A FORMULA FOR ESTIMATING THE MEAN ERROR BASED ON MEASUREMENT SIMULATION

By

O. L'Auné

Department of Survey, Technical University, Budapest

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Presented by Ass. Prof. Dr. F. Sárközy

There exists a method to estimate the maximum error if the mean value of the error series is known:

$$v_{\max} \cong 3\mu$$

Is this formula reversible? What are the conditions for this relation to be valid?

From the distribution function of measurement errors the relationship between magnitude and probability of occurrence of the error may be calculated. Denoting the error by v (deviation from the most reliable value), the probability that only errors less than $x \cdot \mu$ occur, is $P(v)$, where $x = v/\mu$.

The equation of the probability curve is [1]:

$$P(v) = \frac{1}{\mu\sqrt{2\pi}} \int_{-\nu_{\min}}^{+\nu_{\max}} e^{-\frac{x^2}{2}} \, dv = \frac{1}{\mu\sqrt{2\pi}} \int_{-\nu_{\max}}^{+\nu_{\max}} e^{-\frac{v^2}{2\mu^2}} \, dv$$

or simplified:

$$P(v) = \frac{1}{\mu\sqrt{2\pi}} \int_{-\nu_{\max}}^{+\nu_{\max}} \exp\left(-\frac{v^2}{2\mu^2}\right) dv \, .$$

In the series, errors greater than $x \cdot \mu$ also appear by a probability $1 - P(v)$. This is obvious because the sum of the two probabilities is the certitude equalling unity.

The occurrence of an error greater than a given value in a series means more exactly that at least one term in the series is greater than that value.

For a series of n terms (i.e., involving n measurements), this can be expressed as:

$$n[1 - P(v)] \geqq 1$$

expressing n:

$$n \geqq \frac{1}{1 - P(v)}$$

or, in particular:

$$n \geqq \frac{1}{1 - \frac{1}{\mu \sqrt{2\pi}} \int\limits_{-v_{max}}^{+v_{max}} \exp\left(-\frac{v^2}{2\mu^2}\right) dv}$$

or, with shortened notation

$$n \geqq F\left(\frac{v}{\mu}\right).$$

Let us see now how many measurements are needed that at least one error in the series should be greater than 3μ?

$$v_{max} \geqq 3\mu$$

Integration yields: $n = 385$. So many measurements are not made in practice, therefore it can be stated that in any series no maximum error greater than three times the mean error is found. Considering the v/μ value a dependent variable,

$$\frac{v}{\mu} \leqq f(n)$$

let us compute the value of $f(n)$ for different n values. The simplest way of calculation is to apply formula:

$$n \geqq F\left(\frac{v}{\mu}\right)$$

and to use an integral chart from which the values of $f(n)$ for integer n values are recalculated. (The substitution is made at $v = v_{max}$). (Table 1)

Table 1

$f(n)$ values recalculated from an integral chart

n	2	3	5	10	15	20	385
$f(n)$	0.67	1	1.30	1.64	1.85	2.00	3.00

$$\frac{v_{max}}{\mu} \leqq f(n)$$

may also be written in the form

$$\mu \geqq \frac{v_{\max}}{f(n)},$$

an approximate formula for the mean error. It can be used, provided the series of the measurements satisfies certain conditions, such as: uniform reliability of the measurements and normal distribution of errors. The uniform reliability warrants that no measurements giving extreme values are accidentally affected by a coarse error [2].

The sense of sign of inequality in the last formula is justified by the fact that $f(n)$ and $F(\frac{v}{\mu}$ are inverse functions and, in both cases, increasing arguments are associated with increasing functions.

Calculating the data in Table 1 between 2 and 15 one by one yields a chart function from which the lower limit of the mean error may be determined if the range is known, range being the difference between maximum and minimum of the measured values. If this lower limit is close to the mean error then the chart function may be used as a good estimate formula for the mean error. There are two ways to examine how erroneous the mean error is.

In both methods the n value limiting the table validity is sought for. The deviation obviously increases with n.

How great is the error of the approximate function

$$\mu_B \geq \frac{v_{\max}}{f(n)}$$

(i.e., of the chart function denoted by μ_B) for a predetermined value of n? Or better, what is the upper limit for the error in μ_B to keep lower than a predetermined bound?

Mean error of the mean error is considered the error bound. Be the mean error of the approximate mean error σ_B, the approximate mean error μ_B, and the exact, but unknown value of the mean error μ, then:

$$|\mu_B - \mu| \leq \sqrt{2}\,\sigma_B,$$

the difference of two erroneous quantities being equal to the square of the sum of their mean errors. This follows from the law of the propagation of errors, with the approximation that the values of the exact and approximate mean errors have equal mean errors.

Since

$$v_{\max} \leq \frac{\Delta}{2},$$

denoting by Δ the range of the measurement series, the above relationship may be written in detail:

$$\left|\frac{\Delta}{2}\frac{1}{f(n)} - \mu\right| \leq \frac{\sqrt{2}}{\sqrt{2(n-1)}} \cdot \frac{\Delta}{2} \cdot \frac{1}{f(n)}$$

The right-hand side of the inequality is $\sqrt{2}\,\sigma_B$, and

$$\sigma_B = \frac{\mu_B}{\sqrt{2(n-1)}}$$

it being the mean error of the mean error.

If μ is known, n can be computed from this inequality. n represents the number of measurements for which μ_B is practically of the same accuracy as μ. This problem, as has been said above, may be solved in two ways of different accuracy. The principle of the less exact one is as follows.

In lieu of μ, another estimate formula (fully independent of μ_B) is established, where only n and Δ are known.

$$\mu_\Delta = \mu_\Delta(n, d)$$

The independence is stressed, it being a condition for the above statement concerning the mean error of the differences to be valid.

J. SCHÜNKE [3] derived the lower and upper limits a and f, resp., of the mean errors of each result in a series of measurements from the knowledge of the maximum and minimum result as well as of the number of measurements:

$$a = \frac{\Delta}{2}\frac{\sqrt{2}}{\sqrt{n-1}}$$

$$f = \frac{\Delta}{2}\frac{\sqrt{n}}{\sqrt{n-1}}$$

Both formulae have been deduced analytically with considerations fully independent of the deduction of μ_B. While μ_B roots in the theory of probability, a and f are of purely algebraic origin.

Let the arithmetic mean μ_Δ of values a and f be the most reliable value of the unknown mean error. Evidently, this is an approximation, without knowing, however, the exact solution, it seems to be the most obvious. It would be worth while to examine the distribution in the series with limiting values a and f and to establish the most probable value of the series. Actually, however, an approximate formula for μ_A independent of μ_B but not the best approximation is sought for. Be then, for lack of a better one:

$$\mu_A = \frac{a+f}{2} = \frac{\Delta}{2}\,\frac{\sqrt{2}+\sqrt{n}}{2\sqrt{n-1}}$$

The limit of validity of this formula is sought for by the trial and error method, by substituting various n values both sides until the two sides are equal.

For $$n = 12,$$

$$0.17 = 0.17,$$

hence, this is the upper limit where the mean error of μ_A and μ_B is permissible.

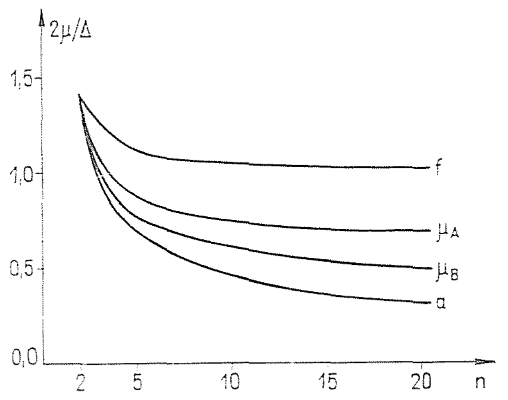

Fig. 1

The four functions a, f, μ_B and μ_A are linear functions of the maximum error, i.e., of the half range $\frac{\Delta}{2}$. Thus, dividing them by $\frac{\Delta}{2}$ yields n-dependent values (Fig. 1).

Now, there are two methods for estimating the mean error; one of them is a formula, the other consists in using a table and solving a formula. To decide upon which of the μ to apply, the aspect of economy is involved, their accuracy being all the same.

In the traditional calculation of the mean error $4n+1$ operations are to be carried out.

By using the formula μ_A, the necessary operations are: 1 subtraction, 2 divisions, 3 multiplications, 2 root extractions; altogether 8 operations.

For the calculation of μ_B one has to do with 1 subtraction, 1 looking up in the table $\frac{1}{2f(n)}$ and 1 multiplication. These are not more than 3 operations, and this number, just as for the calculation of μ_A, is independent of the number of measurements.

Percentage savings in the number of operations characterizing the economy are:

$$g_A\% = \frac{4n+1-8}{4n+1}\,100 \cong 100 - \frac{200}{n}\,,$$

and

$$g_B\% = \frac{4n+1-3}{4n+1}\,100 = 100 - \frac{75}{n}\,.$$

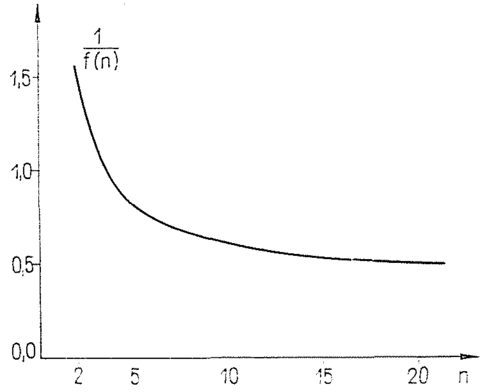

Fig. 2

The economy is seen to be better in the calculation of μ_B, especially if only a few measurements have been made, with the only disadvantage that a table is needed. It is advisable to establish a formula by function adjustment for the approximation of the data of the tabulated function [4].

The formula is required to be simple and sufficiently exact. Plotting the function $\frac{1}{f(n)}$ gives a hyperbola-like curve (Fig. 2). It is the equation of a hyperbola with straight lines

$$n = 1 \quad \text{and} \quad \frac{1}{f(n)} = 0$$

as asymptotes:

$$\frac{1}{f(n)} = \frac{\alpha}{(n-1)^{\beta}}$$

where α and β are parameters to be determined by adjustment.

Linearizing the unknown equation by plotting it to logarithmic scale, the two unknown values can be determined by dual adjustment [4].

The adjustment leads to $\beta = 0.36$; and to make the term $(n - 1)\beta$ accessible to the use of a slide rule, let us write:

$$\beta = 0{,}36 \cong \frac{1}{3}$$

Whence, $\alpha = 1.26$.
Substituting them in the formula for μ_B yields

$$\mu_B = \frac{\Delta}{2f(n)} = \frac{\Delta\alpha}{2(n-1)^\beta}, \quad \text{or} \quad \mu_B = \frac{0{,}63\Delta}{\sqrt[3]{n-1}} .$$

Calculation with this formula requires four operations, but as

$$0{,}63 \cong \frac{1}{\sqrt[3]{4}}$$

the formula will be transformed into

$$\mu_B = \frac{\Delta}{\sqrt[3]{4(n-1)}}$$

where only three operations are needed in case of not too many measurements, because $4(n - 1)$ may be calculated in head.

The formula for μ_B approaches μ from the lower side, thus

$$\mu_B = \mu$$

The formula may be refined by producing a great number of factitious measurements, computing from the series of measurements both the mean error μ and the estimated mean error μ_B. Their quotient is

$$\frac{\mu}{\mu_B} = \chi$$

If χ varies only in a narrow range, its mean value multiplied by μ_B yields μ with a good approximation.

A rather great number of factitious or simulated measurements were produced using an instrument similar to a micrometer microscope [5]. A spacing of 9 mm was halved at a 1 per cent accuracy, read off a graduated drum. Since to halve a spacing by estimate is the same as to hit the bull's eye in a score-card (the score card being here linear), the errors occurring are of normal distribution. Thus, series of measurements of normal distribution arise, remarkably suited for investigating the normal distribution. Production of an error does not need more than 3 seconds, accordingly this method is a very economic one.

The simulated measurement is a model of a real measurement, a model which from the viewpoint of the theory of probability may be characterized and observed as the true one and yields conclusions on the reality. This is the characteristic feature of the Monte-Carlo methods [6]. This more exact procedure mentioned on the second place is applied to refine the formula for μ_B and to expand its range of validity.

The number of the simulated measurements affects the reliability of the conclusions to be drawn from them. It should be taken into account, however, that the reliability must be both *satisfactory* and *identical*. The satisfactory or necessary reliability is defined by the magnitude of the deviations of the mean errors of the simulated measurements from the estimated mean error. For great deviations, reliable conclusions can only be drawn from a great number of measurements. Originally, 400 simulated measurements were to be used, but owing to the small deviations of μ to μ_B, only the first 80 results were utilized. The independence of measurements was safeguarded by utilizing them in their order of succession. The identity between measurement reliabilities was due to nearly equal weights provided by $m \cdot n = const.$ taken arbitrarily as 80 (Table 2).

Table 2

Number m of measurement series of n measurements. Both m and n being naturally integers, $m \cdot n$ is not absolutely 80

n	2	3	4	5	6	7	8	9	10	11	12	20
m	40	24	20	16	12	11	10	8	8	7	6	4

The tests were made by producing the series of measurements and calculating the mean errors both by the exact (μ) and the approximate method (μ_B), from the range of the series. Evidently, also in conformity with the deduction, the values

$$\mu > \mu_B$$

$$\frac{\mu}{\mu_B} = \chi$$

were calculated, showing but slight variations. The values showing the greatest specific deviations have been compiled in Table 3.

Table 3

Test values of exact (μ) and approximate (μ_B) mean errors and of their maximum deviations χ_{max}

n	μ					μ_B					χ_{max}
20	7,1	6.2	7.2	6.6		6.6	5.7	6.3	5.9		1.15
12	9.6	5,5	7.5	4.9	8.1	8.0	5.4	7.4	4.5	7.7	1.20
	6.1					6.2					
11	9.1	7.3	7.2	7.8	7.6	8.1	6.1	7.0	7.7	7.0	1.20
	6.5	6.2				5.9	5.2				
10	9.6	7.1	6.9	6.6	6.7	8.5	6.7	7.3	5.8	6.4	1.14
	7.5	6.4	7.0			7.3	6.7	7.6			
9	9.5	7.8	7.4	6.3	4.7	8.9	7.6	6.4	6.0	4.4	1.19
	8.0	6.4	6.1			8.6	5.4	7.0			
8	8.4	9.4	3.7	8.3	4.9	7.9	8.2	3.0	7.9	4.9	1.23
	6.0	8.0	6.5	6.3	7.7	4.9	8.9	5.6	7.2	8.2	
	7.4	11.3	3.5	7.4	5.9	6.9	9.7	3.1	6.9	5.9	1.19
7	5.6	5.4	8.9	7.0	6.9	5.2	4.5	9.4	5.9	7.6	
	6.1					5.9					
	7.3	12.4	6.4	6.7	8.9	6.9	10.3	5.9	6.1	8.3	
6	6.4	4.2	5.7	9.8	6.8	6.6	4.9	4.8	8.4	5.5	1.23
	6.6	6.1	6.9			6.9	5.9	5.9			
	4.5	11.3	6.9	3.8	3.1	4.4	10.2	5.9	3.3	2.9	
5	9.8	5.7	6.1	5.5	2.9	8.4	5.1	5.5	4.4	2.6	1.19
	8.7	7.1	4.8	8.9	7.0	8.7	6.3	4.8	8.0	6.2	
	6.5					6.6					
	3.7	5.0	12.2	7.4	4.2	4.0	5.2	10.9	7.4	3.9	
4	3.6	7.0	8.0	4.8	3.9	3.5	6.5	7.4	4.4	3.9	1.20
	3.5	0.5	11.5	6.3	7.7	3.5	0.4	11.6	6.1	7.4	
	6.8	4.6	7.8	7.1	4.3	5.7	4.8	7.4	7.4	4.4	
	4.5	7.2	14.8	12.0	6.5	4.5	7.0	14.0	12.0	6.5	
	3.8	4.4	3.6	8.7	7.2	3.5	4.0	3.5	7.5	6.5	
3	7.6	5.0	3.8	4.6	6.2	7.0	5.0	3.5	4.0	6.0	1.20
	0.6	6.0	12.0	4.0	8.7	0.5	6.0	12.0	3.5	8.5	
	6.8	5.9	7.4	8.2	8.6	6.5	7.0	6.0	3.0	8.5	
	5.3					5.0					

As is shown in the table, also χ_{max} is but slightly scattered. The mean of this maximum χ is 1.2. Since $\chi_{min} = 1$, the mean value of χ may be taken as

$$\chi = \frac{\chi_{min} + \chi_{max}}{2} = 1{,}1\,.$$

Be μ_B' the corrected value of μ_B.

$$\mu_B' = \chi \mu_B = \frac{0{,}63 \cdot 1{,}1\,\Delta}{\sqrt[3]{n-1}}$$

thus

$$\mu_B' = \frac{0{,}69\,\Delta}{\sqrt[3]{n-1}}$$

and since

$$0{,}69 \cong \frac{1}{\sqrt[3]{3}}$$

we have

$$\mu_B' = \frac{\Delta}{\sqrt[3]{3(n-1)}}\,.$$

In Table 3 the value $n = 2$ is not included. Firstly, the formula does not give a good result, namely

$$\mu = \frac{\Delta}{\sqrt{2}} = \frac{\Delta}{1{,}41} \quad \text{and} \quad \mu_B = \frac{\Delta}{\sqrt[3]{4}} = \frac{\Delta}{1{,}59}$$

are rather different.

The second reason is that in case of two measurements μ_B is easier to calculate by the usual exact formula

$$\mu_B = \frac{\Delta}{\sqrt{2}}\,.$$

The corrected formula is better also from this point of view, because it can also be used for $n = 2$. Namely, between

$$\mu = \frac{\Delta}{\sqrt{2}} = 0.71\Delta \quad \text{and} \quad \mu_B' = \frac{\Delta}{\sqrt[3]{3}} = 0.69\Delta$$

there is a deviation of only 3 per cent. In general, the corrected formula gives very accurate values. Fig. 3 is a plot of values

$$\frac{2\mu}{\Delta} \quad \text{and} \quad \frac{2\mu'_B}{\Delta}$$

with $\mu = \mu'_B =$ maximum. Even these are seen in Fig. 3 to be very small values.

The advantage of the corrected estimate formula is its wider range of validity. Also this fact may be examined in two ways, either with μ_A or with μ (mean errors with $\mu'_B = \mu =$ maximum among those of the simulated measurements).

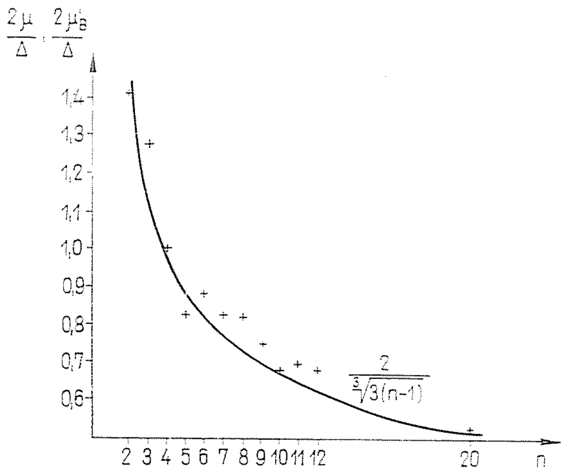

Fig. 3

Comparison with μ_A can be expressed as:

$$\mu'_B - \mu_A \leqq \frac{\sqrt{2}\,\mu'_B}{\sqrt{2(n-1)}} .$$

Simplified and rearranged:

$$\frac{\mu'_B - \mu_A}{\mu'_B} \leqq \frac{1}{\sqrt{n-1}} .$$

In this expression different n values are replaced to find the n values where left-hand and right-hand sides are equal.

For safety's sake $n = 16$ is taken as limiting value, accordingly the range of validity of μ'_B is $2 \leqq n \leqq 16$. The other method is similar, only that here μ'_B is compared with the true mean error computed from the simulated measurements. Obviously, this method is nearer to the reality:

$$\mu - \mu'_B \leq \frac{\sqrt{2}\,\mu'_B}{\sqrt{2(n-1)}}$$

$$\frac{\mu - \mu'_B}{\mu'_B} \leq \frac{1}{\sqrt{n-1}}$$

The formula can be utilized up to $n = 20$.

Economy of this formula is:

$$g_B\% = 100 - \frac{75}{20} \cong 96\%,$$

i.e., only 4 per cent of the operations have to be carried out for obtaining the estimate value.

Summarizing what has been said above, an estimation formula for the mean error has been found:

$$\mu'_B = \frac{\Delta}{\sqrt[3]{3(n-1)}}$$

with an inherent error practically identical to the mean error of the mean error, and may be calculated in the range

$$2 \leq n \leq 20.$$

The formula involves three operations accessible to the use of a slide rule. It has been deduced in conformity with the theory of probability, and its accuracy and range of validity have been increased by using the simulation method.

Summary

Probability considerations permit to set a lower limit for the mean error as a function of number and range of the measurements. The formula can be refined and its range of validity widened by the method of simulation. The advantage of the formula is to need only three elementary operations, and the error involved does not exceed the mean error of the mean error, accordingly, it lends itself to the estimation of the mean error.

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Associate Prof. Ottó L'AUNÉ, Budapest XI., Műegyetem rkp. 3. Hungary