REGULATION OF THE WATER-LEVEL OF A RESERVOIR WITH APPROXIMATELY PERIODICAL WATER RESERVE CHANGES*

by

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Let a reservoir have natural water reserve changes determined primarily by prevailing rainfall and weather conditions affecting smaller natural affluxes (e.g. a major lake). It is supposed that records on the resultant monthly water reserve changes are available for at least 40 to 50 years, to be considered as statistically (approximately) cyclic within a period of one year. The water-level can be regulated (sluiced) by controlling the capacity of the flow into or out of the reservoir. Sluicing instructions (optimum strategy) are to be established to assure a reservoir water level within specified limits. The mathematical model developed to solve the problem makes use of the techniques of inhomogeneous Markov chains and of dynamic programming.

The mathematical model

The change, both of time (with a month as unit) and of water-level (e.g. 5 cm intervals) is considered for technical reasons as discrete. The model is suitable to determine the optimum (monthly changing) sluice regulating instructions for a period of time (e.g. 10 years). A finite time interval has to be supposed because of the inhomogeneity of the Markov chain [1].

Denote the end of the investigated period by τ_0 and the nth time interval (month) counted backwards by τ_n. Let ξ_n be the reservoir water level interval at a time τ_n. The process is to be assumed homogeneous in the state space i.e. the probability of a displacement by h intervals of the water level in the given period is independent of the interval it belonged to at the beginning of the period.

In symbols: $Pr\{n, i, j\} = p(n, j - i)$, where

$$Pr\{n, i, j\} = Pr\{\xi_{n-1} = j \mid \xi_n = i\} ,$$

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and $p(n, h)$ is the displacement probability by h units in the interval $[\tau_n, \tau_{n-1}]$. In the same interval a possible sluice instruction is denoted by $s(n, i)$ if $\xi_n = i$, and a fixed system of instructions by $s_1(n)$:

$$s_1(n) = \{s(n, i);\ i \in I\}$$

where I is the set of the possible states, and $s_1(n)$ a single-step strategy. Similar is the definition of a strategy $s_{n-1}(n-1)$ of "$n-1$ steps" in the interval $[\tau_{n-1}, \tau_0]$.

Any deviation from the specified water-level is undesirable and involves a loss that is great if the deviation is great. Denoting by $v\{s_n(n)\}$ the loss over the interval $[\tau_n, \tau_0]$ for strategy $s_n(n)$, it holds:

$$v\{s_n(n)\} = v\{s_1(n)\} + v\{s_{n-1}(n-1)\} .$$

Denote the conditional expectation

$$M(v\{s_1(n)\} \mid \xi_n = i \cap \xi_{n-1} = j)$$

by $r(n, i, j)$ and the corresponding transition probability when applying the strategy $s(n, i)$ (sluice regulation) by

$$Pr^{s(n,i)}\{\xi_{n-1} = j \mid \xi_n = i\}.$$

Supposing the number of possible states to be $N+1$, from the theorem on the total expected value we have:

$$\left.\begin{aligned} M\big(v\{s_n(n)\} \mid \xi_n = i\big) = \sum_{j=0}^{n} \{r(n, i, j) + \\ + M\big(v\{s_{n-1}(n-1)\} \mid \xi_{n-1} = j\big)\} \times \\ \times p^{s(n,i)}\{\xi_{n-1} = j \mid \xi_n = i\}. \end{aligned}\right\} \tag{1}$$

Introducing the notations:

$$P^{s(n,i)}\{\xi_{n-1} = j \mid \xi_n = i\} = p\{n, s(n, i), i, j\} ,$$

$$M(v\{s_n(n)\} \mid \xi_n = i) = v\{n, s_n(n), i\}$$

and

$$\min_{s_n(n)}.\ v\{n, s_n(n), i\} = v\{n, i\}$$

i.e. the expected loss in the course of the process of n steps minimized by applying the optimum strategy (if $\xi_n = i$), (1) takes the form:

$$v\{n, s_n(n), i\} = \sum_{j=0}^{N} \{r(n, i, j) + v\{n-1,$$

$$s_{n-1}(n-1), j\}\} \cdot p\{n, s(n, i), i, j\} . \tag{2}$$

Considering that $s_n(n) = s_1(n) + s_{n-1}(n-1)$ and the two strategies on the right side can be chosen independently of each other, i.e.

$$\min_{s_n(n)} . \; v\{n, s_n(n), i\} = \min_{s_1(n)} \left[\min_{s_{n-1}(n-1)} v\{n, s_n(n), i\} \right],$$

hence

$$v\{n, i\} = \min_{s(n,i)} \sum_{j=0}^{N} \{r(n, i, j) + v\{n-1, j\}\} \times p\{n, s(n, i), i, j\} \tag{3}$$

a recursive formula delivering the minimized expected losses $v(n, i)$ and the optimum strategies $s(n, i)$ if the "initial losses" $v\{o, j\}$ are given.

Foundations for the computation

Transition probabilities $p(n, h)$ serving as data can be replaced by the available data of frequencies.

Finiteness of the state space (i.e. of the number of occurring states) was assured as follows: A total of states $i = 0, 1, 2, \ldots, N$ are permitted. With regard to actual conditions, states A and B $(1 < A < B < N - 1)$ are marked out. For arbitrary n (month), if $0 \leqslant i \leqslant A$, i.e. the water-level is "too low", the only permitted strategy $s(n, i)$ is that providing the highest rise of the water-level. For a water-level "too high", i.e., $B + 1 \leqslant i \leqslant N$, the only permitted strategy $s(n, i)$ is the one lowering maximally the water-level. For $A + 1 \leqslant i \leqslant B$, any of the "permitted" (considered) strategies $s(n, i)$ can be chosen. When the strategies are chosen in this way, the probability to reach or exceed "limit" states O or N is very small, therefore this latter probability may be neglected, or better, transition probabilities adequately rewritten.

The effect of "permitted" sluice regulating instructions is to rewrite the transient probabilities $p(n, h)$; computerized rewriting can be based on the actual conditions.

The single-step losses $r(n, i, j)$ and the initial losses $v(o, i)$ are given arbitrarily, of course with the restriction that they must reflect correctly the economic target. Two cases will be considered now, emphasizing that the use of other, economically better motivated functions may result in more exact approaches.

I. It is supposed that for every month, a specified optimum water-level interval $\sigma(n)$ is to be kept, where $A + 1 \leqslant \sigma(n) \leqslant B$ and $\sigma(n + 12) = \sigma(n)$ for every n. Furthermore, the initial loss and the single-step loss are supposed to be essentially proportional to the square of the deviation of levels, more exactly:

$$v(0, i) = i - \sigma(0)^2 \tag{4}$$

and

$$r(n, i, j) = \max \{[i - \sigma(n)]^2 , [j - \sigma(n)]^2\} \tag{5}$$

respectively, for each n. The "scope" of this hypothesis is naturally to keep the water-level always near the optimum state.

II. In the second case it is only desired to keep the water-level always in the interval $A + 1$ to B. In this case the loss is wanted to be O, otherwise it would increase very quickly (cubically), more exactly:

$$V(0, i) = \begin{cases} 0 \text{ for } A + 1 \leqslant i \leqslant B \\ (i - B)^3 \quad \text{for } i > B \\ (A + 1 - i)^3 \text{ for } i < A + 1 , \end{cases} \tag{6}$$

and

$$r(n, i, j) = \begin{cases} \max \{ \max. [(A+1-i)^3, 0], \max [(j-B)^3, 0]\} \text{ for } i \geq j \\ r(n, j, i) \qquad \text{for } \quad i > j \end{cases} \tag{7}$$

respectively.

Case II is also interesting because — going reversely — comparison of the minimized expected losses $v(n, i)$ $(i = 0, 1, 2, \ldots, N)$ in the months of the last year demonstrates the state starting from which yields the least expected loss minimized by the optimum strategy i.e. which is the optimum state (proposed to be kept).

This method lends itself to computer use, especially for a big storage one; a program has been prepared for computer Rasdan –3.

Summary

A mathematical model has been presented, based on the theory of Markov chains, inhomogeneous in time, suitable to establish the monthly changing optimum sluice regulation instructions (water quantities to be drawn off) for a finite period. The scope is to minimize losses due to deviation from the optimum level or excess of the determined limits. The model has been analyzed by dynamic programming, assuming different practical alternatives, using the 100-year data series for water reserve changes of the Lake Balaton.

References

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