1 i_2 \ldots i_n}\, e^{2\pi i\left(\sum_1^n i_k x_k\right)}\right)^N \cdot e^{2\pi i\left(\sum_1^n r_k x_k\right)} dx_1 \ldots dx_n. \qquad (17)$$

Whether in a random flight in spaces of n dimensions, lattice points (states of the Markov chain) have transient or recurrent properties is known to depend on whether the series

$$\sum_{N=0}^{\infty} (k)\, P_0^{(N)}$$

is convergent or divergent, where $(k)\mathrm{P}_0^N$ is the probability for the particle flying at random in a k-dimensional space to return to its initial position by N steps. Particulars of this problem have been published in [2]. Notice that for a random flight such that the particle gets by one step from any point of the k-dimensional space to any point of its $(2l+1)$-sided cube-shaped environment (or remains in place) with the same probability, according to (15):

$$(2)\,\mathbf{P}_0^{(N)} = \frac{1}{(2l+1)^{2N}} \cdot \frac{1}{(2\pi)^2} \left[\int_0^{2\pi} \left(\frac{\sin (2l+1)\, {}^{t}/_{2}}{\sin {}^{t}/_{2}} \right)^N dt \right]^2 = [(1)\mathbf{P}_0^{(N)}]^2$$

and, as it is evident from (17):

$$(k)\, P_0^{(N)} = [(1)\, P_0^{(N)}]^k .$$

Applications of our results in industrial quality control will be discussed in a subsequent paper. Let us only remark here that these results have found a statistical application in the theory of sequential sampling developed by Abraham Wald.

Summary

In this paper, random flight is the term for a generalized case of random walk where the particle moving on lattice points of an n-dimensional space can get by one step not only to neighbouring lattice points but to any point of the defined finite neighbourhood of its place, with preassigned probabilities. To compute the N-step transition probabilities, only the spectral decomposition of the cyclic matrix, and for a space of $n > 2$ dimensions, application of the Egerváry—Stéphanos theorem on hypermatrices is needed.

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