APPROXIMATE ANALYSIS OF SUSPENSION BRIDGES WITH CABLES IN INCLINED PLANES

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1. Introduction

At a difference to the usual structural form of suspension highway bridges, industrial steel structures are often designed with cables arranged in non-vertical planes. Thereby the horizontal stiffness to wind loads of the whole suspension girder is increased by the cables in inclined plane helping the stiffening beam to carry the horizontal loads. Due to the geometry of the bridge section, however, the stiffening beam gets twisted and the response of the structure is different from that of the ordinary suspension girders.

A general approximate analysis will be here presented for suspension bridges with cables in inclined planes subject to vertical and horizontal static loads. The method is primarily suitable for narrow suspension tube bridges under uniformly distributed static loads. This assumption is satisfactory for the analysis of horizontal loading due mostly to wind loads. Nevertheless, the method may be generalized to involve other cases of loading as well.

The approximate analysis is based on the energy method and on the deflection theory, considering that I^{st} order theory cannot be applied to suspension structures. A single restriction is made concerning the cross section of the stiffening beam, namely it is supposed to have one axis of symmetry coincident with that of the bridge. Otherwise the cross-sectional form is optional: opened or closed; thin-walled or conventional.

The assumptions for the whole girder system are the same as customary in design: structures of simply supported stiffening beams suspended on two cables of identical geometry.

2. Approximate analysis of vertically loaded suspension structures with inclined cable planes

According to the energy theorem, in case of equilibrium, the potential energy of the system is stationary. The potential energy of an elastic system can be written as

$$\Pi = -(L_K - L_a) \tag{1}$$

where L_K is the work done by the external load with the elastic deflection, L_a is the deformation energy of the whole structure. The deformation energy of the system is obtained by summing the deformation energies of the cables and the beam:

$$L_a = L_c + L_g \tag{2}$$

The deformation energy of the cable is composed of the work of the initial cable tension H_g along the deformation due to the imposed load [1]:

$$L_{cg} = -H_g y_k'' \int_0^l v_k(z) \mathrm{d}z \,, \tag{3}$$

and of the work done by the cable tension increment H_p:

$$L_{cp} = -\frac{1}{2}\int_0^l H_p y_k'' v_k \, dz - \frac{1}{2}\int_0^l H_p v_k'' v_k \, dz - \frac{1}{2}\int_0^l H_g v_k'' v_k \, dz \,. \tag{4}$$

The deformation energy of the stiffening beam is:

$$L_g = \frac{1}{2} EJ_x \int_0^l v''(z) \, dz \,. \tag{5}$$

For practical purposes the ordinates y_k and v_k in the cable-plane can be replaced by corresponding vertical values.

According to Fig. 1:

$$y_k(z) = \frac{y(z)}{\cos \alpha}$$

and:

$$v_k(z) \simeq v(z) \cos\alpha$$

Thus, the total deformation energy can be written as:

$$L_a = -H_g y'' \int_0^l v(z) \, dz - \frac{1}{2} H_p \, y'' \int_0^l v(z) \, dz - \frac{1}{2} H_p \cos^2 \alpha \int_0^l v''(z) \; v(z) \, dz - \\ - \frac{1}{2} H_g \cos^2 \alpha \int_0^l v''(z) \, v(z) \, dz + \frac{1}{2} EJ_x \int_0^l v''^2(z) \, dz \,. \tag{6}$$

The other part of the potential energy is the work done by the external load, composed of the dead load and a uniform vertical imposed load p_1:

$$L_K = (g + p_1) \int_0^l v(z) \mathrm{d}z. \tag{7}$$

Thus the potential energy of the elastic system is

$$\Pi = -H_g y'' \int_0^l v(z)\,dz - \frac{1}{2} H_p y'' \int_0^l v(z)\,dz - \frac{1}{2} H_p \cos^2\alpha \int_0^l v''(z)\,v(z)\,dz - \frac{1}{2} H_g \cos^2\alpha \int_0^l v''(z)\,v(z)\,dz + \frac{1}{2} EJ_x \int_0^l v''^2(z)\,dz - (g+p_1)\int_0^l v(z)\,dz \,. \tag{8}$$

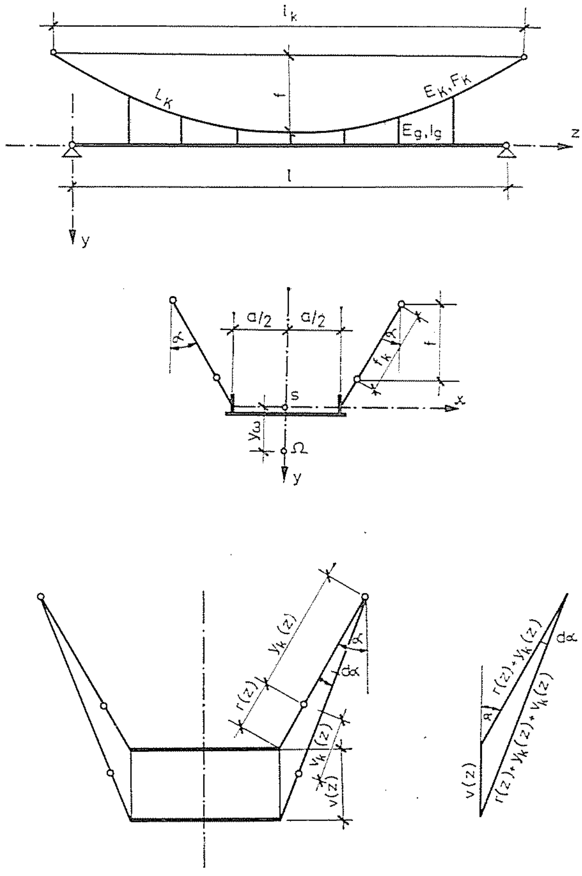

Fig. 1

The expression simplifies, namely:

$$H_g y'' + g = 0$$

and after Klöppel—Lie [2] $H_p \int_0^l v''v \cdot dz \approx 0$, the tensile energy H_p is starting from the initial cable co-ordinates, thus it can be assumed that $y''_{k1} \approx y''_k$. Hence:

$$\Pi = -\frac{1}{2} H_p y'' \int_0^l v dz - \frac{1}{2} H_g \cos^2\alpha \int_0^l v'' v \, dz + \frac{1}{2} EJ_x \int_0^l v''^2 \, dz - p_1 \int_0^l v \, dz. \qquad (9)$$

Applying the Ritz method, the function of deflection is approximated by the first term in the trigonometrical series

$$v(z) \approx v_1 \sin \frac{\pi z}{l} + v_2 \sin \frac{3\pi z}{l} + \dots$$

so that the definite integrals in (9) can be solved. The value of cable tension, written in converted form:

$$H_g = \frac{g l_k^2}{8f} \quad \text{and} \quad y'' = -\frac{8f}{l_k^2}$$

which follows from the original form of the cable, and:

$$H_p = \frac{E_k F_k}{L_k} \left[y'' \int_0^l v \, dz \mp \alpha_T \Delta T \, L_T \right]$$

deduced from the well-known compatibility equation [3] (considering the temperature change, too) substituted into (9) yield:

$$\begin{aligned} \Pi = {} & \frac{E_k F_k}{L_k} \left[\frac{128 f^2 l^2}{\pi^2 l_k^4} v_1^2 \mp \frac{8fl}{\pi l_k^2} \alpha_T \Delta T \, L_T \, v_1 \right] + \\ & + \frac{E_k F_k}{L_k} \left[\mp \frac{\pi^2}{4l} \cos^2\alpha \cdot \alpha_T \Delta T \, L_T \, v_1^2 \right] + \\ & + \frac{g \pi^2 l_k^2}{32 f l} \cos^2\alpha \cdot v_1^2 + \frac{EJ_x \pi^4}{4 l^3} v_1^2 - \frac{2 p_1 l}{\pi} v_1 . \end{aligned} \qquad (10)$$

For practical purposes the following geometrical and stiffness characteristics may be introduced:

$$\frac{l^2}{l_k^2} = \lambda^2 ; \; \frac{f}{l_k} = n ; \quad n_k = \frac{f_k}{l_k} = \frac{f}{l_k} \frac{1}{\cos\alpha} \qquad (11)$$

$$K = \frac{E_k F_k}{L_k} n_k^2 \lambda^2 = \frac{E_k F_k}{L_k} \frac{f^2}{l_k^2} \frac{1}{\cos^2\alpha} \cdot \frac{l^2}{l_k^2} \qquad (12)$$

$$B_x = \frac{EJ_x}{l^3} . \qquad (13)$$

The value of potential energy can be written as follows:

$$\Pi = K\cos^2\alpha\left[\frac{128}{\pi^2}v_1^2 \mp \frac{8}{\lambda\pi n_k}\alpha_T\cdot\Delta T\cdot L_T v_1\right] + + \frac{g\pi^2}{32 n_k\lambda}\cos\alpha\cdot v_1^2 + \frac{\pi^4 B_x}{4}v_1^2 - \frac{2p_1 l}{\pi}v_1 \tag{14}$$

The criterion of equilibrium is:

$$\frac{\partial\Pi}{\partial v_1} = 0 \tag{15}$$

accordingly

$$K\cos^2\alpha\left[\frac{128}{\pi^2}2v_1 \mp \frac{8}{\pi n_k\lambda}\alpha_T\cdot\Delta T\cdot L_T\right] + \frac{g\pi^2}{32\,n_k\lambda}\cos\alpha\cdot 2v_1 + + \frac{\pi^4 B_x}{4}2v_1 = \frac{2p_1 l}{\pi} \tag{16}$$

and from this

$$v_1 = \frac{p_1 l \pm K\cos^2\alpha\dfrac{4}{n_k\lambda}\alpha_T\cdot\Delta T\cdot L_T}{\dfrac{\pi^5 B_x}{4} + \dfrac{128}{\pi}K\cos^2\alpha + \dfrac{g\pi^3}{32\,n_k\lambda}\cos\alpha}\,. \tag{17}$$

Factoring out from the expression (17) the approximate deflection $v_0 = = \frac{4}{\pi^5}\frac{p_1 l}{B_x}$ of the simply supported girder:

$$v_1 = \frac{4}{\pi^5}\frac{p_1 l}{B_x}\cdot\frac{1 \pm \dfrac{4K}{n_k p_1 l\cdot\lambda}\alpha_T\,\Delta T\cdot L_T\cdot\cos^2\alpha}{1 + \dfrac{512}{\pi^6}\dfrac{K}{B_x}\cos^2\alpha + \dfrac{g\cos\alpha}{8\,n_k\lambda\pi^2 B_x}} = v_0\eta_v\,. \tag{18}$$

Consequently, the deflection of the suspension structure with cables in skew planes is obtained by multiplying the deflection of a simply supported stiffening girder by a factor η_v, which contains the stiffness coefficients and geometrical characteristics of both the beam and the cable and the value of dead load as well.

Knowing the deflection function of the girder, the bending moments and the shear forces can be obtained in the usual way:

$$M(z) = -EJ_x v'' \approx v_1\frac{\pi^2}{l^2}EJ_x\sin\frac{\pi z}{l} \tag{19}$$

$$T(z) = \frac{d}{dz} M(z) \approx v_1 \frac{\pi^3}{l^3} EJ_x \cos \frac{\pi z}{l} . \tag{20}$$

The maximum moment at the mid-span:

$$M_{max} = v_{01} \cdot \eta_v \frac{\pi^2}{l^2} EJ_x \approx M_0 \cdot \eta_v \tag{20a}$$

where

$$M_0 = \frac{4}{\pi^3} p_1 l^2 \approx \frac{p_1 l^2}{8} .$$

The maximum shear force at the support:

$$T_{max} = v_{01} \eta_v \frac{\pi^3}{l^3} EJ_x = \frac{4}{\pi^2} p_1 l \eta_v \approx \frac{p_1 l}{2{,}47} \cdot \eta_v$$

These values are in good agreement with those delivered by the transformed form of the well-known Melan differential equation for suspension girders. Using approximate analysis instead of the time-consuming exact solution of the differential equation, good results can be obtained from simpler relationships.

3. Approximate analysis of horizontally loaded suspension structures with cables in inclined planes

The knowledge of the behaviour of suspension structures with cables in skew planes exposed to horizontal load is of importance especially where — e. g. for tube bridges — the beam is too narrow to have a horizontal stiffness to resist horizontal loads. The numerical value of the horizontal stiffening effect of the cable has also to be known so that the excess load due to the horizontal loads in the cables and the internal forces in the stiffening beam can be computed.

The basic assumptions of the approximate analysis based on the energy method — in addition to the usual ones for suspension girders already referred to — are:

a) The girder is subject to a single horizontal uniform load p_2 acting on the stiffening beam at arbitrary depth;

b) in the state previous to horizontal loading only the dead load is acting, and exclusively on the cables;

c) the planes of the hangers and the cables coincide both before and after horizontal loading;

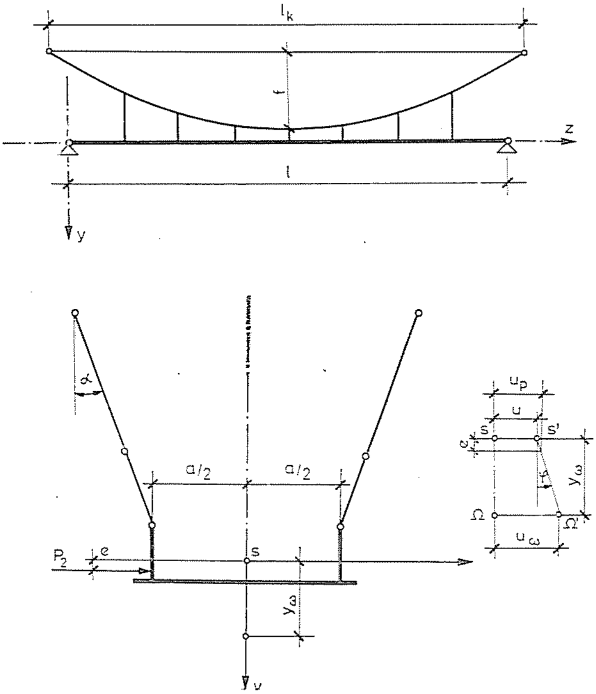

Fig. 2

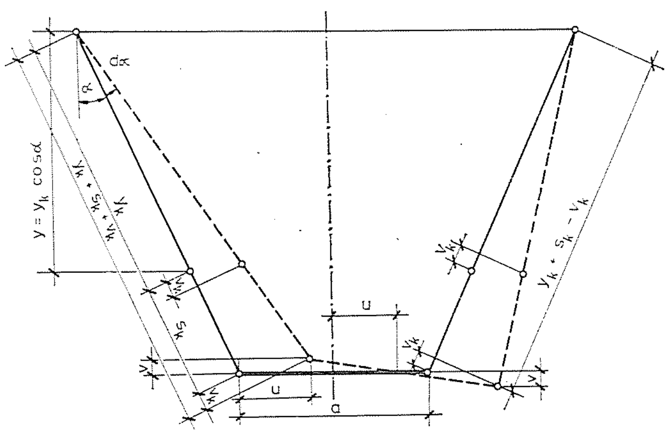

Fig. 3

d) the deformation of the suspension girder due to horizontal load consists of horizontal displacement $u(z)$ and of twist $\varphi(z)$ of the beam cross section (see Fig. 3);

e) the twist causes the suspension points of the beam cross section to be vertically displaced by equal values, but in opposite directions;

f) the stiffening beam is a simply supported girder of constant cross section, supplied at the points of support by a so-called "fork" grip to permit twisting.

By means of the vertical load, the value of potential energy can be written. In the expression for cable deformation energy — according to assumption e) — the energy of the constant load in both cables is equal, but of opposite sign, hence:

$$L_{cg} = 0.$$

Again, according to the assumption e) the energy of the cable tension increment is equal in both cables and so the joint deformation energy of the two cables is

$$L_c = L_{cp} = -\int_0^l H_p y_k'' v_k \, dz - \int_0^l H_p v_k'' v_k \, dz - \int_0^l H_g v_k'' v_k \, dz . \tag{21}$$

The deformation energy of the beam now consists of three parts: energies of horizontal bending moment M_y, of pure torsion moment M_T and of the warping moment W:

$$L_g = \frac{1}{2}\int_0^l M_y \frac{dz}{EJ_y} + \frac{1}{2}\int_0^l M_T^2 \frac{dz}{GJ_T} + \frac{1}{2}\int_0^l W^2 \frac{dz}{EJ_\omega} \tag{22}$$

where EJ_y, GJ_T and EJ_ω are bending, torsional and warping stiffnesses, respectively.

Since

$$\begin{aligned} M_y &= -EJ_y\,(u'' + y_\omega \varphi'') \text{ (see Fig. 2)} \\ M_T &= GJ_T \varphi'(z) \text{ and } W = -EJ_\omega \varphi''(z) \end{aligned} \tag{23}$$

the total deformation energy of the beam is:

$$\begin{aligned} L_g = \frac{EJ_y}{2}\int_0^l u''^2 \, dz + y_\omega EJ_y \int_0^l u''\,\varphi''\,dz + \frac{EJ_y}{2} y_\omega^2 \int_0^l \varphi''^2 \, dz + \\ + \frac{GJ_T}{2}\int_0^l \varphi'^2 \, dz + \frac{EJ_\omega}{2}\int_0^l \varphi''^2 \, dz . \end{aligned} \tag{24}$$

The energy of the external load is, according to Fig. 2:

$$L_K = p_2 \int_0^l u_p \, dz = p_2 \left[\int_0^l u dz + e \int_0^l \varphi \, dz \right] . \tag{25}$$

Now, with the same neglect $H_p \int_0^l v''v\,dz = 0$ as for the vertical load, the potential energy of the system can be written as:

$$\Pi = -y_k'' \int_0^l H_p v_k\,dz - \int_0^l H_g v_k'' v_k\,dz + \frac{EJ_y}{2}\int_0^l u''^2\,dz + y_\omega EJ_y \int_0^l u''\varphi''\,dz + \tag{26}$$

$$+ \frac{GJ_T}{2}\int_0^l \varphi'^2\,dz + \frac{y_\omega^2 EJ_y + EJ_\omega}{2}\int_0^l \varphi''^2\,dz - p_2\int_0^l u\,dz - ep_2\int_0^l \varphi\,dz.$$

The function of deflection for the cable plane $v_k(z)$ can be expressed by that in vertical plane $v(z)$, the twist of the beam and the function of horizontal displacement according to Fig. 3:

$$v_k(z) = u(z)\sin\alpha - v(z)\cos\alpha \tag{27}$$

and

$$v(z) = \frac{a}{2}\varphi(z)$$

so that only two unknown functions $u(z)$ and $\varphi(z)$ occur in the formula of potential energy.

These are substituted for trigonometric series, satisfying the boundary conditions

$$\left.\begin{matrix} x = 0 \\ x = l \end{matrix}\right\} u = 0; \qquad \left.\begin{matrix} x = 0 \\ x = l \end{matrix}\right\} \varphi = 0;\ \varphi' = 0$$

$$u(z) \cong \sum_{i=1}^{m} u_i \sin\frac{i\pi z}{l}\ (i = 1, 3, 5 \ldots m)$$

$$\varphi(z) \cong \frac{1}{2}\sum_{k=1}^{n}\varphi_k\left(1 - \cos\frac{2k\pi z}{l}\right)\ (k = 1, 3, 5 \ldots n). \tag{28}$$

The expression of potential energy can be converted by solving the definite integral in the expression of potential energy with these substituting functions, replacing

$$H_g = \frac{gl_k^2}{8f_k\cos\alpha}$$

$$H_p = \frac{E_kF_k}{L_k}\left(\frac{8f_k}{l_k^2}\int_0^l v_k(z)\,dz \pm \alpha_T\,\Delta TL_T\right)$$

and introducing stiffness characteristics and stiffness ratios:

$$B_x = \frac{EJ_x}{l^3} \qquad \frac{K}{B_y} = \varkappa_y$$

$$B_y = \frac{EJ_y}{l^3} \qquad \frac{C_T}{B_y} = \vartheta_T$$

$$K = \frac{E_k F_k}{L_k} \frac{f^2}{l_k^2} \frac{1}{\cos^2 \alpha} \frac{l^2}{l_k^2} \qquad \frac{C_\omega}{B_y} = \vartheta_\omega \tag{29}$$

$$C_T = \frac{GJ_T}{a^2 l}$$

$$C = \frac{4\pi^2}{l^2} \frac{EJ_\omega}{a^2 l}$$

and two geometry ratios:

$$\frac{y_\omega}{a} = \gamma ; \quad \frac{e}{a} = \varepsilon . \tag{30}$$

The potential energy is expressed by:

$$\begin{aligned} \Pi = {} & 64\, K \left(\frac{2}{\pi} \sin \alpha \sum_{i=1}^{m} \frac{u_i}{i} + \frac{a}{4} \cos \alpha \sum_{k=1}^{n} \varphi_k \right)^2 \mp \\ & \mp \frac{8K_y}{n_k \lambda} \alpha_T \Delta T L_T \left(\frac{2}{\pi} \sin \alpha \sum_{i=1}^{m} \frac{u_i}{i} + \frac{a}{4} \cos \alpha \sum_{k=1}^{n} \varphi_k \right) + \\ & + \frac{\pi g}{8 n_k \lambda B_y} \left(\frac{\pi \sin^2 \alpha}{2 \cos \alpha} \sum_{i=1}^{m} i^2 u_i^2 - 4\, a \sin \alpha \sum_{i=1}^{m} \sum_{k=1}^{n} \frac{i k^2 u_i \varphi_k}{i^2 - 4k^2} + \frac{\pi}{8} a^2 \cos \alpha \sum_{k=1}^{n} k^2 \varphi_k^2 \right) + \\ & + \frac{\pi^4}{4} \sum_{i=1}^{m} i^4 u_i^2 - 4\pi^3 y_\omega \sum_{i=1}^{m} \sum_{k=1}^{n} \frac{i^3 k^2 u_i \varphi_k}{i^2 - 4k^2} + \\ & + \frac{\pi^2 a^2}{4} \vartheta_T \sum_{k=1}^{n} k^2 \varphi_k^2 + \frac{\pi^2 a^2}{4} \vartheta_\omega \sum_{k=1}^{n} k^4 \varphi_k^2 + \\ & + \pi^4 y_\omega^2 \sum_{k=1}^{n} k^4 \varphi_k^2 - \frac{2}{\pi B_y} p_2 l \sum_{i=1}^{m} \frac{u_i}{i} - \frac{l}{2B_y} p_2 l \sum_{k=1}^{n} \varphi_k . \end{aligned} \tag{31}$$

According to the Ritz-theorem, in case of equilibrium of the elastic system, free parameters u_i and φ_j must be of such values as to minimize together the potential energy.

This minimum condition is expressed by a linear equation system with $(m + n)$ unknowns:

$$\frac{\partial \Pi}{\partial u_j} = 0 \qquad (j = 1, 3, 5 \ldots m)$$
$$\frac{\partial \Pi}{\partial \varphi_j} = 0 \qquad (j = 1, 3, 5 \ldots n) \tag{32}$$

resulting in the functions of displacement $u(z)$ and $\varphi(z)$ sought for.

For practical purposes a more favourable form is obtained by converting the equation system so as to include the unknowns in dimensionless form. The unknowns divided by the approximate value $u_0 = \frac{4p_2 l}{\pi^5 B_y}$ of the horizontal mid-span displacement of the simply supported beam results in new dimensionless unknowns forming an unknown displacement vector

$$\Omega = \begin{bmatrix} \bar{u}_1 \\ \bar{u}_3 \\ \bar{u}_5 \\ \vdots \\ \bar{u}_m \\ \bar{\varphi}_1 \\ \bar{\varphi}_3 \\ \bar{\varphi}_5 \\ \vdots \\ \bar{\varphi}_n \end{bmatrix}$$

where

$$\bar{u}_i = \frac{u_i}{u_0} \quad \text{and} \quad \bar{\varphi}_i = \frac{a}{2} \frac{\varphi_i}{u_0} .$$

Eq. (32) can written in matrix form:

$$\mathbf{G} \cdot \mathbf{\Omega} = \mathbf{R} \tag{34}$$

where the elements of the quadratic matrix $\mathbf{G}$ of size $(m + n)$ include the stiffness data of the suspension girder, and $\mathbf{R}$ is the load vector. It is reasonable to partition the matrices by separating the components of the horizontal displacement vector and of the torsion vector. Then a matrix equation with two unknowns is obtained which can be written in the following form:

$$\begin{aligned} \mathbf{A}\,\mathbf{U} + \mathbf{B}\,\mathbf{\Phi} &= \mathbf{A}_1 \\ \mathbf{B}^*\mathbf{U} + \mathbf{D}\,\mathbf{\Phi} &= \mathbf{A}_2 , \end{aligned} \tag{35}$$

where

$$\mathbf{G} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^* & \mathbf{D} \end{bmatrix}, \quad \mathbf{\Omega} = \begin{bmatrix} \mathbf{U} \\ \mathbf{\Phi} \end{bmatrix} \quad \text{and} \quad \mathbf{R} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix}$$

Taking from the function series for the horizontal displacement and for the torsion **m** and **n** members, resp., into consideration, then **U** and Φ will be vectors of dimension **m** and **n**, resp. Now **A** is a quadratic matrix of $m \cdot n$ dimension, **D** one of $n \cdot n$ dimension, nevertheless **B** will be an oblong matrix $m \cdot n$. The hypermatrix **G** is symmetrical and so its fourth element **B*** is the transposed of **B**.

The matrix elements can be expressed as:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \dots a_{1m} \\ a_{21} & a_{22} & a_{23} \dots a_{2m} \\ a_{31} & a_{32} & a_{33} \dots a_{3m} \\ \cdot & & \\ \cdot & & a_{ij} \\ \cdot & & \\ a_{m1} & a_{m2} & a_{m3} \dots a_{mm} \end{bmatrix}$$

$$a_{ij} = \frac{1024}{\pi^6} \frac{1}{(2i-1)(2j-1)} \varkappa_y \sin^2 \alpha + \\ + \delta_{ij} \left[(2i-1)^4 + \frac{g \sin^2 \alpha}{4\pi^2 B_y n_k \lambda \cos \alpha} (2i-1)^2 \right]$$

$$(i = 1, 2, 3 \dots m) \qquad \delta_{ij} = 0, \text{ for } i \neq j$$
$$(j = 1, 2, 3 \dots m) \qquad \delta_{ij} = 1, \text{ for } i = j$$

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \dots b_{1,n} \\ b_{21} & b_{22} & b_{23} \dots b_{2,n} \\ b_{31} & b_{32} & b_{33} \dots b_{3,n} \\ \cdot & & \\ \cdot & & b_{jk} \\ \cdot & & \\ b_{m1} & b_{m2} & b_{m3} \dots b_{m,n} \end{bmatrix} \tag{36}$$

$$b_{jk} = \frac{128}{\pi^5} \frac{1}{2j-1} \varkappa_y \sin 2\alpha - \frac{16}{\pi} \frac{(2j-1)^3 (2k-1)^2}{(2j-1)^2 - 4(2k-1)^2} \gamma - \\ - \frac{2g \sin \alpha}{\pi^3 n_k \lambda B_y} \cdot \frac{(2j-1)(2k-1)^2}{(2j-1)^2 - 4(2k-1)^2} \tag{37}$$

$$j = 1, 2, 3 \dots m$$
$$k = 1, 2, 3 \dots n$$

$$\mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & d_{13} \dots d_{1,n} \\ d_{21} & d_{22} & d_{23} \dots d_{2,n} \\ d_{31} & d_{32} & d_{33} \dots d_{3,n} \\ \cdot & & \\ \cdot & & d_{jk} \\ \cdot & & \\ d_{n1} & d_{n2} & d_{n3} \dots d_{nn} \end{bmatrix}$$

$$d_{jk} = \frac{64}{\pi^4} \varkappa_y \cos^2 \alpha + \left\{ \frac{4}{\pi^2} [(2j-1)^4 \vartheta_\omega + (2j-1)^2 \vartheta_{cs}] + 16(2j-1)^4 \gamma^2 \right\} \delta_{jk} +$$
$$+ \frac{g \cos \alpha}{4\pi^2 n_k \lambda B_y} (2j-1)^2 \delta_{jk}$$
$$j = 1, 2, 3 \dots n \quad \delta_{jk} = 0 \quad \text{for} \quad j \neq k$$
$$k = 1, 2, 3 \dots n \quad \delta_{jk} = 1 \quad \text{for} \quad j = k \tag{38}$$

The solution of the equation system:

$$\mathbf{U} = \mathbf{X}\mathbf{A}_1 + \mathbf{V}\,\mathbf{A}_2$$
$$\mathbf{\Phi} = \mathbf{Y}\,\mathbf{A}_1 + \mathbf{Z}\,\mathbf{A}_2 \tag{39}$$

where

$$\mathbf{X} = (\mathbf{A} - \mathbf{B}\,\mathbf{D}^{-1}\,\mathbf{B}^*)^{-1}$$
$$\mathbf{Y} = -\mathbf{D}^{-1}\,\mathbf{B}^*(\mathbf{A} - \mathbf{B}\,\mathbf{D}^{-1}\,\mathbf{B}^*)^{-1}$$
$$\mathbf{V} = -\mathbf{A}^{-1}\,\mathbf{B}(\mathbf{D} - \mathbf{B}^*\mathbf{A}^{-1}\,\mathbf{B})^{-1}$$
$$\mathbf{Z} = (\mathbf{D} - \mathbf{B}^*\mathbf{A}^{-1}\,\mathbf{B})^{-1}$$

since

$$\mathbf{G}^{-1} = \begin{bmatrix} \mathbf{X} & \mathbf{V} \\ \mathbf{Y} & \mathbf{Z} \end{bmatrix}.$$

Vectors **U** and **Φ** give the parameters of the unknowns u_i and φ_j, respectively, yielding the displacement functions:

$$u(z) = \sum_{i=1}^{m} u_i \sin \frac{i\pi z}{l} \; (i = 1, 3, 5 \dots m)$$

$$\varphi(z) = \sum_{i=1}^{n} \varphi_j \frac{1 - \cos \dfrac{2j\pi z}{l}}{2} \; (j = 1, 3, 5 \dots n).$$

With these, the internal forces of the stiffening beam of the suspension girder can be written:

$$M_y(z) = - EJ_y u''(z) = -EJ_y \sum_{i=1}^{m} u_i \frac{i^2 \pi^2}{l^2} \sin \frac{i\pi z}{l} \tag{40}$$

$$T(z) = \frac{d}{dz} M_y(z) = EJ_y \sum_{i=1}^{m} u_i \frac{i^3 \pi^3}{l^3} \cos \frac{i\pi z}{l} \tag{41}$$

and the horizontal projection of the cable tension can also be obtained from

$$H_p = u_0 \frac{8K}{\lambda n_k} \left[\frac{2 \sin \alpha}{\pi} \sum_{i=1}^{m} \frac{\bar{u}_i}{i} + \frac{\cos \alpha}{2} \sum_{j=1}^{m} \bar{\varphi}_j \right] \mp \frac{K}{\lambda^2 n_k^2} \alpha_T \Delta T L_T \tag{42}$$

where

$$u_0 = \frac{4p_2 l}{\pi^5 B_y}$$

leading to a very simple form for the mid-span moment of the stiffening beam:

$$M_{y,\max} = EJ_y \Sigma u_i \frac{i^2 \pi^2}{l^2} \ (i = 1, 3, 5 \ldots m) \tag{43}$$

or

$$M_{y,\max} = \frac{4p_2 l^2}{\pi^3} (\bar{u}_1 - 9\bar{u}_3 + 25\bar{u}_5 \ldots) \cong M_0 \eta_u^M \tag{44}$$

where

$$\eta_u^M = \bar{u}_1 - 9\bar{u}_3 + 25\bar{u}_5 \ldots$$

in a form similar to (20a).

The results can be simplified by omitting any but the first terms of the unknown functions replacing the trigonometrical series. Though this is at the expence of accuracy, according to our investigations the approximation closeness requirements are more than met, since in all cases the error is within $\pm 12\%$. In this case $m = n = 1$, and the equation simplifies into an algebraic equation with two unknowns:

$$\begin{aligned} a_{11} \bar{u}_1 + b_{11} \bar{\varphi}_1 &= a_{10} \\ b_{11} \bar{u}_1 + d_{11} \bar{\varphi}_1 &= b_{10} \end{aligned} \tag{45}$$

where

$$a_{11} = 1 + \frac{1024}{\pi^6} K_y \sin^2 \alpha$$

$$b_{11} = \frac{128}{\pi^5} K_y \sin 2\alpha + \frac{16}{3\pi} \gamma$$

$$d_{11} = \frac{64}{\pi^4} K_y \cos^2 \alpha + \frac{4}{\pi^2} (\vartheta_T + \vartheta_\omega) + 16 \gamma^2 + \frac{g \cos \alpha}{4\pi^2 n_k \lambda B_y}$$

$$a_{10} = 1 \pm \frac{8K \sin \alpha}{p_2 f_k \lambda} \alpha_T \Delta T L_T$$

$$b_{10} = \frac{\pi}{2} \varepsilon \pm \frac{3\pi K \cos \alpha}{2p_2 f_k \lambda} \alpha_T \Delta T L_T$$

yielding:

$$\bar{u}_1 = \frac{d_{11} a_{10} - b_{11} b_{10}}{a_{11} d_{11} - b_{11}^2}$$

$$\bar{\varphi}_1 = \frac{a_{11} b_{10} - b_{11} a_{10}}{a_{11} d_{11} - b_{11}^2} \, .$$

Carrying out the operations, the twisting and the horizontal mid-span displacement of the stiffening beam are obtained in simple form as:

$$u_1 = u_0 \eta_u \tag{46}$$

$$\varphi_1 = \frac{2}{a} u_0 \eta_0 = \varphi_0 \bar{\eta}_0 \tag{47}$$

where

$$\eta_u = \frac{4\varkappa_y (\cos^2 \alpha - \varepsilon \sin 2\alpha) + \frac{\pi^2}{4} \vartheta + \gamma^2 \pi^4 \left(1 - \frac{\varepsilon}{6\gamma}\right)}{\left(\frac{1024}{\pi^2}\gamma^2 + \frac{256}{\pi^4}\vartheta - 4\right)\varkappa_y \sin^2 \alpha - \frac{256}{3\pi^2}\gamma\varkappa_y \sin 2\alpha + 4\varkappa_y + \frac{\pi^2}{4}\vartheta + \gamma^2\pi^4\left(1 - \frac{16}{9\pi^2}\right)} \tag{48}$$

$$\bar{\eta}_\varphi = \frac{\frac{64}{\pi^2} \varkappa_y (4\varepsilon \sin^2 \alpha - \sin 2\alpha) + \pi^2 \left(\frac{\pi^2}{4}\varepsilon - \frac{8}{3}\gamma\right)}{\left(\frac{1024}{\pi^2}\gamma^2 + \frac{256}{\pi^4}\vartheta - 4\right)\varkappa_y \sin^2 \alpha - \frac{256}{3\pi^2}\gamma\varkappa_y \sin 2\alpha + 4\varkappa_y + \frac{\pi^2}{4}\vartheta + \gamma^2\pi^4\left(1 - \frac{16}{9\pi^2}\right)} \tag{49}$$

For stiffening box beams where $\gamma = \varepsilon = 0$, the expressions simplify further:

$$\eta_u = \frac{4K_y \cos^2 \alpha + \frac{\pi^2}{4} \vartheta}{\left(\frac{256}{\pi^4} \vartheta - 4\right) K_y \sin^2 \alpha + 4K_y + \frac{\pi^2}{4} \vartheta} \tag{48a}$$

and

$$\bar{\eta}_\varphi = - \frac{\frac{64}{\pi^2} K_y \sin 2\alpha}{\left(\frac{256}{\pi^4} \vartheta - 4\right) K_y \sin^2 \alpha + 4K_y + \frac{\pi^2}{4} \vartheta} \, . \tag{49a}$$

According to the presented approximate structural analysis, displacement and internal forces of the suspension structures can be obtained in closed form in the case of horizontal loads, too, namely introducing the factors η_u and η_φ reduce them to the corresponding values of a simply supported beam like those for vertical loads. Our subsequent investigations are concerned with the determination of the optimum skewness of the cable planes to provide adequate stiffness to both vertical and horizontal loads.

Summary

An approximate analysis for the current problem of narrow structures suspended on cables in skew planes — such as tube bridges — is presented. The analysis is based on the energy method and involves the usual simplifying assumptions. Accuracy is within $\pm$ 12%.

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