# FLEXURAL ANALYSIS OF SECTOR SHELLS CUT OF A SINGLE-SHELL HYPERBOLOID OF REVOLUTION*

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## 1. Introduction

A calculation method will be presented for the flexural analysis of forces acting in sector shells cut out of a single-shell hyperboloid of revolution. These shells are highly convenient as shallow shell roofs, taking into consideration also the aspects of construction. It follows that shell forces are decisively affected by snow and dead loads. Consequently, in what follows, these two load types will be considered. In previous papers [9—11] the same problem was treated by the membrane theory and by the theory of geometry, respectively. The presented analysis suits to rather exactly determine internal forces in shells and besides, it lends itself to check the earlier approximation methods if a digital computer is used.

## 2. Derivation and features of the shell surface

In the co-ordinate system $\bar{x},\bar{y},z$ (Fig. 1) the part cut out of a single-shell hyperboloid of revolution with axis $\bar{y}$ defined by:

$$\frac{\bar{x}^2+z^2}{r_0^2}-\frac{\bar{y}^2}{\bar{b}^2}=1 \tag{1}$$

by two planes passing through the axis and including an acute central angle lends itself as a shell roof over rectangular floor plan. $\bar{b}$ in Eq. (1) is the half length of the fictitious axis of the hyperbola in the meridian principal section.

The so derived sector — surface part of hyperboloid of revolution — is confined by two circular arcs of radius $r_1$ in the vertical plane and by two hyperbolic edges of skew plane (Fig. 1). Because of the skew-plane hyperbolic edges, the basis under this part differs from a rectangle. The difference is, however, rather unimportant in case of shallow shells derived from a hyper-

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boloid of revolution with a relatively great radius and short axis, and can still be reduced for even shallower shells. If a vertical gable wall is required, a conoid part may be added to the skew-plane hyperbola.

Surface in Fig. 1 is seen to have all sections in planes normal to or coincident with the axis of revolution with negative or positive curvatures, respectively. Because of different signs of curvatures for each section family,

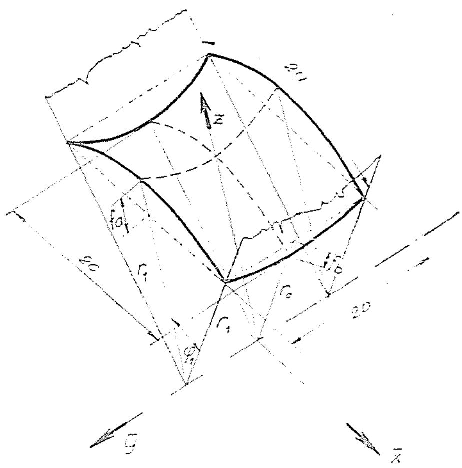

*Fig. 1*

the Gaussian multiplication curvature is negative in any point of the shell, so from differential geometry aspects, any point is hyperbolic. Besides, the surface is of constant curvature in one direction, and can be constructed with straight generatrices: the resulting highly favourable geometry facilitates its use in practice.

## 3. Basic equations

### *3.1 Assumptions. The applied co-ordinate system*

In general, spatial stress state prevails in shell points. There are ten internal forces to be considered for the stress state, namely two normal forces, two shear forces, two tangential forces, two flexural moments and two torsional moments (Fig. 3). An analysis starting from this model involves a highly complex calculation problem, involving some steps practically inaccessible to conventional calculation even after simplifications inherent with the shell shallowness. This fact urged the development of the algorithm to be presented, solved by means of a digital computer. This algorithm yielded numerically the order of magnitude of the flexural stresses, occurring also inside the shell because of the rectilinear generatrix of the shell surface, not only on the boundaries where edge beams do not follow shell deformations.

Assumptions for the fundamental equations are the same as usual in the theory of elasticity of shell structures:

1. The shell material (reinforced concrete) is homogeneous, isotropic, ideally elastic and follows the Hooke law;
2. the shell is thin as compared to other shell dimensions;
3. the stress component $\sigma_z$ normal to the shell medium surface is negligible as compared to other stress components;
4. the Navier hypothesis is valid;
5. deformations are small as compared to the shell thickness;
6. deformations due to flexural shear are negligible;
7. shell surface is shallow, i.e. rise to span ratio is less than 0.2 in either direction;
8. in the shell equations the higher derivatives are significantly greater than are lower derivatives, hence these latter can be considered negligible for sake of convenience.

Idealizing the shell geometry, its medium surface halving the interspace of two boundary surfaces will be indicated in a convenient co-ordinate system defined as follows: It is endeavoured to formulate the problem in a co-ordinate system leading to equations similar in form — with unavoidable deviations — to results relating to translation shells considered in a plane orthogonal co-ordinate system. Such a co-ordinate system consists of a cylindrical surface, coaxial and tangential to the shell surface, i.e. a cylindrical co-ordinate surface matched to the least circular arc of the surface, the co-ordinate lines of which are the directrix circle (arc length $x$ or central angle $\varphi$), the generatrix ($y$ axis) and the normal ($z$ axis).

The sector shell surface cut out of a hyperboloid of revolution together with the co-axial cylindrical co-ordinate surface, the co-ordinate lines and an infinitesimal part with sides $dx$, $dy$ and $ds_x$, $ds_y$ belonging to an arbitrary point with co-ordinates $x$, $y$, $z$ are shown in Fig. 2. In the described co-axial co-ordinate system the shell surface equation can be written by means of the equation of the cross-wise hyperbola

$$z = r_0 - r_0 \sqrt{1 + \frac{r_1^2 - r_0^2}{r_0^2} \cdot \frac{y^2}{b^2}} \,. \tag{2}$$

### 3.2 Equilibrium equations

The infinitesimal part (Fig. 2) and its projection in the co-ordinate surface are shown enlarged in Fig. 3, together with their stresses in positive sense. Stresses acting in the surface part

$$(N_x, N_{xy}, Q_x, M_x, M_{xy} \text{ and } N_y, N_{yx}, Q_y, M_y, M_{yx})$$

are *real stresses*, while their projections in the co-ordinate surface

$$(\bar{N}_x, \bar{N}_{xy}, \bar{Q}_x, \bar{M}_x, \bar{M}_{xy} \text{ and } \bar{N}_y, \bar{N}_{yx}, \bar{Q}_y \bar{M}_y, \bar{M}_{yx})$$

are *redd stresses*.

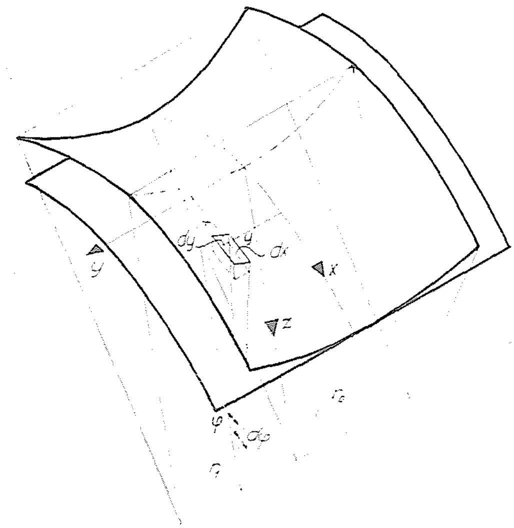


*Fig. 2*

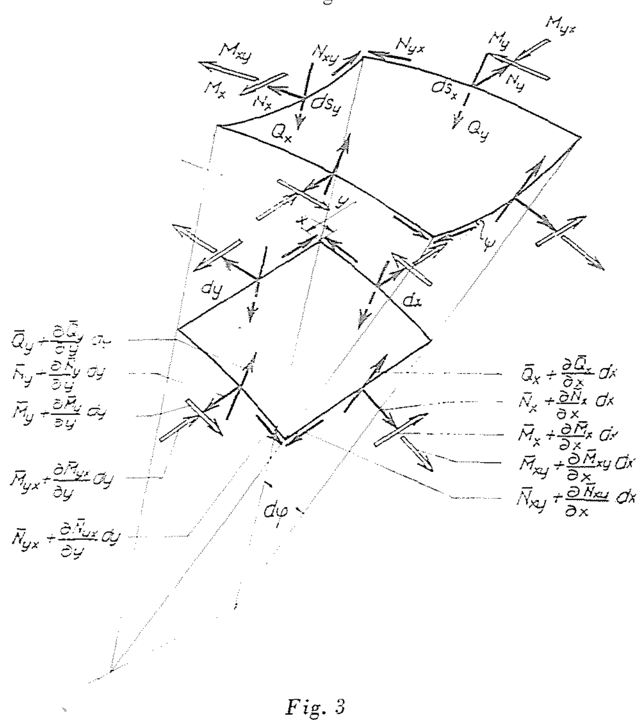


*Fig. 3*

The surface part and its projection in the co-ordinate surface are connected by the geometrical relationship:

$$\left.\begin{aligned} dx &= \frac{r_0}{r}\, ds_x \\ dy &= ds_y \cos\psi \end{aligned}\right\} \tag{3}$$

where $r_0$ radius of the least circle (Fig. 2);
$r$ radius belonging to an arbitrary point of the hyperboloid; and
$\psi$ angle included between the tangent to the hyperbola and the generatrix $y$ of the co-ordinate surface (Fig. 3).

Six equations expressing the equilibrium of the infinitesimal part referred to the co-ordinate surface will be written by means of the reduced stresses. Since, however, relationships for the real stresses on the surface part are needed, correlations between reduced and real stresses have to be known. These can be written by means of projection (Fig. 3) and Eqs (3) as follows:

$$\left.\begin{aligned} \bar{N}_x &= \frac{N_x}{\cos\psi} & \bar{N}_y &= \frac{r}{r_0}(N_y\cos\psi - Q_y\sin\psi) \\ \bar{N}_{xy} &= N_{xy} - Q_x \operatorname{tg}\psi & \bar{N}_{yx} &= \frac{r}{r_0} N_{yx} \\ \bar{Q}_x &= Q_x + N_{xy}\operatorname{tg}\psi & \bar{Q}_y &= \frac{r}{r_0}(Q_y\cos\psi - N_y\sin\psi) \\ \bar{M}_x &= M_x & \bar{M}_y &= \frac{r}{r_0} M_y \\ \bar{M}_{xy} &= \frac{M_{xy}}{\cos\psi} & \bar{M}_{yx} &= \frac{r}{r_0} M_{yx}\cos\psi\,. \end{aligned}\right\} \tag{4}$$

Applying neglects usual and permissible for shallow shells, the following approximations will be introduced:

$$\cos\psi \approx 1.0; \qquad \sin\psi \approx \operatorname{tg}\psi \tag{5}$$

and from (3) $\qquad dy \approx ds_y$.

Terms $Q_x \operatorname{tg}\psi$ and $Q_y \sin\psi$ in expressions for $\bar{N}_{xy}$ and $\bar{N}_y$, resp., Eqs (4), can be omitted, since flexural shear stresses $Q_x$ and $Q_y$ are much lower than are membrane stresses and besides, they are multiplied by the tangent of the small angle $\psi$. Right-side column of Eqs (4) contains the ratio $r/r_0$, that can also be written as follows, assuming matching of the infinitesimal part to the section $y = 0$:

$$\frac{r}{r_0} = \frac{r_0 + \Delta s_y \sin\psi}{r_0} = 1 + \frac{\Delta y}{r_0}\,\mathrm{tg}\,\psi = 1 + \frac{\Delta y}{r_0}\frac{\Delta z}{\Delta y} = 1 + \frac{\Delta z}{r_0}\,. \tag{6}$$

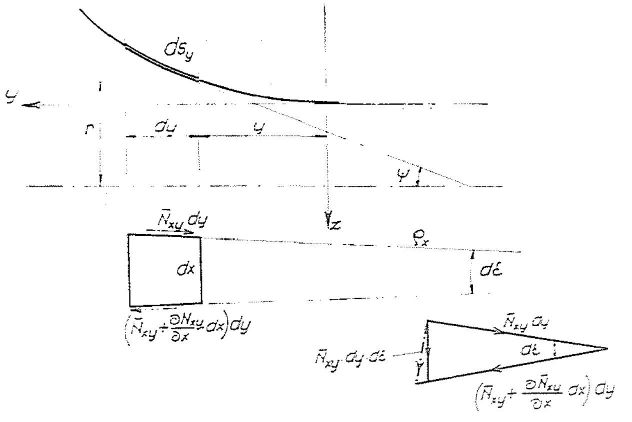

Fig. 4

The shell being a shallow one, term $\Delta z/r_0$ can be omitted with respect to unity. Introducing the above simplifications, Eqs (4) will be of the form:

$$\left.\begin{array}{ll} \bar{N}_x \approx N_x & \bar{N}_y \approx N_y \\ \bar{N}_{xy} \approx N_{xy} & \bar{N}_{yx} \approx N_{yx} \\ \bar{Q}_x = Q_x + N_{xy}\cdot \mathrm{tg}\ \psi & \bar{Q}_y \approx Q_y + N_y \cdot \mathrm{tg}\ \psi \\ \bar{M}_x = M_x & \bar{M}_y \approx M_y \\ \bar{M}_{xy} \approx M_{xy} & \bar{M}_{yx} \approx M_{yx}. \end{array}\right\} \tag{7}$$

Taking into consideration relationships

$$\left.\begin{array}{l} \mathrm{d}\varphi = \dfrac{\mathrm{d}x}{r_0}; \qquad \mathrm{d}\varepsilon = \dfrac{\mathrm{d}x \sin\psi}{r_0} \\ \bar{N}_{xy}\,\mathrm{d}y\,\mathrm{d}\varepsilon = \bar{N}_{xy}\,\mathrm{d}x\,\mathrm{d}y\,\dfrac{\sin\psi}{r_0} \cong \bar{N}_{xy}\,\mathrm{d}x\,\mathrm{d}y\,\dfrac{\mathrm{tg}\,\psi}{r_0} \end{array}\right\} \tag{8}$$

related to Figs 3 and 4 as well as Figs 5$a$ and $b$, the formula for $M_x$ similar to Eq. (8) and approximations

$$\cos\frac{d\varepsilon}{2} \approx 1{,}0\,; \qquad \cos\frac{\mathrm{d}\varphi}{2} \approx 1{,}0\,,$$

projection equations for the three axes expressing the equilibrium of the infinitesimal part, and moment equations also for the three axes will be of the form:

$$
\left.
\begin{aligned}
&\frac{\partial \bar{N}_x}{\partial x} + \frac{\partial \bar{N}_{yx}}{\partial y} + \bar{N}_{xy}\frac{\operatorname{tg}\psi}{r_0} + \bar{Q}_x\frac{1}{r_0} + X = 0\\
&\frac{\partial \bar{N}_y}{\partial y} + \frac{\partial \bar{N}_{xy}}{\partial x} + Y = 0\\
&\frac{\bar{N}_x}{r_0} - \frac{\partial \bar{Q}_x}{\partial x} - \frac{\partial \bar{Q}_y}{\partial y} - Z = 0\\
&\frac{\partial \bar{M}_y}{\partial y} + \frac{\partial \bar{M}_{xy}}{\partial x} + \bar{Q}_y - \bar{M}_x\frac{\operatorname{tg}\psi}{r_0} = 0\\
&\frac{\partial \bar{M}_x}{\partial x} + \frac{\partial \bar{M}_{yx}}{\partial y} + \bar{Q}_x = 0\\
&\bar{N}_{xy}\frac{r}{r_0} - \bar{N}_{yx} + \frac{\bar{M}_{xy}}{r_0} = 0
\end{aligned}
\right\} \qquad (9)
$$

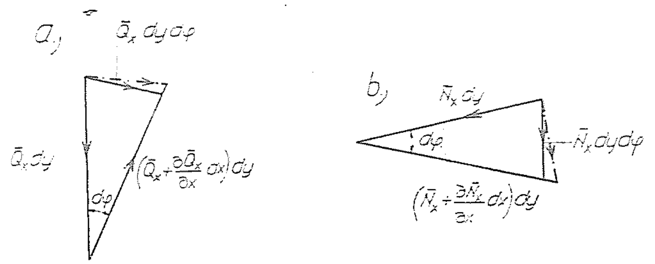

*Fig. 5*

where $X$, $Y$ and $Z$ are load function components for the co-ordinate surface. The last equation of Eqs (9) corresponds to the known theorem of reciprocity of plane problems. This one will not be made use of.

Eqs (7) will be applied to convert reduced stresses of Eqs (9) into real stresses. The fourth and fifth equations of this system of equations contain only flexural stresses, and in order to avoid disturbance from flexural stresses, reduction of their importance, approximations $\bar{Q}_x \approx Q_x$ and $\bar{Q}_y \approx Q_y$ will be introduced in Eqs (7) (see [3], p. 99).

In conformity with the above, after operations indicated in the first and third equations of Eqs (9) and arranging:

$$
\left.
\begin{aligned}
&\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} + 2N_{xy}\,\frac{\operatorname{tg}\psi}{r_0} + Q_x\frac{1}{r_0} + X = 0\\
&\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} + Y = 0\\
&\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + z'' N_y - N_x\frac{1}{r_0} - Y\operatorname{tg}\psi - Z = 0\\
&\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} + Q_y - M_x\frac{\operatorname{tg}\psi}{r_0} = 0\\
&\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} + Q_x = 0.
\end{aligned}
\right\} \qquad (10)
$$

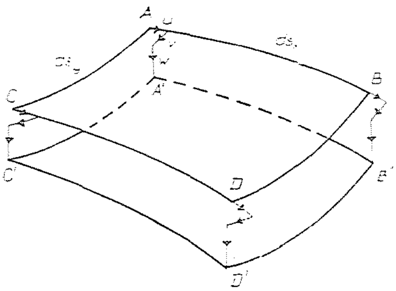


*Fig. 6*

### 3.3 Geometrical equations

There exist several references, e.g. [3] and [4], on geometrical equations for an infinitesimal part cut out of a shell surface of arbitrary form plotted in a surface co-ordinate system.

Original and deformed shapes of the infinitesimal part cut out in Fig. 2 are shown in Fig. 6.

Approximations applied when deducing Eqs (10) expressing the equilibrium of the infinitesimal shell surface are considered valid in relation to the geometrical equations too.

From approximations (5) and (6) it follows that $\partial s_x$ and $\partial s_y$; and $\partial x$ and $\partial y$ can be exchanged. The geodesic curvature pertaining to normal sections $z$—$y$ and distorsion of the surface $(1/r_{xy})$ are zero. The geodesic curvature belonging to the normal sections $z$—$x$, can, however, be considered zero, if the slightly trapezoidal form of the shell part is ignored, in accordance with the above.

Accordingly, the geometrical equations of the shell part are the following:

Specific strains in directions $x$ and $y$, rotation between the two directions and rotations for the length of arch are:

$$\left.\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} - \frac{w}{r_x} \\ \varepsilon_y &= \frac{\partial v}{\partial y} + \frac{w}{r_y} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{aligned}\right\} \quad (11) \qquad \left.\begin{aligned} \chi_x &= \frac{\partial w}{\partial x} + \frac{u}{r_x} \\ \chi_y &= \frac{\partial w}{\partial y} - \frac{v}{r_y} \\ \chi_z &= \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \end{aligned}\right\} \quad (12)$$

Curvature changes expressed by means of Eqs (11) and (12):

$$\left.\begin{aligned} \varkappa_x &= \frac{\partial \chi_x}{\partial x} \\ \varkappa_y &= \frac{\partial \chi_y}{\partial y} \\ \varkappa_{xy} &= \frac{1}{2}\left[\frac{\partial \chi_y}{\partial x} + \frac{\partial \chi_x}{\partial y} + \frac{\gamma_{xy}}{2}\left(\frac{1}{r_x} - \frac{1}{r_y}\right)\right]. \end{aligned}\right\} \quad (13)$$

Here $u$, $v$ and $w$ are displacements in directions $x$, $y$ and $z$, and $1/r_x$, $1/r_y$ are curvatures of normal sections. Using the first two equations of (12) and the third one of (11), curvature changes (13) can be expressed as:

$$\left.\begin{aligned} \varkappa_x &= \frac{\partial^2 w}{\partial x^2} + \frac{1}{r_x}\frac{\partial u}{\partial x} \\ \varkappa_y &= \frac{\partial^2 w}{\partial y^2} - \frac{\partial}{\partial y}\left(\frac{1}{r_y} v\right) \\ \varkappa_{xy} &= \frac{\partial^2 w}{\partial x\,\partial y} + \frac{1}{2}\left[\frac{\partial}{\partial y}\left(\frac{u}{r_x}\right) - \frac{1}{r_y}\frac{\partial v}{\partial x} + \right. \\ &\quad \left. + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\left(\frac{1}{r_x} - \frac{1}{r_y}\right)\right]. \end{aligned}\right\} \quad (14)$$

All of the Eqs (14) are composed of two parts. As compared to main terms containing $w$, all other terms are negligible in conformity with assumption

3.1/8. Thus, unit deformations referred to the medium surface of the infinitesimal shell part can be expressed as:

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} - \frac{w}{r_x} & \varkappa_x &= \frac{\partial^2 w}{\partial x^2} \\ \varepsilon_y &= \frac{\partial v}{\partial y} + \frac{w}{r_y} & \varkappa_y &= \frac{\partial^2 w}{\partial y^2} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & \varkappa_{xy} &= \frac{\partial^2 w}{\partial x \, \partial y} \end{aligned} \qquad (15)$$

Stresses in a point of arbitrary ordinate $z$ can be expressed by specific strain units for the same point. Hence, strain units (15) are to be replaced by deformation units for the surface of ordinate $z$. On the other hand, the assumption made for shallow shells permits to apply simplifications, so that, as a final result. stresses can be expressed by means of strain units for the medium surface. Purely to illustrate the transformations, let us see specific strain in direction $x$:

$$\varepsilon_x^{(z)} = \frac{\varepsilon_x - z\varkappa_x}{1 - \frac{z}{r_x}} \approx \varepsilon_x - z\left(\varkappa_x - \frac{\varepsilon_x}{r_x}\right) \qquad (16)$$

Here the approximate expression results from taking into consideration the first two terms of the power series $1/\left(1 - \frac{z}{r_x}\right)$ since:

$$\frac{1}{1 - \frac{z}{r_x}} \approx 1 + \frac{z}{r_x}$$

and for the term $z\varkappa_x$ the approximation $1/-\frac{z}{r_x} \approx 1.0$ is admissible, since it only corrects the term $\varepsilon_x$.

### 3.4 Physical equations

Notations:

| | |
|---|---|
| $E$ | modulus of elasticity of reinforced concrete |
| $\mu$ | Poisson's coefficient of reinforced concrete |
| $\delta$ | shell thickness |
| $D = \dfrac{E\delta}{1-\mu^2}$ | tensile or compressive rigidity |
| $K = \dfrac{E\delta^3}{12(1-\mu^2)}$ | flexural rigidity. |

Hooke's law in the theory of elasticity is of the general form:

$$\left.\begin{aligned}\sigma_x^{(z)} &= \frac{E}{1-\mu^2}\left(\varepsilon_x^{(z)} + \mu\varepsilon_y^{(z)}\right)\\ \sigma_y^{(z)} &= \frac{E}{1-\mu^2}\left(\varepsilon_y^{(z)} + \mu\varepsilon_x^{(z)}\right)\\ \tau_{yx}^{(z)} &= \frac{E}{2(1+\mu)}\gamma_{xy}^{(z)}.\end{aligned}\right\} \tag{17}$$

Three stress components expressed in (17) help to write all ten functions of internal forces in Fig. 3. For instance, using the stress component $\sigma_x$:

$$\left.\begin{aligned}N_x &= \int_{-\delta/2}^{+\delta/2} \sigma_x^{(z)}\left(1+\frac{z}{r_y}\right)dz\\ M_x &= \int_{-\delta/2}^{+\delta/2} \sigma_x^{(z)} z\left(1+\frac{z}{r_y}\right)dz.\end{aligned}\right\} \tag{18}$$

Substituting $\sigma_x^{(z)}$ from (17) into (18), accomplishing operations and arranging yields for $N_x$ and $M_x$:

$$\left.\begin{aligned}N_x &= D(\varepsilon_x + \mu\varepsilon_y) + \frac{K}{r_x^2}\left(1+\frac{r_x}{r_y}\right)(\varepsilon_x - r_x\varkappa_x)\\ M_x &= -K\left[\varkappa_x + \mu\varkappa_y - \frac{1}{r_x}\left(1+\frac{r_x}{r_y}\right)\varepsilon_x\right].\end{aligned}\right\} \tag{19}$$

Expression (19) for the normal force $N_x$ consists of two parts, just as do those for internal forces $N_{xy}$, $N_y$ and $N_{yx}$, namely the parts multiplied by $D$ and by $K/r_x^2$ and $K/r_y^2$, respectively. Since $D \gg K/r_x^2 : K/r_y^2$, terms multiplied by $K$ are omissible as compared to those multiplied by $D$.

Also the expressions (19) for the bending moment $M_x$ and for moments $M_{xy}$, $M_y$ and $M_{yx}$ consist of two parts. One part contains the effect of terms $\varkappa$ expressing the curvature changes and the other the effect of the specific strain or angular distortion. The latter is in any case multiplied by a curvature, so this magnitude can be omitted as compared to the first part. Accordingly, the approximate physical equations of the problem are of the form:

$$\left.\begin{aligned}N_x &= D(\varepsilon_x + \mu\varepsilon_y)\\ N_y &= D(\varepsilon_y + \mu\varepsilon_x)\\ N_{xy} &= N_{yx} = D\frac{1-\mu}{2}\gamma_{xy}\\ M_x &= -K(\varkappa_x + \mu\varkappa_y)\\ M_y &= -K(\varkappa_y + \mu\varkappa_x)\\ M_{xy} &= M_{yx} = -K(1-\varkappa)\mu_{xy}.\end{aligned}\right\} \tag{20}$$

## 4. Deduction of shell equations

### 4.1 Load function definitions

Dead load and snow load distribution diagrams plotted in planes $x$—$z$ and $y$—$z$ are seen in Fig. 7, together with the resultant diagrams. In load functions contained in the system of equilibrium equations (10), the load component in direction $y$ equals zero ($Y = 0$) in view of the load projections

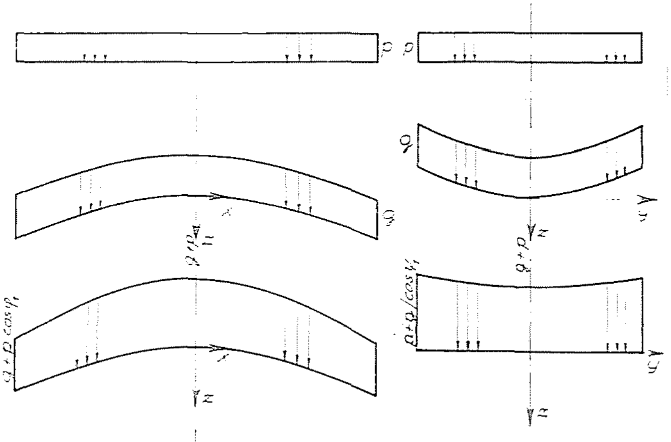

*Fig. 7*

in the co-ordinate surface, while load components in directions $x$ and $z$ can be expressed as:

$$\left.\begin{aligned} X &= \left(\frac{g}{\cos\psi} + p\cos\varphi\right)\frac{r}{r_0}\sin\varphi \\ Z &= \left(\frac{g}{\cos\psi} + p\cos\varphi\right)\frac{r}{r_0}\cos\varphi\,. \end{aligned}\right\} \tag{21}$$

### 4.2 The first shell equation

From Eqs (10) the terms containing $Y$ vanish, at the same time (assuming the appropriate load case) the equation system differs from the equilibrium equations of shallow translation shells referred to an orthogonal co-ordinate system by its first equation containing also terms $2N_{xy}\ \mathrm{tg}\,\psi/r_0$ and $Q_x/r_0$ and the fourth one including also the term $M_x\,\mathrm{tg}\,\psi/r_0$. These deviating terms can be interpreted as:

Term $M_x\,\mathrm{tg}\,\psi/r_0$ in the fourth equation can be omitted as compared to the other terms, $M_x$ being not only small but also multiplied by a small curvature and the tangent of $\psi$, a small angle. Term $2N_{xy}\ \mathrm{tg}\,\psi/r_0$ in the first equation will not be omitted, though its multipliers ($\mathrm{tg}\,\psi \cdot 1/r_0$) are small and so it cannot be of importance. Neither the term $Q_x/r_0$ will be omitted. The procedure will be simplified by making it iterative, that is, by solving Eqs (10) in two steps. The first step will consist in solving the equation system — from which the term $(2N_{xy}\ \mathrm{tg}\,\psi + Q_x)/r_0$, has been omitted — for load functions $X$ and $Z$, then term $(2N_{xy}\ \mathrm{tg}\,\psi + Q_x)/r_0$ is established with values $N_{xy}$ and $Q_x$ obtained in the first step. The second step consists in solving the problem again for the effect of this term as load acting in direction $x$. In spite of the iterative character of the problem, after the fundamental step at most one accessory step is needed. Taking the above into consideration, in the first step Eqs (10) assume the form:

$$\left.\begin{aligned}
&\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} + X = 0\\
&\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0\\
&\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial x} + z'' N_y - \frac{1}{r_0} N_x - Z = 0\\
&\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} + Q_y = 0\\
&\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} + Q_x = 0 .
\end{aligned}\right\} \tag{22}$$

Substituting relationships

$$\left.\begin{aligned}
N_x &= \frac{\partial^2 F}{\partial y^2} - \int_{x_0}^{x} X\,dx\\
N_y &= \frac{\partial^2 F}{\partial x^2}\\
N_{xy} &= -\frac{\partial^2 F}{\partial x\,\partial y}
\end{aligned}\right\} \tag{23}$$

known from the literature into the first three equations of (22), the first two equations are identically satisfied, while the third one assumes the form:

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + z''\frac{\partial^2 F}{\partial x^2} - \frac{1}{r_0}\frac{\partial^2 F}{\partial y^2} = Z - \frac{1}{r_0}\int_0^x X\,dx = P_1(x, y) . \tag{24}$$

In view of (15), the last three equations of (20) are:

$$\left.\begin{aligned} M_x &= -K\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right) \\ M_y &= -K\left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\right) \\ M_{xy} = M_{yx} &= -K(1-\mu)\frac{\partial^2 w}{\partial x\,\partial y}\,. \end{aligned}\right\} \tag{25}$$

Substituting Eqs (25) into the last two equations of (22) yields for the shear forces:

$$\left.\begin{aligned} Q_x &= K\frac{\partial}{\partial x}(\Delta w) \\ Q_y &= K\frac{\partial}{\partial y}(\Delta w) \end{aligned}\right\} \tag{26}$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\,. \tag{27}$$

Substituting (26) into (24) delivers the first shell equation:

$$K\,\Delta\Delta w - z''\frac{\partial^2 F}{\partial x^2} - \frac{1}{r_0}\frac{\partial^2 F}{\partial y^2} = P_1(x, y)\,. \tag{28}$$

This is a relationship between displacement function $w$, stress function $F$ and the loads.

### 4.3 *The second shell equation*

By means of Eqs (11), a compatibility equation devoid of displacement components $u$ and $v$ can be written:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x\,\partial y} = \frac{1}{r_y}\frac{\partial^2 w}{\partial x^2} - \frac{1}{r_x}\frac{\partial^2 w}{\partial y^2} \tag{29}$$

omitting small terms of second order.

First three equations of (20) yield relationships:

$$\left.\begin{aligned} \varepsilon_x &= \frac{1}{E\delta}(N_x - \mu N_y) \\ \varepsilon_y &= \frac{1}{E\delta}(N_y - \mu N_x) \\ \gamma_{xy} &= \frac{2(1+\mu)}{E\delta} N_{xy}\,. \end{aligned}\right\} \tag{30}$$

By means of (23) and (30) the left-hand side of (29) can be written as:

$$\frac{1}{E\delta}\left\{\left[\frac{\partial^4 F}{\partial x^4} + 2\frac{\partial^4 F}{\partial x^2\,\partial y^2} + \frac{\partial^4 F}{\partial y^4}\right] + \left[\mu\frac{\partial X}{\partial x} - \int_{x_0}^{x}\frac{\partial^2 X}{\partial y^2}\,dx\right]\right\} =$$

$$= \frac{1}{E\delta}\left\{\Delta\Delta F + P_2(x, y)\right\}. \tag{31}$$

(29) and (31) deliver the other shell equation:

$$\Delta\Delta F - E\delta\left[\frac{1}{r_y}\frac{\partial^2 w}{\partial x^2} - \frac{1}{r_x}\frac{\partial^2 w}{\partial y^2}\right] = -P_2(x, y) \tag{32}$$

where

$$\Delta\Delta = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\,\partial y^2} + \frac{\partial^4}{\partial y^4}\,. \tag{33}$$

This is another relationship between displacement function $w$, stress function $F$ and loads.

### 4.4 *Determination of the right-hand sides of shell equations*

Load functions in form (21) are inconvenient to integration and so they are from the aspect of boundary conditions, hence they will be approximated by an appropriate function. The variation along the $y$ axis of both functions is described by the cosine of the angle $\psi$ and by the factor $r/r_0$. Practically, this variation is easy to follow by an integrable hyperbolic cosine function of a single wave. Thus, exact load functions (21) will be replaced by:

$$\left.\begin{aligned} X &= \mathrm{ch}\,\gamma\, y(g \sin\alpha\, x + p \sin\alpha\, x \cos\alpha\, x) \\ Z &= \mathrm{ch}\,\gamma\, y(g \cos\alpha\, x + p \cos^2\alpha\, x) \end{aligned}\right\} \tag{34}$$

with $\gamma =$ constant, obtained from:

$$\operatorname{ch} \gamma b = \frac{q_3}{q_0} \tag{35}$$

where

$$q_0 = g + p\,; \quad q_3 = \left(\frac{g}{\cos \psi_1} + p\right) \frac{r_1}{r_0} \quad \text{and} \quad \alpha = \frac{1}{r_0}\,. \tag{36}$$

Accordingly, right-hand sides of shell equations (28) and (32) can be determined as:

$$P_1(x, y) = Z - \alpha \int_0^x X \,\mathrm{d}x = \operatorname{ch} \gamma y \left[-g - \frac{p}{2} + 2g \cos \alpha x - 1.5\, p \cos^2 \alpha x\right] \tag{37}$$

$$P_2(x, y) = \mu \frac{\partial X}{\partial x} - \int_0^x \frac{\partial^2 X}{\partial y^2} \mathrm{d}x = \operatorname{ch} \gamma y \left[-\frac{\gamma^2}{\alpha} g - \left(\mu\alpha + \frac{\gamma^2}{2\alpha}\right) p + \right.$$
$$\left. + \left(\mu\alpha + \frac{\gamma^2}{\alpha}\right) g \cos \alpha x + \left(2\mu\alpha + \frac{\gamma^2}{2\alpha}\right) p \cos^2 \alpha x \right]. \tag{38}$$

Functions (37) and (38) are of similar character, both including a constant of two terms and a term multiplied by $\cos \alpha x$ and $\cos^2 \alpha x$ each. Functions (37) and (38) are indicated by full thick lines in the section $y = 0$ in Figs 8 and 9, respectively.

The two diagrams show a rather similar shape for both functions. For (37) and (38) as right-hand sides of (28) and (32), resp., the shell equation system cannot be solved with the appropriate boundary conditions, but the variation along the $x$ axis of both expressions can be followed by a second-order parabola expressed by the equation, in case of e.g. function (37):

$$q_1 + q_2 \left(1 - \frac{x^2}{a^2}\right). \tag{39}$$

Constant $q_1$ and term $q_2(1 - x^2/a^2)$ can, however, be separately expanded in simple trigonometric series. This expansion is done according to the principle of load distribution proposed and applied by HRUBAN for the solution of hyperbolic paraboloids [5]. That is, the load (shaded area in Fig. 8) that cannot be expanded into a Fourier series and affects a narrow band of the edge $x = \pm a$

will be assumed to be absorbed directly by the shell edge (entity of edge beam and thickened shell edge) as boundary disturbances easy to determine in view of the constant curvature along the $x$ axis.

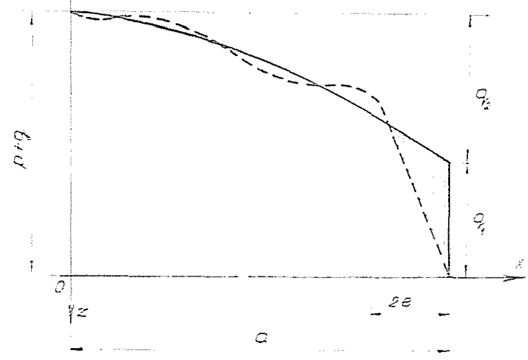


*Fig. 8*

And since, with respect to the above,

$$q_1 \cong \frac{4}{\pi^2}\,\frac{q_1\,a}{e}\sum_n (-1)^{(n-1)/2}\,\frac{1}{n^2}\sin\frac{n\pi e}{a}\cos\vartheta_n x$$

$$q_2\left(1-\frac{x^2}{a^2}\right) \approx \frac{32}{\pi^3}\,q_2\sum_n (-1)^{(n-1)/2}\,\frac{1}{n^3}\cos\vartheta_n x$$

the Fourier series for (37) and (38) are:

$$\left.\begin{aligned} P_1(x,y) &\approx \operatorname{ch}\gamma y \sum_n A_n \cos\vartheta_n x \\ P_2(x,y) &\approx \operatorname{ch}\gamma y \sum_n B_n \cos\vartheta_n x \end{aligned}\right\} \tag{40}$$

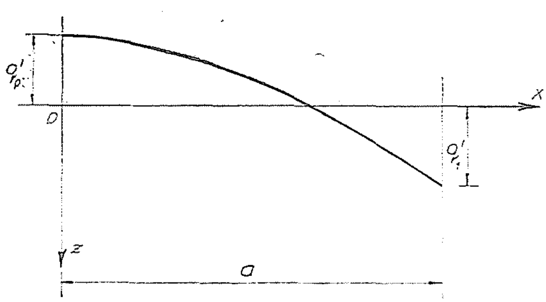


*Fig. 9*

with constants delivered by:

$$
\left.
\begin{aligned}
A_n &= \frac{4}{\pi^2}(-1)^{(n-1)/2}\frac{1}{n^2}\left[\frac{q_1 a}{e}\sin\frac{n\pi e}{a}+\frac{8q^2}{n\pi}\right]\\
B_n &= \frac{4}{\pi^2}(-1)^{(n-1)/2}\frac{1}{n^2}\left[\frac{q_1' a}{e}\sin\frac{n\pi e}{a}+\frac{8q_2'}{n\pi}\right]\\
q_1 &= -g-\frac{p}{2}+2g\cos\alpha a+1.5\,p\cos^2\alpha a\\
q_2 &= q_0-q_1;\quad q_0{}'=\mu\alpha q_0\\
q_1' &= -\frac{\gamma^2}{\alpha}g-\left(\mu\alpha+\frac{\gamma^2}{2\alpha}\right)p+\left(\mu\alpha+\frac{\gamma^2}{\alpha}\right)g\cos\alpha a+\\
&\quad+\left(2\mu\alpha+\frac{\gamma^2}{2\alpha}\right)p\cos^2\alpha a\\
q_2' &= q_0'-q_1';\qquad e=0,76\sqrt{r_0\,\delta}\\
\vartheta_n &= \frac{n\pi}{2a}\qquad\text{and}\qquad n=1,3,5,7,\dots
\end{aligned}
\right\}
\tag{41}
$$

In view of assumption 3.1/7, $1/r_x$ in (32) can be replaced by $1/r_0$, $1/r_y \approx z''$ while surface equation (2) can be approximated by the first term of the power series of the term under root sign, hence:

$$
z=-\frac{r_1^2-r_0^2}{2r_0\,b^2}\,y^2=-\frac{\beta}{2}\,y^2
$$

where

$$
\beta=\frac{r_1^2-r_0^2}{r_0\,b^2}\,. \tag{42}
$$

In the following, $z''$ will be replaced by $|z''|$, the negative sign in (42) being already reckoned with in the deductions. Because of (42) and making use of (40), shell equations (28) and (32) can be written as:

$$
\begin{aligned}
K\Delta\Delta w+\beta\frac{\partial^2 F}{\partial x^2}-\alpha\frac{\partial^2 F}{\partial y^2} &= \operatorname{ch}\gamma y\sum_n A_n\cos\vartheta_n x\\
\Delta\Delta F-E\delta\left(\beta\frac{\partial^2 w}{\partial x^2}-\alpha\frac{\partial^2 w}{\partial y^2}\right) &= -\operatorname{ch}\gamma y\sum B_n\cos\vartheta_n x
\end{aligned}
\tag{43}
$$

## 5. Determination of the functions of stress and displacement

### *5.1 Boundary conditions. Solution principle*

Edges formed along the shell boundaries are of low rigidity normally to their plane, therefore it is expedient to set absence of lateral pressure as boundary condition, thus:

$$\text{at boundary } x = \pm a \qquad N_x = 0$$
$$\text{at boundary } y = \pm b \qquad N_y = 0. \tag{44}$$

Another boundary condition permitting to determine the unknown constants and approximating the effective forces acting in the shell structure is as follows:

$$\text{at boundary } x = \pm a \qquad w = 0; \qquad M_x = 0 \tag{45}$$
$$\text{at boundary } y = \pm b \qquad w = 0; \qquad M_y = 0; \quad Q_y = 0.$$

Boundary conditions (44) and (45) assume the shell to be simply supported along its edges so that part of the external load is transmitted in form of tangential forces by the edge beams to the supporting structure, and the rest as shear forces but only through edge arches $x = \pm a$.

In view of the nature of the right-hand side of shell equations (43), there is a possibility to find a solution satisfying boundary conditions for edge $x = \pm a$ while there is not for $y = \pm b$. Nevertheless, there are altogether eight unsatisfied boundary conditions to determine eight unknown constants.

Solution for $F$ and $w$ of equation system (43) can be composed of two parts, namely a particular solution of the inhomogeneous equation system, and a general solution of the homogeneous one. Both solutions can be made to satisfy the equation system, but the boundary conditions not. These unsatisfied boundary conditions are satisfied by superposing the two solutions so that the arbitrary constants in the general solution are determined by collocation at the boundary $y = \pm b$.

### *5.2 One particular solution of inhomogeneous shell equations*

Particular solution of the inhomogeneous shell equation system (43) is sought for in the form:

$$F_p = \sum_n F_n \operatorname{ch} \gamma y \cos \vartheta_n x$$
$$w_p = \sum_n w_n \operatorname{ch} \gamma y \cos \vartheta_n x . \tag{46}$$

$F_n$ and $w_n$ being constants of the Fourier series. Substituting expressions (46) into (43), we obtain for $F_n$ and $w_n$:

$$K w_n(\vartheta_n^4 - 2\vartheta_n^2\gamma^2 + \gamma^4) - F_n(\beta\vartheta_n^2 + \alpha\gamma^2) = A_n$$
$$F_n(\vartheta_n^4 - 2\vartheta_n^2\gamma^2 + \gamma^4) + E\delta w_n(\beta\vartheta_n^2 + \alpha\gamma^2) = -B_n$$

yielding expressions

$$F_n = \frac{1}{\beta\vartheta_n^2 + \alpha\gamma^2}\left[K w_n(\vartheta_n^2 - \gamma^2)^2 - A_n\right],$$
$$w_n = \frac{A_n(\vartheta_n^2 - \gamma^2)^2 - B_n(\beta\vartheta_n^2 + \alpha\gamma^2)}{K(\vartheta_n^2 - \gamma^2)^2 + E\delta(\beta\vartheta_n^2 + \alpha\gamma^2)^2}. \tag{47}$$

*5.3 General solution of the homogeneous shell equations*

Since left-hand side of Eqs (43) contains even derivatives throughout, since furthermore shell loads are symmetrical, functions $F_h$ and $w_h$ as possible solutions for the homogeneous part of the equation system may be written in the form:

$$F_h = \sum_n G_n e^{\omega_n y} \cos\vartheta_n x$$
$$w_h = \sum_n H_n e^{\omega_n y} \cos\vartheta_n x. \tag{48}$$

Here $\vartheta_n$ is as defined in (41), while $G_n$, $H_n$ and $\omega_n$ are unknown constants. The former can be determined from the boundary conditions, while $\omega_n$ can be obtained as follows: Replacing (48) into the homogeneous part of (43), the algebraic equation system containing also unknowns $G_n$ and $H_n$ will be:

$$K(\omega_n^4 - 2\omega_n^2\vartheta_n^2 + \vartheta_n^4) H_n - (\alpha\omega_n^2 + \beta\vartheta_n^2) G_n = 0$$
$$(\omega_n^4 - 2\omega_n^2\vartheta_n^2 + \vartheta_n^4) G_n + E\delta(\alpha\omega_n^2 + \beta\vartheta_n^2) H_n = 0 \tag{49}$$

Replacing $G_n$ expressed from one equation into the other one; and dividing by $H_n$ yields for $\omega_n$ an equation system of eighth order, containing but even powers:

$$\omega_n^8 - 4\vartheta_n^2\omega_n^6 + \left(6\vartheta_n^4 + \frac{E\delta}{K}\alpha^2\right)\omega_n^4 + \left(2\frac{E\delta}{K}\alpha\beta\vartheta_n^2 - 4\vartheta_n^6\right)\omega_n^2 =$$
$$= -\left(\frac{E\delta}{K}\beta^2\vartheta_n^4 + \vartheta_n^8\right). \tag{50}$$

In this expression $E\,\delta$, $K$, $\alpha$, $\beta$ and $\vartheta_n$ being positive and its right-hand side being in fact negative, $\omega_n$ may be a complex number. The eight complex roots conjugated in pairs can be written in the form:

$$\begin{aligned}
\omega_{1n} &= \sigma_{1n} + \tau_{1n}\, i & \qquad \omega_{5n} &= -\sigma_{1n} + \tau_{1n}\, i \\
\omega_{2n} &= \sigma_{1n} - \tau_{1n}\, i & \qquad \omega_{6n} &= -\sigma_{1n} - \tau_{1n}\, i \\
\omega_{3n} &= \sigma_{2n} + \tau_{2n}\, i & \qquad \omega_{7n} &= -\sigma_{2n} + \tau_{2n}\, i \\
\omega_{4n} &= \sigma_{2n} - \tau_{2n}\, i & \qquad \omega_{8n} &= -\sigma_{2n} - \tau_{2n}\, i
\end{aligned} \tag{51}$$

It is expendient here to deal with the practical calculation of these roots, Eq. (50) being rather tedious to calculate either exactly or approximately by manual means. Therefore it is advisable to apply a computer method. To this aim we developed the following algorithm, which proved upon programming and then running to be rather helpful. Computation process will be written for $n = 1$. The problem can also be solved for any $n$ upon input "for statement".

Eq. (50) can also be written as:

$$\omega^8 + a_3\,\omega^6 + a_2\,\omega^4 + a_1\,\omega^2 + a_0 = 0$$

Introducing symbol $\omega^2 = m$, the equation can be reduced from eighth to fourth order:

$$m^4 + a_3 m^3 + a_2 m^2 + a_1 m + a_0 = 0 \tag{52}$$

or, in factorized form:

$$(m^2 + p_1 m + p_2)\,(m^2 + p_3 m + p_4) = 0 \tag{53}$$

where

$$\left.\begin{aligned}
p_4 &= \frac{-(a_3 p_3 - p_3^2 - a_2) \pm \sqrt{(a_3 p_3 - p_3^2 - a_2)^2 - 4a_0}}{2} \\
p_3 &= \frac{(a_3 p_4 - a_1)\, p_4}{p_4^2 - a_0} \\
p_2 &= \frac{a_0}{p_4} \\
p_1 &= a_3 - p_3.
\end{aligned}\right\} \tag{54}$$

From Eqs (54), constants of factors of the second order of (53) can be obtained by iteration (e.g. starting from $p_3 = 0$). If equations of second order are known, $m_1$ and $m_3$ can be obtained from

$$m^2 + p_1 m + p_2 = 0, \text{ and}$$
$$m^2 + p_3 m + p_4 = 0,$$

respectively, then roots $\omega_1 = \sqrt{m_1}$ and $\omega_3 = \sqrt{m_3}$ yield all roots of (50).

If roots (51) are known, $F_h$ in (48) assumes the form:

$$F_h = \sum_n \left[ e^{\sigma_{1n} y} (G_{1n} e^{i\tau_{1n} y} + G_{2n} e^{-i\tau_{1n} y}) + e^{\sigma_{2n} y} (G_{3n} e^{i\tau_{2n} y} + \right.$$
$$+ G_{4n} e^{-i\tau_{2n} y}) + e^{-\sigma_{1n} y} (G_{5n} e^{i\tau_{1n} y} + G_{6n} e^{-i\tau_{1n} y}) +$$
$$\left. + e^{-\sigma_{2n} y} (G_{7n} e^{i\tau_{2n} y} + G_{8n} e^{-i\tau_{2n} y}) \right] \cos \vartheta_n x \tag{55}$$

first subscripts of constants $G$ referring to subscripts of roots in (51).

In view of

$$G_{1n} e^{i\tau_{1n} y} + G_{2n} e^{-i\tau_{1n} y} = (G_{1n} + G_{2n}) \cos \tau_{1n} y +$$
$$+ i(G_{1n} - G_{2n}) \sin \tau_{1n} y$$

following from the Euler relationship, applicable to the sense to all other expressions with inside brackets, Eq. (55) is transformed into:

$$F_h = \sum_n \left[ e^{\sigma_{1n} y} (C_{1n} \cos \tau_{1n} y + C_{2n} \sin \tau_{1n} y) + \right.$$
$$+ e^{\sigma_{2n} y} (C_{3n} \cos \tau_{2n} y + C_{4n} \sin \tau_{2n} y) +$$
$$+ e^{-\sigma_{1n} y} (C_{5n} \cos \tau_{1n} y + C_{6n} \sin \tau_{1n} y) +$$
$$\left. + e^{-\sigma_{2n} y} (C_{7n} \cos \tau_{2n} y + C_{8n} \sin \tau_{2n} y) \right] \cos \vartheta_n x \tag{56}$$

where

$$\left.\begin{aligned} C_{1n} &= G_{1n} + G_{2n} & C_{5n} &= G_{5n} + G_{6n} \\ C_{2n} &= i(G_{1n} - G_{2n}) & C_{6n} &= i(G_{5n} - G_{6n}) \\ C_{3n} &= G_{3n} + G_{4n} & C_{7n} &= G_{7n} + G_{8n} \\ C_{4n} &= i(G_{3n} - G_{4n}) & C_{8n} &= i(G_{7n} - G_{8n}). \end{aligned}\right\} \tag{57}$$

Function (56) is certainly symmetrical with respect to the $y$ axis because of the nature of the cos $\vartheta_n x$ function, and its symmetry with respect to the $x$ axis can be warranted by satisfying the requirement

$$F_h(+y) = F_h(-y).$$

By meeting this condition, i.e., substituting $+y$ and $-y$ into (56) and equalizing the obtained expressions, the number of eight constants according to (57) is halved, thereby:

$$\begin{aligned} C_{1n} &= C_{5n} & C_{3n} &= C_{7n} \\ C_{2n} &= -C_{6n} & C_{4n} &= -C_{8n}. \end{aligned} \tag{58}$$

In view of (58), of the relationships

$$e^{\sigma_n y} + e^{-\sigma_n y} = 2\text{ch}\,\sigma_n y$$
$$e^{\sigma_n y} - e^{-\sigma_n y} = 2\text{sh}\,\sigma_n y$$

and of the analogy between functions $F_h$ and $w_h$, yielding for $w_h$ an expression differing from (56) only by its constants, general solution of the homogeneous part of equation system (43) consists of the function sums:

$$\begin{aligned} F_h = 2\sum_n \big[&C_{1n}\,\text{ch}\,\sigma_{1n}\,y\cos\tau_{1n}\,y + C_{2n}\,\text{sh}\,\sigma_{1n}\,y\sin\tau_{1n}\,y + \\ &+ C_{3n}\,\text{ch}\,\sigma_{2n}\,y\cos\tau_{2n}\,y + C_{4n}\,\text{sh}\,\sigma_{2n}\,y\sin\tau_{2n}\,y\big]\cos\vartheta_n x \\ w_h = 2\sum_n \big[&K_{1n}\,\text{ch}\,\sigma_{1n}\,y\cos\tau_{1n}\,y + K_{2n}\,\text{sh}\,\sigma_{1n}\,y\sin\tau_{1n}\,y + \\ &+ K_{3n}\,\text{ch}\,\sigma_{2n}\,y\cos\tau_{2n}\,y + K_{4n}\,\text{sh}\,\sigma_{2n}\,y\sin\tau_{2n}\,y\big]\cos\vartheta_n x \end{aligned} \tag{59}$$

where

$$\begin{aligned} K_{1n} &= H_{1n} + H_{2n} & K_{3n} &= H_{3n} + H_{4n} \\ K_{2n} &= i(H_{1n} - H_{2n}) & K_{4n} &= i(H_{3n} - H_{4n}). \end{aligned} \tag{60}$$

In view of the characteristic equation system (49), if $\omega_n$ is known, $H_n$ and $G_n$ can be related as:

$$H_n = R_n G_n \tag{61}$$

where

$$R_n = \frac{\varkappa\omega_n^2 + \beta\vartheta_n^2}{K(\omega_n^2 - \vartheta_n^2)^2}\,. \tag{62}$$

Then (61), (60) and (57) help to express the $K_n$ by means of the $C_n$ values, as follows:

$$\begin{aligned} K_{1n} &= \frac{R_{1n} + R_{2n}}{2}C_{1n} - \frac{R_{1n} - R_{2n}}{2i}C_{2n} \\ K_{2n} &= \frac{R_{1n} + R_{2n}}{2}C_{2n} - \frac{R_{1n} - R_{2n}}{2i}C_{1n} \\ K_{3n} &= \frac{R_{3n} + R_{4n}}{2}C_{3n} - \frac{R_{3n} - R_{4n}}{2i}C_{4n} \\ K_{4n} &= \frac{R_{3n} + R_{4n}}{2}C_{4n} - \frac{R_{3n} - R_{4n}}{2i}C_{3n}. \end{aligned} \tag{63}$$

The $C_n$ are in turn expressed by means of the $K_n$:

$$\begin{aligned}
C_{1n} &= \frac{R_{1n} + R_{2n}}{R_{1n}^2 + R_{2n}^2} K_{1n} - i \frac{R_{1n} - R_{2n}}{R_{1n}^2 + R_{2n}^2} K_{2n} \\
C_{2n} &= \frac{R_{1n} + R_{2n}}{R_{1n}^2 + R_{2n}^2} K_{2n} - i \frac{R_{1n} - R_{2n}}{R_{1n}^2 + R_{2n}^2} K_{1n} \\
C_{3n} &= \frac{R_{3n} + R_{4n}}{R_{3n}^2 + R_{4n}^2} K_{3n} - i \frac{R_{3n} - R_{4n}}{R_{3n}^2 + R_{4n}^2} K_{4n} \\
C_{4n} &= \frac{R_{3n} + R_{4n}}{R_{3n}^2 + R_{4n}^2} K_{4n} - i \frac{R_{3n} - R_{4n}}{R_{3n}^2 + R_{4n}^2} K_{3n} \,.
\end{aligned} \tag{64}$$

No computation reasons require the numerical determination of the $G_n$ and the $H_n$ in the right-hand sides of (57) and (60), respectively, nevertheless it is important to know that they are complex numbers conjugated in pairs (e.g. $G_{1n}$ and $G_{2n}$). Thus, both their sum and their difference multiplied by the imaginary number $i$ are real numbers. Hence, constants $C_n$ and $K_n$ sought for are real numbers.

Finally, the displacement function and the stress function, general solutions of the inhomogeneous differential equation system (43), can be written as sum of (46) and (59).

## 6. Stress determinations

### 6.1 Stress functions

In view of relationships (23), (25), (26) and (37), as well as of functions (46) and (59), shell stress formulae can be written as:

$$\begin{aligned}
N_x = {} & \gamma^2 \operatorname{ch} \gamma y \sum_n F_n \cos \vartheta_n x + \\
& + 2 \sum_n \big[ C_{1n} (\eta_{1n} \operatorname{ch} \sigma_{1n} y \cos \tau_{1n} y - 2\sigma_{1n} \tau_{1n} \operatorname{sh} \sigma_{1n} y \sin \tau_{1n} y) + \\
& + C_{2n} (\eta_{1n} \operatorname{sh} \sigma_{1n} y \sin \tau_{1n} y + 2\sigma_{1n} \tau_{1n} \operatorname{ch} \sigma_{1n} y \cos \tau_{1n} y) - \\
& + C_{3n} (\eta_{2n} \operatorname{ch} \sigma_{2n} y \cos \tau_{2n} y - 2\sigma_{2n} \tau_{2n} \operatorname{sh} \sigma_{2n} y \sin \tau_{2n} y) - \\
& + C_{4n} (\eta_{2n} \operatorname{sh} \sigma_{2n} y \sin \tau_{2n} y + 2\sigma_{2n} \tau_{2n} \operatorname{ch} \sigma_{2n} y \cos \tau_{2n} y) \big] \cos \vartheta_n x + \\
& \qquad + \frac{1}{\alpha} \operatorname{ch} \gamma y \left( - g - \frac{p}{2} + g \cos \alpha x + \frac{p}{2} \cos^2 \alpha x \right)
\end{aligned} \tag{65}$$

$$\begin{aligned}
N_y = {} & - \operatorname{ch} \gamma y \sum_n \vartheta_n^2 F_n \cos \vartheta_n x - \\
& - 2 \sum_n \vartheta_n^2 \big[ C_{1n} \operatorname{ch} \sigma_{1n} y \cos \tau_{1n} y + C_{2n} \operatorname{sh} \sigma_{1n} y \sin \tau_{1n} y + \\
& + C_{3n} \operatorname{ch} \sigma_{2n} y \cos \tau_{2n} y + C_{4n} \operatorname{sh} \sigma_{2n} y \sin \tau_{2n} y \big] \cos \vartheta_n x
\end{aligned} \tag{66}$$

$$N_{xy} = \gamma \operatorname{sh} \gamma y \sum_n \vartheta_n F_n \sin \vartheta_n x +$$
$$+ 2 \sum_n \vartheta_n \big[C_{1n} (\sigma_{1n} \operatorname{sh} \sigma_{1n} y \cos \tau_{1n} y - \tau_{1n} \operatorname{ch} \sigma_{1n} y \sin \tau_{1n} y) +$$
$$+ C_{2n} (\sigma_{1n} \operatorname{ch} \sigma_{1n} y \sin \tau_{1n} y + \tau_{1n} \operatorname{sh} \sigma_{1n} y \cos \tau_{1n} y) +$$
$$+ C_{3n} (\sigma_{2n} \operatorname{sh} \sigma_{2n} y \cos \tau_{2n} y - \tau_{2n} \operatorname{ch} \sigma_{2n} y \sin \tau_{2n} y) +$$
$$+ C_{4n} (\sigma_{2n} \operatorname{ch} \sigma_{2n} y \sin \tau_{2n} y + \tau_{2n} \operatorname{sh} \sigma_{2n} y \cos \tau_{1n} y)\big] \sin \vartheta_n x \qquad (67)$$

$$M_x = - K \;\Big\{\operatorname{ch} \gamma y \sum_n w_n (\mu\gamma^2 - \vartheta_n^2) \cos \vartheta_n x +$$
$$+ 2 \sum_n \big[K_{1n} (\eta_{3n} \operatorname{ch} \sigma_{1n} y \cos \tau_{1n} y - 2\mu \sigma_{1n} \tau_{1n} \operatorname{sh} \sigma_{1n} y \sin \tau_{1n} y) +$$
$$+ K_{2n} (\eta_{3n} \operatorname{sh} \sigma_{1n} y \sin \tau_{1n} y + 2\mu\sigma_{1n} \tau_{1n} \operatorname{ch} \sigma_{1n} y \cos \tau_{1n} y) +$$
$$+ K_{3n} (\eta_{4n} \operatorname{ch} \sigma_{2n} y \cos \tau_{2n} y - 2\mu \sigma_{2n} \tau_{2n} \operatorname{sh} \sigma_{2n} y \sin \tau_{2n} y) +$$
$$+ K_{4n} (\eta_{4n} \operatorname{sh} \sigma_{2n} y \sin \tau_{2n} y + 2\mu\sigma_{2n} \tau_{2n} \operatorname{ch} \sigma_{2n} y \cos \tau_{2n} y)\big] \cos \vartheta_n x\Big\} \qquad (68)$$

$$M_y = - K \;\Big\{\operatorname{ch} \gamma y \sum_n w_n (\gamma^2 - \mu\vartheta_n^2) \cos \vartheta_n x +$$
$$+ 2 \sum_n \big[K_{1n} (\eta_{5n} \operatorname{ch} \sigma_{1n} y \cos \tau_{1n} y - 2\sigma_{1n} \tau_{1n} \operatorname{sh} \sigma_{1n} y \sin \tau_{1n} y) +$$
$$+ K_{2n} (\eta_{5n} \operatorname{sh} \sigma_{1n} y \sin \tau_{1n} y + 2\sigma_{1n} \tau_{1n} \operatorname{ch} \sigma_{1n} y \cos \tau_{1n} y) +$$
$$+ K_{3n} (\eta_{6n} \operatorname{ch} \sigma_{2n} y \cos \tau_{2n} y - 2\sigma_{2n} \tau_{2n} \operatorname{sh} \sigma_{2n} y \sin \tau_{2n} y) +$$
$$+ K_{4n} (\eta_{6n} \operatorname{sh} \sigma_{2n} y \sin \tau_{2n} y + 2\sigma_{2n} \tau_{2n} \operatorname{ch} \sigma_{2n} y \cos \tau_{2n} y)\big] \cos \vartheta_n x\Big\} \qquad (69)$$

$$M_{xy} = M_{yx} = K(1 - \mu) \Big\{\gamma \operatorname{sh} \gamma y \sum_n w_n \vartheta_n \sin \vartheta_n x +$$
$$+ 2 \sum_n \vartheta_n \big[K_{1n} (\sigma_{1n} \operatorname{sh} \sigma_{1n} y \cos \tau_{1n} y - \tau_{1n} \operatorname{ch} \sigma_{1n} y \sin \tau_{1n} y) +$$
$$+ K_{2n} (\sigma_{1n} \operatorname{ch} \tau_{1n} y \sin \tau_{1n} y + \tau_{1n} \operatorname{sh} \sigma_{1n} y \cos \tau_{1n} y) +$$
$$+ K_{3n} (\sigma_{2n} \operatorname{sh} \sigma_{2n} y \cos \tau_{2n} y - \tau_{2n} \operatorname{ch} \sigma_{2n} y \sin \tau_{2n} y) +$$
$$+ K_{4n} (\sigma_{2n} \operatorname{ch} \sigma_{2n} y \sin \tau_{2n} y + \tau_{2n} \operatorname{sh} \sigma_{2n} y \cos \tau_{2n} y)\big] \sin \vartheta_n x\Big\} \qquad (70)$$

$$Q_x = K \;\Big\{\operatorname{ch} \gamma y \sum_n w_n \vartheta_n (\vartheta_n^2 - \gamma^2) \sin \vartheta_n x +$$
$$+ 2 \sum_n \vartheta_n \big[K_{1n} (\eta_{1n} \operatorname{ch} \sigma_{1n} y \cos \tau_{1n} y + 2\sigma_{1n} \tau_{1n} \operatorname{sh} \sigma_{1n} y \sin \tau_{1n} y) +$$
$$+ K_{2n} (\eta_{7n} \operatorname{sh} \sigma_{1n} y \sin \tau_{1n} y - 2\sigma_{1n} \tau_{1n} \operatorname{ch} \sigma_{1n} y \cos \tau_{1n} y) +$$
$$+ K_{3n} (\eta_{8n} \operatorname{ch} \sigma_{2n} y \cos \tau_{2n} y + 2\sigma_{2n} \tau_{2n} \operatorname{sh} \sigma_{2n} y \sin \tau_{2n} y) +$$
$$+ K_{4n} (\eta_{8n} \operatorname{sh} \sigma_{2n} y \sin \tau_{2n} y - 2\sigma_{2n} \tau_{2n} \operatorname{ch} \sigma_{2n} y \cos \tau_{2n} y)\big] \sin \vartheta_n x\Big\} \qquad (71)$$

$$\begin{aligned}
Q_y = K \{ & \gamma \operatorname{sh} \gamma y \sum_n w_n (\gamma^2 - \vartheta_n^2) \cos \vartheta_n x + \\
& + 2 \sum_n [K_{1n} (\eta_{9n} \operatorname{sh} \sigma_{1n} y \cos \tau_{1n} y - \eta_{10n} \operatorname{ch} \sigma_{1n} y \sin \tau_{1n} y) + \\
& + K_{2n} (\eta_{9n} \operatorname{ch} \sigma_{1n} y \sin \tau_{1n} y + \eta_{10n} \operatorname{sh} \sigma_{1n} y \cos \tau_{1n} y) + \\
& + K_{3n} (\eta_{11n} \operatorname{sh} \sigma_{2n} y \cos \tau_{2n} y - \eta_{12n} \operatorname{ch} \sigma_{2n} y \sin \tau_{2n} y) + \\
& + K_{4n} (\eta_{11n} \operatorname{ch} \sigma_{2n} y \sin \tau_{2n} y + \eta_{12n} \operatorname{sh} \sigma_{2n} y \cos \tau_{2n} y)] \cos \vartheta_n x \}
\end{aligned} \tag{72}$$

where

$$\left.\begin{aligned}
&\eta_{1n} = \sigma_{1n}^2 - \tau_{1n}^2 && \eta_{3n} = \mu \eta_{1n} - \vartheta_n^2 \\
&\eta_{2n} = \sigma_{2n}^2 - \tau_{2n}^2 && \eta_{4n} = \mu \eta_{2n} - \vartheta_n^2 \\
&\eta_{5n} = \eta_{1n} - \mu \vartheta_n^2 && \eta_{7n} = \vartheta_n^2 - \eta_{1n} \\
&\eta_{6n} = \eta_{2n} - \mu \vartheta_n^2 && \eta_{8n} = \vartheta_n^2 - \eta_{2n} \\
&\eta_{9n} = \sigma_{1n} (\eta_{1n} - 2\tau_{1n}^2 - \vartheta_n^2) \\
&\eta_{10n} = \tau_{1n} (\eta_{1n} + 2\sigma_{1n}^2 - \vartheta_n^2) \\
&\eta_{11n} = \sigma_{2n} (\eta_{2n} - 2\tau_{2n}^2 - \vartheta_n^2) \\
&\eta_{12n} = \tau_{2n} (\eta_{2n} + 2\sigma_{2n}^2 - \vartheta_n^2).
\end{aligned}\right\} \tag{73}$$

Some of these stress functions have to satisfy boundary conditions (44) and (45), a requirement already met or to be met after duly choosing the unknown constants, except the last term of the normal force expression (67). Namely, this term fails to meet boundary condition $N_x$ in (44). This deviation can, however, be disregarded, since the practically favourable shell slope of $\varphi = 30°$ is accompanied by a lateral pressure not higher than to be absorbed by an edge arch of the anyhow required size. The more shallow the shell, the lesser is the deviation.

### *6.2 Determination of the unknown constants*

Because of the mentioned symmetry with respect to the $x$ axis, the number of unknown constants diminishes to four, in conformity with (58), and so does the number of unsatisfied boundary conditions (44) and (45) hence this is a statically determined problem.

As it has been mentioned earlier, the unknown constants will be determined by boundary collocation. There being $4 \times n$ unknows where $n$ is arbitrary, the required number of collocation points is also $4 \times n$. Converting constants $C_n$ in (66) to constants $K_n$ according to (63), and making use of relationships (46), (59), (69) and (72), a system of $4 \times n$ inhomogeneous linear equations containing $4 \times n$ unknowns, of the following type, can be written:

$$
\left.
\begin{aligned}
&\sum_n \left[K_{1n} \operatorname{ch} \sigma_{1n} b \cos \tau_{1n} b + K_{2n} \operatorname{sh} \sigma_{1n} b \sin \tau_{1n} b + \right. \\
&\quad \left. + K_{3n} \operatorname{ch} \sigma_{2n} b \cos \tau_{2n} b + K_{4n} \operatorname{sh} \sigma_{2n} b \sin \tau_{2n} b \right] \cos \vartheta_n x = \\
&\qquad = - \frac{\operatorname{ch} \gamma b}{2} \sum_n w_n \cos \vartheta_n x \\[2ex]
&\sum_n \vartheta_n^2 \left[ K_{1n} \left( \frac{R_{1n} + R_{2n}}{R_{1n}^2 + R_{2n}^2} \operatorname{ch} \sigma_{1n} b \cos \tau_{1n} b - i \frac{R_{1n} - R_{2n}}{R_{1n}^2 + R_{2n}^2} \operatorname{sh} \sigma_{1n} b \sin \tau_{1n} b \right) + \right. \\
&\quad + K_{2n} \left( \frac{R_{1n} + R_{2n}}{R_{1n}^2 + R_{2n}^2} \operatorname{sh} \sigma_{1n} b \sin \tau_{1n} b + i \frac{R_{1n} - R_{2n}}{R_{1n}^2 + R_{2n}^2} \operatorname{ch} \sigma_{1n} b \cos \tau_{1n} b \right) + \\
&\quad + K_{3n} \left( \frac{R_{3n} + R_{4n}}{R_{3n}^2 + R_{4n}^2} \operatorname{ch} \sigma_{2n} b \cos \tau_{2n} b - i \frac{R_{3n} - R_{4n}}{R_{3n}^2 + R_{4n}^2} \operatorname{sh} \sigma_{2n} b \sin \tau_{2n} b \right) + \\
&+ K_{4n} \left. \left( \frac{R_{3n} - R_{4n}}{R_{3n}^2 + R_{4n}^2} \operatorname{sh} \sigma_{2n} b \sin \tau_{2n} b - i \frac{R_{3n} - R_{4n}}{R_{3n}^2 + R_{4n}^2} \operatorname{ch} \sigma_{2n} b \cos \tau_{2n} b \right) \right] \cos \vartheta_n x = \\
&\qquad = - \frac{\operatorname{ch} \gamma b}{2} \sum_n \vartheta_n^2 F_n \cos \vartheta_n x \\[2ex]
&\sum_n \left[ K_{1n} (\eta_{5n} \operatorname{ch} \sigma_{1n} b \cos \tau_{1n} b - 2\sigma_{1n} \tau_{1n} \operatorname{sh} \sigma_{1n} b \sin \tau_{1n} b) + \right. \\
&\quad + K_{2n} (\eta_{5n} \operatorname{sh} \sigma_{1n} b \sin \tau_{1n} b + 2\sigma_{1n} \tau_{1n} \operatorname{ch} \sigma_{1n} b \cos \tau_{1n} b) + \\
&\quad + K_{3n} (\eta_{6n} \operatorname{ch} \sigma_{2n} b \cos \tau_{2n} b - 2\sigma_{2n} \tau_{2n} \operatorname{sh} \sigma_{2n} b \sin \tau_{2n} b) + \\
&\quad + \left. K_{4n} (\eta_{6n} \operatorname{sh} \sigma_{2n} b \sin \tau_{2n} b + 2\sigma_{2n} \tau_{2n} \operatorname{ch} \sigma_{2n} b \cos \tau_{2n} b) \right] \cos \vartheta_n x = \\
&\qquad = - \frac{\operatorname{ch} \gamma b}{2} \sum_n w_n (\gamma^2 - \mu \vartheta_n^2) \cos \vartheta_n x \\[2ex]
&\sum_n \left[ K_{1n} (\eta_{9n} \operatorname{sh} \sigma_{1n} b \cos \tau_{1n} b - \eta_{10n} \operatorname{ch} \sigma_{1n} b \sin \tau_{1n} b) + \right. \\
&\quad + K_{2n} (\eta_{9n} \operatorname{ch} \sigma_{1n} b \sin \tau_{1n} b + \eta_{10n} \operatorname{sh} \sigma_{1n} b \cos \tau_{1n} b) + \\
&\quad + K_{3n} (\eta_{11n} \operatorname{sh} \sigma_{2n} b \cos \tau_{2n} b - \eta_{12n} \operatorname{ch} \sigma_{2n} b \sin \tau_{2n} b) + \\
&\quad + K_{4n} (\eta_{11n} \operatorname{ch} \sigma_{2n} b \sin \tau_{2n} b + \eta_{12n} \operatorname{sh} \sigma_{2n} b \cos \tau_{2n} b \left[ (\cos \vartheta_n x = \right. \\
&\qquad = - \frac{\gamma \operatorname{sh} \gamma b}{2} \sum_n w_n (\gamma^2 - \vartheta_n^2) \cos \vartheta_n x
\end{aligned}
\right\} \quad (74)
$$

Besides of Eq. (50), equation system (74) is also one inaccessible to conventional calculation methods, these being rather tedious even for the insufficient $n = 1$ case. A digital computer has to and can be used without difficulty, since equation system (74) is an ordinary linear one, for which there are subroutines available.

### 6.3 Next step of iteration

In conformity with item 4, using $N_{xy}$ and $Q_x$ obtained in the first step involving (67) and (71), an expression of the character of a load acting in direction $x$ can be established:

$$X_1 = 2N_{xy}\frac{\operatorname{tg}\psi}{r_0} + Q_x\frac{1}{r_0} = \alpha(2\beta y N_{xy} + Q_x). \tag{75}$$

Since (75) cannot be used in this form, it will be replaced by a close approximate function, convenient both from integration and computation aspects. The common function $\sin \vartheta_n x$ expressing the variation along the $x$ axis of both stresses in (75) is convenient also for the next steps. Nevertheless, expressions of the variation of stresses along the $y$ axis are rather complex and differ from each other. Expression of $Q_x$ according to (71) as a function of $y$ consists purely of terms with even functions, while that of $N_{xy}$ according to (67) has terms of odd functions. This latter becomes even upon multiplication by $y$ in (75).

In view of the above, the best approximate function form of (75) is that where the right-hand side of the shell functions is not or little transformed in the second step, or that differing only by constants $A_n$, $B_n$ and $\gamma$ in (40). Hence, the imaginary load function (75) depending on $y$ will be replaced by a function of form $\operatorname{ch}\gamma_1 y$, where constant $\gamma_1$ can be determined as:

$$\operatorname{ch}\gamma_1 b = \frac{q_3'}{q_0''} . \tag{76}$$

Here $q_0''$ and $q_3'$ are the values of function $X_1$ in (75) at $y = 0$ and $x = x_1$ const., and at $y = b$ and $x = x_1$, respectively. If values of $N_{xy}$ and $Q_x$ obtained in the first step are known, both $q_0''$ and $q_3'$ can be determined.

$\gamma_1$ being known, expedient form of approximate function $X_1$ is:

$$X_1 \approx \operatorname{ch}\gamma_1 y \sum_n E_n \sin \vartheta_n x \tag{77}$$

the $E_n$ being constant, with values delivered by a linear equation system having coefficients obtained from values at the matching points of the approximate expression (77), right-hand sides being delivered by the exact (75). If several terms are reckoned with, this problem is also to be computerized. Unknown constants determined according to the above, as well as according to (77), the right-hand side of (28) in the second step can be expressed as:

$$P_1'(x, y) = -\frac{1}{r_0}\int_0^x X_1\,dx \approx \operatorname{ch}\gamma_1 y \sum_n A_n' \cos\vartheta_n x \tag{78}$$

where

$$A_n' = \varkappa \frac{E_n}{\vartheta_n}\left[1 - \frac{4}{\pi}(-1)^{(n-1)/2}\frac{1}{n}\right]. \tag{79}$$

And the right-hand side of Eq. (32):

$$p_2'(x, y) = \mu\frac{\partial X_1}{\partial x} - \int_0^x \frac{\partial^2 X_1}{\partial y^2}\,dx = \operatorname{ch}\gamma_1 y \sum_n B_n' \cos\vartheta_n x \tag{80}$$

where

$$B_n' = \mu E_n \vartheta_n + \gamma_1^2 \frac{E_n}{\vartheta_n}\left[1 - \frac{4}{\pi}(-1)^{(n-1)/2}\frac{1}{n}\right]. \tag{81}$$

Functions (78) and (80) are seen to be formally identical with the right-hand side of the equation system (43), hence the equation system can be solved the same way in the second step as in the first one.

Since practically, the first, not more than two steps may be of importance, the final stresses are obtained as the true to sign sum of corresponding stresses obtained in the first two steps.

## 7. Numerical example

Notations are the same as in Fig. 1.

### 7.1 Geometry

$b = 5.0$ m $\qquad \delta = 0.05$ m

$c = 10.0$ m $\qquad \varphi_1 = 30°$

$$\sin\varphi_1 = 0.5\ ; \qquad r_0 = \frac{c}{\sin\varphi_1} = 20.0 \text{ m}$$

$$\cos\frac{\varphi_1}{2} = 0.9659\ ; \qquad r_1 = \frac{r_0}{\cos\frac{\varphi_1}{2}} = 20.706 \text{ m}$$

$$\varphi_1 = 0.5236;\ a = \varphi_1 r_0 = 10.472 \text{ m}$$

$$\alpha = \frac{1}{r_0} = 0.05/\text{m}\ : \qquad \beta = \frac{r_1^2 - r_0^2}{r_0 b^2} = 0.0566/\text{m}$$

$k = 2;\qquad f_a = 20.0 - 20 \cdot 0.866 = 2.68$ m

$\operatorname{tg}\varphi_1 = 0.577\ ;\ f_b = r_1 - r_0 = 0.706$ m.

## 7.2 *Material constants and rigidity data*

$E = 275\,000$ kp/cm$^2$; $\mu = 1/6$

$E\delta = 137\,500\,000$ kp/m ; $K = \dfrac{E\delta^3}{12(1-\mu^2)} = 29\,464.285$ kpm.

## 7.3 *Load values*

$g = 1.1\,(0.05 \cdot 2400 + 25) = 160$ kp/m$^2$

$p = 1.4 \cdot 80 = 112$ kp/m$^2$

impermeable layers weighing 25 kp/m$^2$.

## 7.4 *Computation technique*

The description of the computation method involved two partial problems (*5.3* and *6.2*) suggesting the use of a computer. In addition, however, auxiliary computations for establishing the equation system in item *6.2*, as well as evaluation of stress functions in item *6.1* in knowledge of integration constants — even for a low number of terms and nodal points — requires a lot of computation work, utmost tedious and time consuming for manual calculation.

The possibility to use the computer "Ural—2" of the University Computing Center allowed us to computerize nearly the whole process.

Steps of the computation involved three stages each.

The first stage involved auxiliary computations for writing the linear equation system (74) including solution of Eq. (50) of eighth order. The second stage was that of the solution of Eqs (74), and the third one the evaluation of stress functions described in item *6.1*.

In knowledge of stress functions $N_{xy}$ and $Q_x$ obtained in the first step, the value set of the imaginary load function (75) was established manually, and so were the constant $\gamma_1$ and the linear equation system needed for the determination of constants $E_n$, both in imaginary approximate load function (77). This linear equation system was solved by a computer, then the second step consisted in calculating the stresses by repeating the above procedure in three stages.

## 7.5 *Stress values obtained in the first step*

Shell stress values obtained in the first step for $n = 11$ are compiled in Table 1. Tables 2 to 5 contain some stresses of importance, again from the first step, reflecting convergence conditions of the problem. All tabulated values refer to kilopond and meter units.

**Table 1**

Shell stress values in the first step for $n = 11$

| | $y$ \ $x/a$ | 0 | 0.20 | 0.40 | 0.50 | 0.75 | 0.90 | 1.00 |
|---|---|---|---|---|---|---|---|---|
| $N_x$ Normal force | 0 | −10 621 | −10 208 | − 7 701 | − 5 643 | − 1 589 | − 851 | − 709 |
| | 2.5 | − 9 700 | − 8 802 | − 6 835 | − 5 738 | − 2 876 | − 1 464 | − 719 |
| | 5.0 | − 9 342 | − 5 386 | − 3 362 | − 4 956 | − 4 247 | − 602 | − 751 |
| $N_y$ Normal force | 0 | − 4 662 | − 4 379 | − 2 176 | − 268 | − 3 370 | 2 450 | 0 |
| | 2.5 | − 4 139 | − 2 838 | − 948 | − 304 | 1 868 | 1 897 | 0 |
| | 5.0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $N_{xy}$ Tangential force | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 2.5 | 0 | − 1 637 | − 4 015 | − 4 468 | − 2 463 | − 1 304 | − 1 096 |
| | 5.0 | 0 | − 5 238 | − 4 370 | − 4 022 | − 6 548 | − 4 551 | − 3 455 |
| $M_x$ Flexural moment | 0 | 12.21 | 1.81 | −9.28 | −9.12 | 5.82 | −0.45 | 0 |
| | 2.5 | −4.72 | 5.66 | 4.43 | −5.77 | −12.20 | 0.56 | 0 |
| | 5.0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $M_y$ Flexural moment | 0 | 7.10 | −5.74 | −14.22 | −9.25 | 1.40 | 0.83 | 0 |
| | 2.5 | −7.65 | 0.70 | 1.26 | −6.00 | −13.13 | −4.63 | 0 |
| | 5.0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $M_{xy}$ Torque | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 2.5 | 0 | −7.03 | 0.52 | 3.73 | 1.56 | 2.53 | 3.32 |
| | 5.0 | 0 | 4.49 | −8.57 | −10.73 | 8.29 | 10.21 | 8.77 |
| $Q_x$ Shear force | 0 | 0 | 14.58 | −1.21 | −5.90 | −3.58 | −0.86 | 0.97 |
| | 2.5 | 0 | −9.11 | 10.91 | 16.07 | −11.43 | −7.09 | −1.20 |
| | 5.0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $Q_y$ Shear force | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 2.5 | 13.87 | −4.54 | −12.95 | −3.09 | 7.58 | 0.19 | 0 |
| | 5.0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

**Table 2**

Normal force $N_x$

| $y$ | $n$ \ $x/a$ | 0 | 0.5 | 0.9 | 1.0 |
|---|---|---|---|---|---|
| 0 | 5 | −10 786.47 | −5 758.55 | −682.07 | −708.72 |
| | 7 | −10 611.10 | −5 634.55 | −838.55 | −708.73 |
| | 9 | −10 622.74 | −5 642.78 | −849.82 | −708.73 |
| | 11 | −10 621.91 | −5 643.37 | −850.65 | −708.73 |
| | 13 | −10 621.93 | −5 643.36 | −850.66 | −708.73 |
| 2.5 | 5 | −9 626.01 | −5 683.83 | −1 540.35 | −719.15 |
| | 7 | −9 706.88 | −5 741.02 | −1 468.29 | −719.15 |
| | 9 | −9 701.36 | −5 737.11 | −1 462.84 | −719.15 |
| | 11 | −9 700.13 | −5 737.98 | −1 464.06 | −719.15 |
| | 13 | −9 700.51 | −5 737.71 | −1 464.40 | −719.15 |
| 5.0 | 5 | −7 806.77 | −4 100.47 | −2 830.95 | −750.72 |
| | 7 | −9 628.74 | −5 388.79 | −1 207.55 | −750.68 |
| | 9 | −9 179.06 | −5 070.82 | −763.39 | −750.68 |
| | 11 | −9 342.05 | −4 955.55 | −602.40 | −750.67 |
| | 13 | −9 310.03 | −4 978.17 | −573.89 | −750.67 |

**Table 3**

Normal force $N_y$

| $y$ | $n$ \ $x/a$ | 0 | 0.50 | 0.90 | 1.0 |
|---|---|---|---|---|---|
| 0 | 5 | —4 588.92 | —299.60 | 2 095.33 | 0.01 |
| | 7 | —4 754.33 | —416.57 | 2 242.72 | 0.02 |
| | 9 | —4 603.29 | —309.77 | 2 391.89 | 0.02 |
| | 11 | —4 662.39 | —267.98 | 2 450.36 | 0.02 |
| | 13 | —4 652.41 | —275.04 | 2 459.16 | 0.02 |
| 2.5 | 5 | —3 639.86 | —52.21 | 1 042.59 | 0.01 |
| | 7 | —4 281.98 | —506.25 | 1 614.73 | 0.02 |
| | 9 | —4 067.48 | —304.58 | 1 826.59 | 0.02 |
| | 11 | —4 138.51 | —304.35 | 1 896.75 | 0.02 |
| | 13 | —4 127.38 | —312.22 | 1 906.66 | 0.03 |
| 5.0 | 5—13 | 0 | 0 | 0 | 0 |

**Table 4**

Tangential force $N_{xy}$

| $y$ | $n$ \ $x/a$ | 0 | 0.50 | 0.90 | 1.0 |
|---|---|---|---|---|---|
| 0 | 5—13 | 0 | 0 | 0 | 0 |
| 2.5 | 5 | | —4 271.15 | —1 184.49 | —804.66 |
| | 7 | | —4 451.13 | —1 300.04 | —1 059.18 |
| | 9 | | —4 472.53 | —1 304.77 | —1 089.44 |
| | 11 | | —4 467.74 | —1 303.71 | —1 096.22 |
| | 13 | 0 | —4 467.16 | —1 303.34 | —1 097.03 |
| 5.0 | 5 | | —5 189.93 | —5 232.61 | —5 351.51 |
| | 7 | | —4 183.16 | —4 586.26 | —3 927.74 |
| | 9 | | —3 935.26 | —4 531.42 | —3 577.16 |
| | 11 | | —4 021.59 | —4 550.52 | —3 455.08 |
| | 13 | 0 | —4 037.83 | —4 560.95 | —3 432.12 |

*7.6 Stresses from the second step*

Values of exact, imaginary load function $X_1 = \alpha(2\beta\, yN_{xy} + Q_x)$ are compiled in Table 6.

If matching is done in sections $x = 0.75$, $y = 5.0$ and $x = 0.75$, $y = 0$ then, according to (76):

$$\operatorname{ch} \gamma_1 b = \frac{185.30}{0.18} = 1029.44 \sim \frac{e^{\gamma_1 b}}{2}\,; \quad \gamma_1 = 1.526.$$

**Table 5**

Flexural moment $M_y$

| $y$ | $x/a$ \ $n$ | 0 | 0.50 | 0.90 | 1.0 |
|---|---|---|---|---|---|
| 0 | 5 | 4.29 | −11.35 | 3.40 | 0 |
| | 7 | 7.22 | −9.28 | 0.79 | 0 |
| | 9 | 7.18 | −9.30 | 0.75 | 0 |
| | 11 | 7.10 | −9.25 | 0.83 | 0 |
| | 13 | 7.12 | −9.26 | 0.84 | 0 |
| 2.5 | 5 | −4.63 | −4.09 | −8.33 | 0 |
| | 7 | −8.01 | −6.49 | −5.31 | 0 |
| | 9 | −7.48 | −6.12 | −4.79 | 0 |
| | 11 | −7.65 | −6.00 | −4.63 | 0 |
| | 13 | −7.62 | −6.02 | −4.60 | 0 |
| 5.0 | 5—13 | 0 | 0 | 0 | 0 |

**Table 6**

| $a/x$ \ $y$ | 0 | 0.2 | 0.4 | 0.5 | 0.75 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0.73 | −0.06 | −0.30 | −0.18 | −0.04 | −0.05 |
| 2.5 | 0 | −23.62 | −56.27 | −62.42 | −35.43 | −18.80 | −15.57 |
| 5.0 | 0 | −148.23 | −123.68 | −113.81 | −185.30 | −128.78 | −97.78 |

According to (77), making use of values in Table 6 for section $y = 5.0$, and for $n = 1, 3, 5, 7, 9, 11$, the following equation system can be written for constants $E_n$:

$$\begin{aligned}
-0.4653 &= E_1 + 2.6191\,E_3 + 3.2362\,E_5 + 2.6181\,E_7 + E_9 - E_{11}\\
-0.2039 &= E_1 + 1.6180\,E_3 + 0 - 1.6180\,E_7 - E_9 + E_{11}\\
-0.1561 &= E_1 + E_3 - E_5 - E_7 + E_9 + E_{11}\\
-0.1945 &= E_1 - 0.4142\,E_3 - 0.4142\,E_5 + E_7 - E_9 + 0.4142\,E_{11}\\
-0.1249 &= 0.9877\,E_1 - 0.8910\,E_3 + 0.7071\,E_5 - 0.4540\,E_7 + 0.1564\,E_9 + 0.1564\,E_{11}\\
-0.0948 &= E_1 - E_3 + E_5 - E_7 + E_9 - E_{11}
\end{aligned}$$

having as roots:

$$E_1 = -0.1741; \quad E_3 = -0.0495; \quad E_5 = -0.0223;$$
$$E_7 = -0.0391; \quad E_9 = +0.0096; \quad E_{11} = -0.0034$$

to yield an approximate, imaginary load function:

$$X_1 = \text{ch}\, 1.526\, y \left(-0.1741 \sin \frac{\pi}{2a} x - 0.0495 \sin \frac{3\pi}{2a} x - 0.0223 \sin \frac{5\pi}{2a} x - 0.0391 \sin \frac{7\pi}{2a} x + 0.0096 \frac{9\pi}{2a} x + - 0.0034 \sin \frac{11\pi}{2a} x\right).$$

Its value set is compiled in Table 7.

**Table 7**

| $y$ \ $x/a$ | 0 | 0.2 | 0.4 | 0.5 | 0.75 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | −0.14 | −0.12 | −0.11 | −0.18 | −0.12 | −0.09 |
| 2.5 | 0 | −3.26 | −2.72 | −2.50 | −4.08 | −2.84 | −2.15 |
| 5.0 | 0 | −147.93 | −123.22 | −113.55 | −184.99 | −128.58 | −97.59 |

Constants $E_n$ lead to $A'_n$ and $B'_n$ values according to (79) and (81), respectively:

$$A'_1 = 0.0159; \quad A'_3 = -0.0078; \quad A'_5 = -0.0011;$$
$$A'_7 = -0.0022; \quad A'_9 = 0.0003; \quad A'_{11} = -0.0001.$$
$$B'_1 = 0.7344; \quad B'_3 = -0.3686; \quad B'_5 = -0.0544;$$
$$B'_7 = -0.1092; \quad B'_9 = -0.0120; \quad B'_{11} = 0.0061.$$

In their knowledge, right-hand sides of shell equations (43) are:

$$P'_1(x, y) = \text{ch}\, 1.526\, y \left(0.0159 \cos \frac{\pi}{2a} x - 0.0078 \cos \frac{3\pi}{2a} x - 0.0011 \cos \frac{5\pi}{2a} x - 0.0022 \cos \frac{7\pi}{2a} x + 0.0003 \cos \frac{9\pi}{2a} x - 0.0001 \cos \frac{11\pi}{2a} x\right)$$

$$P'_2(x, y) = \text{ch}\, 1.526\, y \left(0.7344 \cos \frac{\pi}{2a} x - 0.3686 \cos \frac{3\pi}{2a} x - 0.0544 \cos \frac{5\pi}{2a} x - 0.1092 \cos \frac{7\pi}{2a} x - 0.0120 \cos \frac{9\pi}{2a} x - 0.0061 \cos \frac{11\pi}{2a} x\right).$$

Shell stresses from the second step (complementary stresses) are compiled in Table 8, in the same order as in Table 1:

Table 8

| $y$ \ $x/a$ | 0 | 0.20 | 0.40 | 0.50 | 0.75 | 0.90 | 1.00 |
|---|---|---|---|---|---|---|---|
| 0 | −28 | −19 | 4 | 15 | 14 | 5 | 0 |
| 2.5 | −21 | −15 | −7 | 4 | 3 | 3 | 0 |
| 5.0 | −114 | −213 | −353 | −374 | −260 | −110 | 0 |
| 0 | −30 | −18 | 6 | 16 | 16 | 6 | 0 |
| 2.5 | −18 | −8 | 1 | 2 | 10 | 9 | 0 |
| 5.0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2.5 | 0 | −18 | −22 | −15 | 9 | 11 | 10 |
| 5.0 | 0 | 23 | 13 | −8 | −56 | −50 | −46 |
| 0 | 0.20 | −0.07 | −0.06 | −0.08 | −1.6 | −0.12 | 0 |
| 2.5 | 0.11 | 0.09 | −0.40 | −0.14 | −0.06 | 0.04 | 0 |
| 5.0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0.10 | −0.03 | −0.02 | −0.02 | −0.12 | −0.10 | 0 |
| 2.5 | 0.10 | 0.06 | −0.04 | −0.10 | −0.04 | 0.01 | 0 |
| 5.0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2.5 | 0 | −0.07 | −0.02 | −0.02 | −0.02 | 0.08 | 0.12 |
| 5.0 | 0 | −0.04 | −0.13 | −0.10 | 0.11 | 0.02 | −0.02 |
| 0 | 0 | 0.12 | −0.02 | −0.02 | −0.05 | −0.14 | −0.20 |
| 2.5 | 0 | 0.05 | 0.14 | 0.10 | −0.14 | 0 | 0.06 |
| 5.0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2.5 | 0.11 | −0.03 | −0.03 | 0.05 | −0.11 | −0.16 | 0 |
| 5.0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

## 8. Conclusions and evaluation

The presented method lends itself for shallow shell structures of any slope over no special floor plan.

This kind of treatment, leading to a practical method, relied upon a mobile co-ordinate system matching the least circle of the hyperboloid of revolution where the sector shell to be used as a shell roof was described.

Even so, the problem is a rather complex one, therefore, when developing this relatively exact method, already in the formulation stage — in writing the subsequent equation systems — neglects resulting from the usual simplifications have been taken into consideration.

This method offers the most of advantages if a computer is used. The solution may be a mixed one if only stresses of the characteristic points are computed, namely by computerizing the characteristic equation of eighth order and the equation system of boundary collocation, and doing other calculations manually. This is of course more difficult and lengthy than a fully computerized solution.

The second step of the procedure is seen to little depend on shear $Q_x$ in the imaginary load function, it having the only aim to lead to an approximate function, similar in form to that in the first step.

Comparison between value sets of the imaginary load function and of the approximate function in Tables 6 and 7, respectively, shows the approximation to gradually roughen from the boundary with rather high values towards the centre of the domain with much lower values. This deviation due to approximation is, however, of little importance, as demonstrated by numerical values in Table 8. Namely, tabulated values are quite unimportant, except the $N_x$, $N_y$ and $N_{xy}$ values in Table 8 allowing some correction, but this is unimportant in itself. The iteration may be continued at will, so that the load fraction omitted in the step before is taken into consideration.

In conformity with the above statements as well as with the numerical solution of a shell structure steeper sloping than that in the numerical example, it can be stated that provided the floor plan data and load values of the shell structure differ by not too much from the data of the numerical example, up to the limit of concreting without top shuttering ($< 35°$) the second step is unnecessary.

Significance of this statement is pointed out by the fact that the computation volume for the second step is equal to that for the first one.

## Summary

A method based on the flexural theory has been developed to determine stresses acting on a shallow sector shell surface cut out of a single-shell hyperboloid of revolution over rectangular floor plan, of arbitrary proportions, taking into consideration both the dead load and the snow load.

This analysis leading to a practical procedure was allowed by the description of the sector shell problem in a mobile co-ordinate system matched to the least circle of the hyperboloid of revolution.

This problem being a complex one, it is most advantageously solved by a computer soon delivering final results, as it appears from the numerical example.

This relatively exact method lends itself to check approximate methods published in the literature.

## References


1. MENYHÁRD, I.: Analysis and construction of shell structures.* Műszaki Könyvkiadó Budapest, 1966.
2. SZMODITS, K.: Design of shell structures.* ÉTI Tudományos Közlemények, Budapest, 1965.
3. BELES, A.—SOARE, M.: Paraboloidul Eliptic Si Hiperbolic In Constructii. Editura Academiei R. P. Romine, 1964.
4. Власов, В. З.: Общая теория оболочек и ее приложения в технике. Гостехиздат, Москва, 1949.
5. HRUBAN, K.: Biegetheorie der Translationsflächen und ihre Anwendung im Hallenbau. Acta Technica Acad. Sci. Hung. **VII**, V (1953).
6. BÖLCSKEI, E.: General theory of flexural shells.* Magyar Építőipar **XI**. (1959).
7. FLÜGGE, B.: Statik und Dynamik der Schalen. Berlin, Springer-Verlag, 1957.
8. SZABÓ, J.: Structural uses of digital computers.* ÉKME Tud. Közl. **XIII**, (1967).
9. BRAJANNISZ, TH.: Approximate calculation of toroidal shell roofs of negative Gaussian curvature. IASS Symposium, Budapest, 1965. Sept.
10. BRAJANNISZ, TH.: Approximate analysis of sector shells cut out of a hyperboloid of revolution.* ÉKME Tud. Közl. Vol. XIII (1967).
11. BRAJANNISZ, TH.: Statical analysis of a sector shell ocut ut of a hyperboloid of revolution.* Candidate's Thesis.

* In Hungarian.


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