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A Universal Recursive Analytical Solution of the Nonlinear Storage Equation for Flood Control Reservoir Sizing

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Abstract
The nonlinear storage equation to arbitrary inflow rates is solved for the outflow time-series by recursively employing the analytical solution of the linear storage equation evaluated at each time-increment with an updated storage coefficient value. As the procedure is fully analytical it avoids any problems a numerical solution may present.

Keywords
nonlinear storage equation, universal recursive analytical solution, discrete linear cascade model, recursion, flood control reservoir

1 Introduction

With the emergence of climate change, the occurrence of flashfloods has significantly increased across Europe [1, 2]. Flood mitigation by flood control reservoirs is a common practice in civil engineering [3–8]. Among the different approaches to reservoir sizing (for a concise review, see Pirone et al. [4]), one objective is common: finding a solution to the nonlinear storage equation (NLSE) in terms of the outflow rates provided inflow rates (in the form of a flood hydrograph or a complete time series of discharge values) and basic physical characteristics of the reservoir and control structure are known. In special cases, when the inflow hydrograph is zero or it follows a certain well-defined shape, there exist analytical solutions to the NLSE (see Pirone et al. [4] for a list of such studies), otherwise it is solved numerically. Numerical solutions however create their own problems in terms of stability, accuracy and time of execution.

In this study we offer a recursive approach to the NLSE made up of piece-wise analytical solutions [9–14] of the linear storage equation (LSE). The advantage of such an approach over a numerical one is that it is simple (no need for learning and meeting the requirements of a numerical solver), always stable, and its accuracy depends simply on the applied time increment, while the time of execution is a non-issue. It is hoped that such a solution will aid future domestic as well as international flood control reservoir sizing efforts as an adequate engineered response to increased flashflood risks reported globally.

2 Methods

Following the recent study of Pirone et al. [4], the outlet discharge (Q) of the flood control reservoir can be described by a rating curve in the form of:

$$Q(t) = c\left[h(t)\right]^n . \tag{1}$$

In Eq. (1) h is the water level (stage) in the reservoir relative to the spillway crest elevation, t is a time reference, and c, n are constants, determined by the outlet type and its geometry. For a rectangular spillway $n = 1.5$, $c = \mu L(2g)^{0.5}$ where μ is the discharge coefficient, L the length of the spillway, and g the gravitational acceleration [4]. The water volume (S) stored in the reservoir at any given time t is given by the stage-storage relationship [4] as:

$$S(t) = a\left[h(t)\right]^m , \tag{2}$$

where a and m are another constants, determined predominantly by the topography of the land in the vicinity and upstream of the spillway. m has a typical range of 1–4.5 [4].

When Eqs. (1) and (2) are inserted into the lumped continuity (i.e., storage) equation:

$$\frac{dS(t)}{dt} = I(t) - Q(t), \tag{3}$$

where $I(t)$ is the inflow time series to the reservoir, one obtains an inhomogeneous, nonlinear, first-order ordinary differential equation in the form of:

$$\upsilon\kappa\left[Q(t)\right]^{\upsilon-1}\frac{dQ(t)}{dt}+Q(t)=I(t), \tag{4}$$

where $\upsilon = m / n$ and $\kappa = a / c^{\upsilon}$ [4]. Equation (4) when subjected to arbitrary inflows can generally be solved numerically as done below by the 'ode45' solver [15, 16] for comparison with the recursive analytical solution described next.

Equation (4) can be linearized over arbitrary dt time-steps by treating the last value of $\upsilon\kappa[Q(t)]^{\upsilon-1}$ constant, K, for the dt interval, which after rearrangement becomes:

$$\frac{dQ(t)}{dt}=-kQ(t)+kI(t), \tag{5}$$

where $k = K^{-1}$ now. Note that K must have a dimension of time (T), therefore k a dimension of (T^{-1}). Equation (5) over an interval of dt is the linear storage equation (i.e., Eq. (3)) of a linear reservoir with storage coefficient k, so that $Q = kS$, but it is also the continuous, spatially discrete form of the linear kinematic wave equation:

$$\frac{\partial Q(x,t)}{\partial t}+C\frac{\partial Q(x,t)}{\partial x}=0, \tag{6}$$

when written for a single characteristic channel reach (of no lateral inflow) of length Δx by employing a backward difference scheme [9, 10]. Here x is the spatial coordinate along the flow direction and C the constant wave celerity. Note that k in Eq. (5) equals $C / \Delta x$ when derived from Eq. (6). When the outflow Q is the inflow to the next characteristic reach, one obtains a homogeneous (each reach having the same k value) cascade of characteristic reaches, or equivalently, a homogeneous cascade of linear reservoirs. A temporally and spatially discrete solution of such a system of ordinary differential equations (made up of Eq. (5) for each element of the cascade of altogether N linear reservoirs) given in a state-space form was first published by Szöllösi-Nagy [9]:

$$\begin{bmatrix} Q_1(t+dt) \\ Q_2(t+dt) \\ Q_3(t+dt) \\ \vdots \\ Q_N(t+dt) \end{bmatrix} = e^{-kdt}\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ kdt & 1 & 0 & \cdots & 0 \\ (kdt)^2/2! & kdt & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ (kdt)^{N-1}/(N-1)! & \cdots & (kdt)^2/2! & kdt & 1 \end{bmatrix}\begin{bmatrix} Q_1(t) \\ Q_2(t) \\ Q_3(t) \\ \vdots \\ Q_N(t) \end{bmatrix} + \begin{bmatrix} \Gamma(1,kdt)/\Gamma(1) \\ \Gamma(2,kdt)/\Gamma(2) \\ \Gamma(3,kdt)/\Gamma(3) \\ \vdots \\ \Gamma(N,kdt)/\Gamma(N) \end{bmatrix} I(t), \tag{7}$$

where the Γ terms (i.e., the unit-step-response functions of storage for the increasing-order cascade evaluated at dt) in the $N \times 1$ input-transition vector contain the regularized incomplete gamma function [9, 17]. The $N \times N$ state-transition (lower-triangular Toeplitz band) matrix is made up of the impulse response functions of storage for the increasing order cascade. This solution assumes that I measured at t is kept constant (this is the so-called classical pulsed data system framework [9]) over the dt interval.

When Eq. (7) is applied for the single linear reservoir it becomes [14]:

$$Q(t+dt)=e^{-kdt}Q(t)+\left(1-e^{-kdt}\right)I(t), \tag{8}$$

where the identity:

$$\frac{\Gamma(i,kdt)}{\Gamma(i)}=1-\sum_{j=0}^{i-1}\frac{(kdt)^j}{j!}e^{-kdt}, \tag{9}$$

was evoked [10, 17].

When I is let to change linearly from its value measured at t to the one measured at $t + dt$, Eq. (8) becomes [10, 11]:

$$Q(t+dt)=e^{-kdt}Q(t)+\left(1-e^{-kdt}\right)\left[\frac{1}{kdt}-\frac{e^{-kdt}}{1-e^{-kdt}}\right]I(t)+\left(1-e^{-kdt}\right)\left[1-\frac{1}{kdt}+\frac{e^{-kdt}}{1-e^{-kdt}}\right]I(t+dt). \tag{10}$$

Here the first term on the right-hand-side of the equation includes the impulse response function of storage, the second term the unit down-ramp response, while the third, the unit up-ramp response [10, 11], all evaluated at dt. Equation (10) reduces to Eq. (8) with the $I(t + dt) = I(t)$ choice. Equation (8) is appropriate in a forecasting situation when the future value of the inflow rate is not yet known at the time of the forecast. It however is not the case for reservoir design purposes.

With the help of Eq. (10) the recursive solution of Eq. (4) becomes:

$t=0;\; k=\left\{\upsilon\kappa\left[Q(0)\right]^{\upsilon-1}\right\}^{-1};\; I(0)=0;$

Start

$Q(t+dt)$ from Eq. (10);

$k=\left\{\upsilon\kappa\left[Q(t+dt)\right]^{\upsilon-1}\right\}^{-1};\; t=t+dt;$

Return

This simple recursive solution of the NLSE is compared to a numerical one next, as well as to a special case of an existing analytical solution of the homogeneous non-linear storage equation.

3 Results and discussion

For testing the above recursive solution, the following parameter values were prescribed: $c = 6$; $n = 1.5$; and m taken from the range of 1–4.5 (Fig. 1). The largest peak outflow value belongs to $m = 1$ (representing a canyon with vertical walls), the smallest to $m = 4.5$ (wide valley with gentle slopes). Here $a = 30{,}000$, $dt = 10$ s, and $Q(0) = 0.1$ m^3/s.

It is seen that the two solutions remain close together throughout the modeling period. Solutions of the NLSE are also displayed in Fig. 2 for a more realistic looking inflow hydrograph where $a = 5{,}000$, dt and $Q(0)$ are the same as in Fig. 1. The inflow hydrograph is defined [12] as: $I(t) = I_{max}\, e^{\sigma(D-t)}[1 - \cos(\omega t)] \;/\; [1 - \cos(\omega D)]$ with $I_{max} = 60$ m^3/s; $D = 4{,}000$ s; $\omega = 2\pi / T$ where $T = 21{,}600$ s; and $\sigma = \omega\cot(\omega D / 2)$. The largest peak outflow value belongs to $m = 2$, while the smallest to $m = 3.5$.

The numerical solver obtains complex values occasionally during its execution for three out of the four cases in Fig. 2, as well as it slows down considerably in its execution. Of course, this can never happen with the analytical recursion.

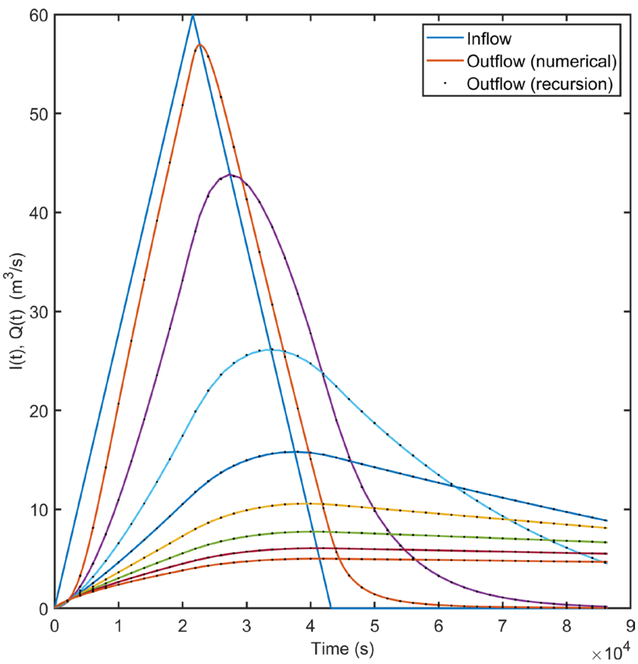

Fig. 1 Comparison of the recursive solution [Eq. (10) plotted at 200 dt s time increments] of Eq. (4), to one obtained by numerical integration for different values of m incremented by 0.5

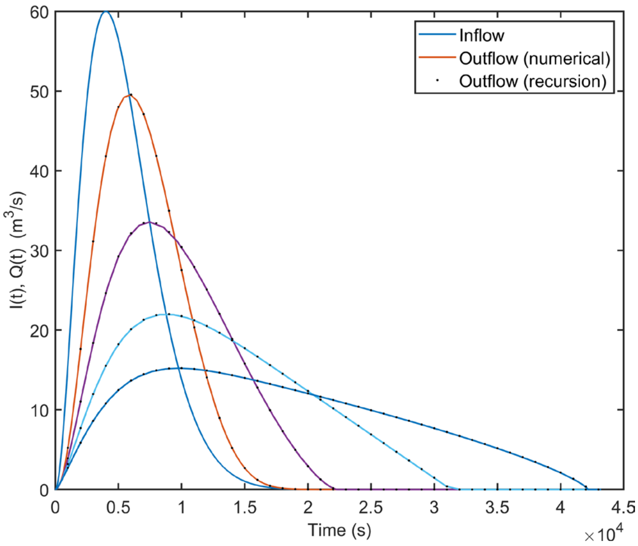

Fig. 2 Comparison of the recursive solution [Eq. (10) plotted at 100 dt s time increments] of Eq. (4) to one obtained by numerical integration for different values of m incremented by 0.5

In special cases there exist closed-form analytical solutions of Eq. (4). Such a solution is available [18] when I is zero, and Eq. (4) is in the form:

$$2\left[Q(t)\right]^{-2}\frac{dQ(t)}{dt} + Q(t) = 0, \tag{11}$$

as

$$Q(t) = \frac{Q(0)}{\sqrt{1 + t\left[Q(0)\right]^2}}. \tag{12}$$

Here the constant (i.e., 2) must have the necessary $[L^6\, T^{-1}]$ dimensions. Fig. 3 compares this special analytical solution to the one obtained by the recursion for different dt increments. As expected, the recursive solution blends into the closed-form one with refinement of the time step of calculations.

4 Conclusions

The proposed recursion employs piecewise (i.e., over dt intervals) analytical solutions of the linear storage equation written in a state-space formulation and it explicitly accounts for data sampling. The storage coefficient, k, is reevaluated in each dt step of the calculations and kept constant over the dt interval, making possible the application of the analytical solution of the linear storage equation. The method is simple, unconditionally stable, and naturally very fast as no iterations are involved characteristic of numerical integrations. Its simple structure and stability makes it an ideal tool for the inclusion of any kind

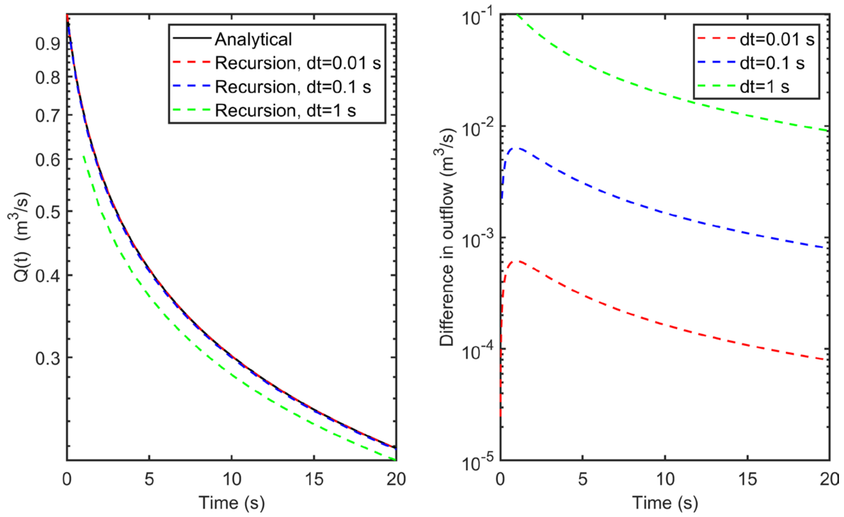

Fig. 3 Comparison of an existing closed-form analytical and the recursive solution for different *dt* time-increment of calculations (here $Q(0) = 1$ m^3/s)

of real-life flood control reservoir sizing procedure where the nonlinear storage equation is required to be solved repeatedly with trial values of the reservoir and/or control structure parameters. It is also recommended for teaching and/or demonstration purposes in institutes of higher education for civil/structural/infrastructural engineering where the students can spend more time on the engineering aspect of the problem rather than on the computational side of how to set up a numerical solver properly.

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