SOLUTION OF DISC PROBLEMS BY AN ELECTRICAL ANALOGUE NETWORK

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1. Introduction

There exist several methods to analyze plates loaded in their plane (discs). Numerical approximate methods markedly developed since the event of digital computers. The two methods most in use are: 1. Production of the value set of the Airy stress function by finite difference approximation of the homogeneous biharmonic differential equation; 2. establishment of displacement functions by the method of finite elements.

In spite of the rapid development of numerical methods, the interest in modelling the problems subsisted, both to facilitate simulation of special conditions and to make use of illustrative, "engineering" approximation, of variation possibilities. Modelling is done primarily by laboratory small-scale specimens or by photoelasticity means. Electric modelling is, however, much more convenient.

Any modelling is based on the similitude between physical processes. Analogy between two processes can be described by a common mathematical model. Theoretically, for each approximate mathematical model of the outlined problem, an electrical circuit can be developed, likely to simulate it. Several such solutions have been suggested or applied in the last 20 to 25 years, such as the analogue network proposed by GUTENMAHER in 1943, likely to produce the Airy stress function by modelling the biharmonic differential equation

$$\Delta\Delta F = 0.$$

Several varieties of this model have been described e.g. by PREDTECHENSKY (USSR), LIEBMANN (U. K.) and BOSCHER (France).

Another well-known model type is based on the displacement method to produce displacement function values, a special application of the method of finite elements. This idea had been published in the USA by KRON in 1944 [1], while his co-worker, CARTER demonstrated its application [2]. In 1959, GOLOVKO directed the construction of such a model to analyze wall slabs in the USSR [3]. This was easier to use, more convenient to simulate boundary conditions and more illustrative than the former. At the same time, however, the required equipment was rather complicated and costly.

To case application, an equipment of the latter type has been constructed at the Department of Civil Engineering Mechanics of the Budapest Technical University permitting to analyze discs with multiply connected domains, confined by rectangular edges acted upon by arbitrary forces or displacements, for arbitrary boundary conditions.

2. Fundamentals of the analogue network

2.1 Notations

x, y	— rectangular co-ordinates of place
Δx, Δy	— network divisions along co-ordinate axes
h	— plate thickness
m, n	— number of elements in directions x and y, resp.
i, j	— consecutive subscripts of elements and nodal points in directions x and y
u, v	— displacement in direction x or y
ε_x, ε_y	— strains
$\gamma_{xy} = \gamma_{yx}$	— distortion
α	— inflection of skew straight lines
A	— cross-sectional area
A_x, A_y	— cross-sectional area in the plane including normal x or y
P	— concentrated force
P_x, P_y	— concentrated forces parallel to the respective axes
p_x, p_y	— intensity of distributed force systems parallel to axes
q_y	— density
Q_y	— dead load of an element
σ_x, σ_y	— normal stresses
$\tau_{xy} = \tau_{yx}$	— shear stresses
E	— modulus of elasticity
G	— modulus of elasticity in shear
μ	— Poisson's ratio
λ, λ'	— Lamé constants
I	— current intensity
U, V	— electric potentials
R, r	— ohmic resistances.

2.2 Elasticity relationships

Let us determine the displacement functions of a plate subject to in-plane forces (eventually displacements). Displacement functions u and v in the orthogonal x—y co-ordinate directions will be obtained from the static equilibrium equations.

$$\frac{\partial \sigma_x}{\partial x} + \frac{\sigma \tau_{yx}}{\partial y} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\sigma y} + q_y = 0. \tag{1}$$

Substituting the displacement functions leads to the Lamé partial differential equation system:

$$
\begin{aligned}
&(\lambda' + 2G)\frac{\partial^2 u}{\partial x^2} + G\frac{\partial^2 u}{\partial y^2} + (\lambda' + G)\frac{\partial^2 v}{\partial x \partial y} = 0 \\
&(\lambda' + 2G)\frac{\partial^2 v}{\partial y^2} + G\frac{\partial^2 v}{\partial x^2} + (\lambda' + G)\frac{\partial^2 u}{\partial x \partial y} + q_y = 0
\end{aligned}
\qquad (2)
$$

with
$$\lambda' = \frac{1 - 2\mu}{1 - \mu}\lambda = \frac{2\mu}{1 - \mu}G$$

i.e. value of the Lamé coefficient for plane stress state.

Besides, functions have to satisfy two boundary conditions of either

a) displacement type (given u and v values along the boundaries); or

b) force type (edge loads);

in this case functions u and v along the boundaries have to satisfy the following two conditions:

$$
\begin{aligned}
&(\lambda' + 2G)\cos\alpha\frac{\partial u}{\partial x} + \lambda'\cos\alpha\frac{\partial v}{\partial y} + G\sin\alpha\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = p_x \\
&(\lambda' + 2G)\sin\alpha\frac{\partial v}{\partial y} + \lambda'\sin\alpha\frac{\partial u}{\partial x} + G\cos\alpha\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = p_y
\end{aligned}
\qquad (3)
$$

α — direction angle of the normal to the limiting curve;
p_x, p_y — components in directions x and y of the specific value of the edge load;

c) some combination of the former two kinds.

2.3 *Definition of the problem*

Restrictions are:

a) The examined domain is outlined by straight runs parallel to the perpendicular co-ordinate axes x and y.

b) Curved or rectilinear outlines including skew angles with the axes can only be approximated by staggered lines parallel to co-ordinate axes.

c) The domain can be multiply connected. Inner edges conform to items *a)* and *b)*.

d) The disc will be modelled by decomposition into rectangular elements of finite dimensions. Accordingly, displacement function values are to be produced in a finite number of points.

e) The disc is of isotropic material.

f) The plate may be braced in co-ordinate axis directions.

2.4 Conversion to elements of finite dimensions

The designated domain is divided into rectangular elements of Δx, Δy, h dimensions. Let the division be of constant value in one direction (Fig. 1).

Let the network lines numbered from 0 to m or n. Indicate nodal points by double subscripts and the elements by their corner of the least subscript.

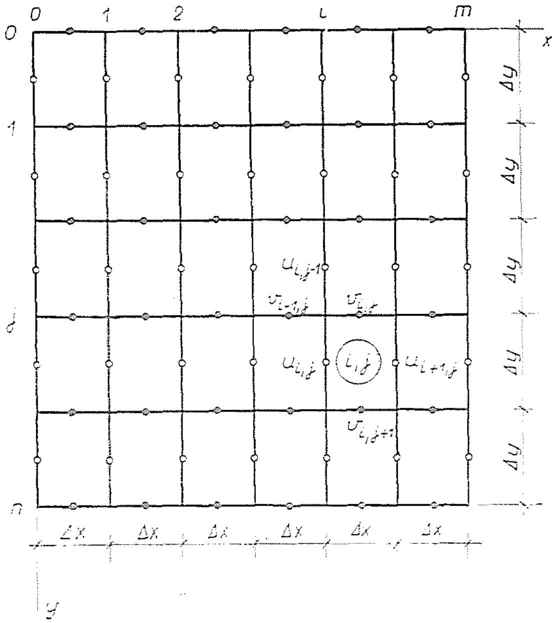

Fig. 1

Let us determine normal stresses of co-ordinate axis direction acting in the midpoint of each element e.g. $(\sigma_x)_{i,j}$ and $(\sigma_y)_{i,j}$ for the element i, j. Stress distribution in an interval is considered approximately uniform in any direction. By other words, elements of finite size are treated as elementary elastic bodies acted upon by central forces of co-ordinate axis direction. With this approximation, normal stresses or strains $(\varepsilon_x)_{i,j}$ and $(\varepsilon_y)_{i,j}$ can be expressed from displacements at the edge mid-points of the element:

$$
\begin{aligned}
(\varepsilon_x)_{i,j} &= \left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{\Delta x} \\
(\varepsilon_y)_{i,j} &= \left(\frac{\partial v}{\partial y}\right)_{i,j} = \frac{v_{i,j+1} - v_{i,j}}{\Delta y}\,.
\end{aligned}
\tag{4}
$$

Subscript of the displacement function values will be the same as that of the element edges converging at the nodal point of the given notation, as seen in Fig. 1.

At point i, j of the network (i.e. at corner points of elements meeting there) also the value of the shear stress $(\tau_{xy})_{i,j} = (\tau_{yx})_{i,j}$ and the distortion can be expressed from the displacement values at edge mid-points:

$$(\gamma_{xy})_{i,j} = (\gamma_{yx})_{i,j} = \left(\frac{\partial u}{\partial y}\right)_{i,j} + \left(\frac{\partial v}{\partial x}\right)_{i,j} = \frac{u_{i,j} - u_{i,j-1}}{\Delta y} + \frac{v_{i,j} - v_{i-1,j}}{\Delta x} . \qquad (5)$$

It is obvious from Eqs (4) and (5) as well as Fig. 1 that function values v and u are to be produced along edges in directions x and y, at $(n+1) \cdot m$ and $n \cdot (m+1)$ points, respectively. Equilibrium equations (2) are to be written in finite form, with respect to these places. One unknown value at one point being sought, a single equation suffices. Equilibrium equations in directions x and y may be written for u and v, respectively. In conformity with the applied approximation, equilibrium conditions are thus met only in the mid-points of the element edges, different conditions for edges in directions x or y, but both conditions in points though different but spaced apart by Δx and Δy, respectively.

Differential quantities in Eqs (1) will be replaced by finite quantities corresponding to the model. Stress increments between points spaced apart by Δx are:

$$\Delta x \left(\frac{\partial \sigma_x}{\partial x}\right)$$

or

$$\Delta x \left(\frac{\partial \tau_{xy}}{\partial x}\right)$$

while for Δy

$$\Delta y \left(\frac{\partial \sigma_y}{\partial y}\right)$$

and

$$\Delta y \left(\frac{\partial \tau_{yx}}{\partial y}\right) .$$

Taking into consideration a uniform stress distribution, the finite form of Eqs (1) at point i, j for a disc of constant h thickness:

$$\begin{aligned} \Delta x \Delta y\, h \left(\frac{\partial \sigma_x}{\partial x}\right)_{i,j} + \Delta x \Delta y\, h \left(\frac{\partial \tau_{yx}}{\partial y}\right)_{i,j} &= 0 \\ \Delta x \Delta y\, h \left(\frac{\partial \tau_{xy}}{\partial x}\right)_{i,j} + \Delta x \Delta y\, h \left(\frac{\partial \sigma_y}{\partial y}\right)_{i,j} + \Delta x \Delta y\, h\, q_y &= 0 \end{aligned} \qquad (6)$$

where $\Delta x \Delta y\, h\, q_y = Q_y$ is the total dead weight of an element.

Let Eqs (2) be transformed according to the same principles:

$$(\lambda' + 2G)\, \Delta x \Delta y\, h \left(\frac{\partial^2 u}{\partial x}\right)_{i,j} + G\, \Delta x \Delta y\, h \left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j} +$$
$$+ (\lambda' + G)\, \Delta x \Delta y\, h \left(\frac{\partial^2 v}{\partial x \partial y}\right)_{i,j} = 0$$

$$(\lambda' + 2G)\, \Delta x \Delta y\, h \left(\frac{\partial^2 v}{\partial y^2}\right)_{i,j} + G\, \Delta x \Delta y\, h \left(\frac{\partial^2 v}{\partial x^2}\right)_{i,j} +$$
$$+ (\lambda' + G)\, \Delta x \Delta y\, h \left(\frac{\partial^2 u}{\partial x \partial y}\right)_{i,j} + Q_y = 0$$

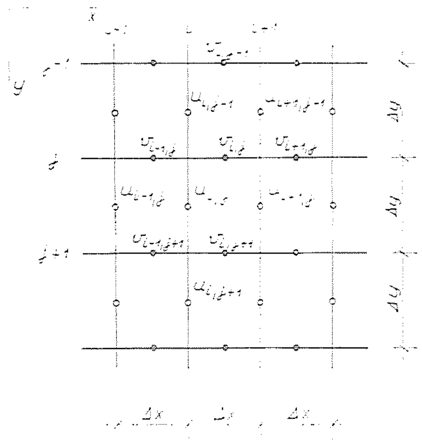

Fig. 2

With notations in Fig. 2. differential quantities can be expressed by finite differences as:

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{\Delta x^2}$$

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{\Delta y^2}$$

$$\left(\frac{\partial^2 v}{\partial x \partial y}\right)_{i,j} = \frac{v_{i-1,j} - v_{i,j} - v_{i-1,j+1} + v_{i,j+1}}{\Delta x \Delta y}$$

$$\left(\frac{\partial^2 v}{\partial y^2}\right)_{i,j} = \frac{v_{i,j-1} - 2v_{i,j} + v_{i,j+1}}{\Delta y^2}$$

$$\left(\frac{\partial^2 v}{\partial x^2}\right)_{i,j} = \frac{v_{i-1,j} - 2v_{i,j} + v_{i+1,j}}{\Delta x^2}$$

$$\left(\frac{\partial^2 u}{\partial x \partial y}\right)_{i,j} = \frac{u_{i,j-1} - u_{i+1,j-1} - u_{i,j} + u_{i-1,j}}{\Delta x \Delta y}.$$

Finite form of the Lamé equations is, by substitution:

$$\left.\begin{aligned}
&(\lambda' + 2G)\frac{\Delta y}{\Delta x} h\,(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) + G\frac{\Delta x}{\Delta y} h(u_{i,j-1} - 2u_{i,j} + u_{i,j+1}) + \\
&+ (\lambda' + G)\,h\,(v_{i-1,j} + v_{i,j+1}) + (\lambda' + G)\,h\,(-v_{i,j} - v_{i-1,j+1}) = 0 \\
&(\lambda' + 2G)\frac{\Delta x}{\Delta y} h\,(v_{i,j-1} - 2v_{i,j} + v_{i,j+1}) + G\frac{\Delta y}{\Delta x}\, h(v_{i-1,j} - 2v_{i,j} + v_{i+1,j}) + \\
&+ (\lambda' + G)\,h\,(u_{i,j-1} + u_{i+1,j}) + (\lambda' + G)\,h\,(-u_{i-1,j-1} - u_{i,j}) + Q_y = 0\,.
\end{aligned}\right\} \quad (7)$$

2.5 Fundamentals of the electric network analogue

The presented problem will be analyzed by an electric network. The method is based on the analogy between the solid behaving elastically and the electric stationary flow field. In mathematical form, this analogy is given by the confrontation of the Hooke law and the Ohm law. The linear flow in a single conductor of an electric network is analogous to the behaviour of an elastic straight bar subject to an axial load. Mutually corresponding quantities in physical processes are:

elastic solid:	*electrical conductor:*
displacement (u)	potential (U)
force (P)	current (I)
rigidity $\left(\frac{EA}{l}\right)$	conductivity $\left(\frac{1}{R}\right)$.

The two analogous laws are:

$$P = \frac{EA}{\Delta x}\Delta u \qquad\qquad I = \frac{1}{R}\Delta U$$

where: Δu — total strain of the bar of length Δx
ΔU — potential difference between conductor ends.

Of course, values of analogous quantities are only proportional, but not identical. This proportion can be expressed by proportionality factors, determined from the condition of analogy. Quantities in the above relationships are made dimensionless as:

$$\bar{P} = \overline{\left(\frac{EA}{\Delta x}\right)} \, \overline{\Delta u} \left[\left(\frac{EA}{\Delta x}\right)_0 \frac{\Delta u_0}{P_0}\right] \qquad \bar{I} = \frac{1}{\bar{R}} \overline{\Delta U} \left[\frac{1}{R_0} \frac{\Delta u_0}{I_0}\right]$$

indicating by top line the dimensionless quantities and by subscript 0 the dimension multipliers. The analogy has as condition:

$$\left(\frac{EA}{\Delta x}\right)_0 \frac{\Delta u_0}{P_0} = \frac{1}{R_0} \frac{\Delta U_0}{I_0}.$$

Relationship between proportionality factors can be expressed from condition (8):

$$\frac{I_0}{P_0} = \frac{\dfrac{1}{R_0}}{\left(\dfrac{EA}{\Delta x}\right)_0} \frac{\Delta U_0}{\Delta u_0}$$

$$m_I = \frac{1}{m_R} m_U \tag{9}$$

where:

$m_I = \dfrac{I_0}{P_0} [A/\mathrm{kp}]$ — proportionality factor of currents

$m_U = \dfrac{\Delta U_0}{\Delta u_0} [V/\mathrm{cm}]$ — proportionality factor of the potential

$m_R = \dfrac{R_0}{\left(\dfrac{\Delta x}{EA}\right)_0} [\Omega/(\mathrm{cm/kp})]$ — proportionality factor of conductor resistances.

The further relationships of the analogy between both physical processes are those between the elastic solid modelled by the set of finite elements and the network composed of elementary conductors.

A network of several conductors permits to electrically simulate the physical equations of the theory of elasticity (generalized Hooke law).

Concentrated forces at defined points of the body replacing distributed inner load systems satisfy the finite form of Eqs (1), in short:

$$\sum_{(k)} P_{kx} = 0 \qquad \text{and} \qquad \sum P_{ky} = 0.$$

It is evident from item *2.4* that these two conditions are to be satisfied in different points. On the electric model two interdependent networks will be developed, simulating strength quantities in directions x and y, respectively. Currents flowing in and out of nodal points of the two networks satisfy the Kirchhoff nodal law:

$$\sum_{(k)} I_k = 0$$

Currents flowing in the conductors being the analogies of mechanical forces, nodal law in the two networks is the analogy of the statical equilibrium equations.

Similarly, it can be verified that for a duly constructed network, the Kirchhoff loop law represents the compatibility equation of the theory of strength that will not be treated here in detail.

2.6 Development of the electric network

In item *2.5* it was seen that two networks were to be built. Nodal points of the one belong to those points of the domain where displacement values in direction x are sought, hence to mid-points of element edges parallel to the

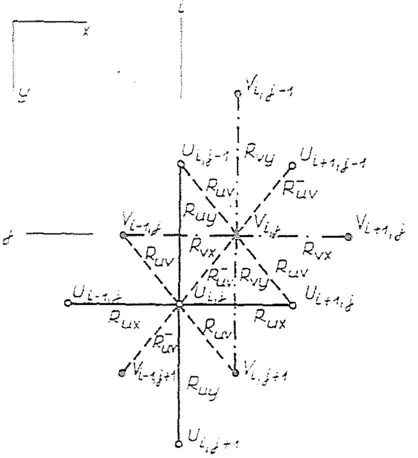

Fig. 3

y axis, and vice versa. Adjacent nodal points of both networks (i.e. on edges of identical elements) are also connected by conductors.

A detail of the networks is shown in Fig. 3, representing by a continuous line the conductors of the network U in direction x, by a dash-and-dot line those of the network V in direction y and by a dashed line the conductors connecting both networks. Resistances are denoted as:

R_{ux} of direction x in network U
R_{uy} of direction y in network U

R_{vx} of direction x in network V
R_{vy} of direction y in network V
R_{uv} between two networks.

Potentials on networks U and V are denoted by $U_{i,j}$ and $V_{i,j}$, respectively.

Let us write the nodal law for points i, j of networks U and V, respectively. To maintain trueness to sign, potentials are considered as to increase in the positive direction of the co-ordinate axes and in the positive direction of x between two points of different networks:

$$\begin{aligned}
&\frac{1}{R_{ux}}(U_{i+1,j}-U_{i,j})-\frac{1}{R_{ux}}(U_{i,j}-U_{i-1,j})+\frac{1}{R_{uy}}(U_{i,j+1}-U_{i,j})-\\
&-\frac{1}{R_{uy}}(U_{i,j}-U_{i,j-1})-\frac{1}{R_{uv}}(U_{i,j}-V_{i-1,j})+\frac{1}{R_{uv}}(V_{i,j}-U_{i,j})-\\
&-\frac{1}{R_{uv}}(U_{i,j}-V_{i-1,j+1})+\frac{1}{R_{uv}}(V_{i,j+1}-U_{i,j})=0
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{R_{vy}}(V_{i,j+1}-V_{i,j})-\frac{1}{R_{vy}}(V_{i,j}-V_{i,j-1})+\frac{1}{R_{vx}}(V_{i+1,j}-V_{i,j})-\\
&-\frac{1}{R_{vx}}(V_{i,j}-V_{i-1,j})-\frac{1}{R_{uv}}(V_{i,j}-U_{i,j-1})+\frac{1}{R_{uv}}(U_{i+1,j-1}-V_{i,j})-\\
&-\frac{1}{R_{uv}}(V_{i,j}-U_{i,j})+\frac{1}{R_{uv}}(U_{i+1,j}-V_{i,j})=0.
\end{aligned}$$

Equations are arranged so as to be formally identical to Eqs (7) to be modelled:

$$\left.\begin{aligned}
&\frac{1}{R_{ux}}(U_{i-1,j}-2U_{i,j}+U_{i+1,j})+\frac{1}{R_{uy}}(U_{i,j-1}-2U_{i,j}+U_{i,j+1})+\\
&+\frac{1}{R_{uv}}(V_{i-1,j}+V_{i,j+1})-\frac{1}{R_{uv}}(-V_{i,j}-V_{i-1,j+1})=0\\
&\frac{1}{R_{vy}}(V_{i,j-1}-2V_{i,j}+V_{i,j+1})+\frac{1}{R_{vx}}(V_{i-1,j}-2V_{i,j}+V_{i+1,j})+\\
&+\frac{1}{R_{uv}}(U_{i,j-1}+U_{i+1,j})-\frac{1}{R_{uv}}(-U_{i-1,j}-U_{i,j})+I_y=0.
\end{aligned}\right\} \qquad (10)$$

Confrontation of Eqs (7) and (10) yields the value of network resistances. With due regard to the proportionality factor:

$$R_{ux}=\frac{1}{\lambda'+2G}\,\frac{\Delta x}{\Delta y}\,\frac{1}{h}\,m_R=\frac{1-\mu^2}{Eh}\,\frac{\Delta x}{\Delta y}\,m_R$$

$$R_{uy}=\frac{1}{G}\,\frac{\Delta y}{\Delta x}\,\frac{1}{h}\,m_R=\frac{2(1+\mu)}{Eh}\,\frac{\Delta y}{\Delta x}\,m_R$$

$$R_{vy} = \frac{1}{\lambda' + 2G} \frac{\Delta y}{\Delta x} \frac{1}{h} m_R = \frac{1-\mu^2}{Eh} \frac{\Delta y}{\Delta x} m_R \tag{11}$$

$$R_{vx} = \frac{1}{G} \frac{\Delta x}{\Delta y} \frac{1}{h} m_R = \frac{2(1+\mu)}{Eh} \frac{\Delta x}{\Delta y} m_R$$

$$R_{uv} = \frac{1}{(\lambda' + G)\,h} m_R = \frac{2\,(1-\mu)}{Eh} m_R\,.$$

Both expressions contain one negative term. A model analogous to Eqs (7) can only be produced by inserting "negative resistances" in the place of the corresponding conductors. As to the origin of negative terms, "negative resistances" turn out to be placed in those branches denoted R_{uv}^{-} in Fig. 3 connecting the two networks which run along the skew straight lines including a negative angle with the x axis. Along these conductors the potential has to grow instead of to decrease, as indicated by the denomination. Thus, in fact, "negative resistance" is a supply to be adjusted so as to produce a potential increase of the same absolute value as the decrease along resistances R_{uv}.

Current I_y represents the dead weight Q_y. $I_y = Q_y m_1$.

For a domain decomposed into square elements ($\Delta x = \Delta y$), $R_{ux} = R_{vy}$ and $R_{uy} = R_{vx}$.

For a ribbed plate, values Δxh or Δyh may be replaced by the cross-sectional area calculated with the rib. Be A_x and A_y the cross-sectional areas of an element each in planes of normals x and y. Transformation of Eqs (2) yields:

$$(\lambda' + 2G)\,\Delta x\,A_x \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} + G\,\Delta_y\,A_y \left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j} + [\mu(\lambda' + 2G)\,\Delta x\,A_x +$$

$$+ G\,\Delta y\,A_y] \left(\frac{\partial^2 v}{\partial x\,\partial y}\right)_{i,j} = 0$$

$$(\lambda' + 2G)\,\Delta y\,A_y \left(\frac{\partial^2 v}{\partial y^2}\right)_{i,j} + G\,\Delta x\,A_x \left(\frac{\partial^2 v}{\partial x^2}\right)_{i,j} + [\mu(\lambda' + 2G)\,\Delta y\,A_y +$$

$$+ G\,\Delta x A_x] \left(\frac{\partial^2 u}{\partial x\,\partial y}\right)_{i,j} + Q_y = 0.$$

Accordingly, network resistances are:

$$R_{ux} = \frac{1-\mu^2}{E} \frac{\Delta x}{A_x} m_R$$

$$R_{uy} = \frac{2(1+\mu)}{E} \frac{\Delta y}{A_y} m_R$$

$$R_{vy} = \frac{1-\mu^2}{E} \frac{\Delta y}{A_y} m_R$$

$$R_{vx} = \frac{2(1+\mu)}{E} \frac{\Delta x}{A_x} m_R$$

$$R_{uv} = \frac{2(1-\mu^2)}{E} \left[\frac{\Delta y}{2\mu A_x} + \frac{\Delta x}{(1-\mu) A_y} \right] m_R .$$

Of course, other cases of orthotropy may also be modelled by duly selecting resistance values.

2.7 Boundary conditions

2.7.1 Edge displacements. If displacement function values along the edge are known, then this can be expressed by appropriate potentials:

$$U_{i,j} = m_U u_{i,j} \qquad \text{and} \qquad V_{i,j} = m_U v_{i,j},$$

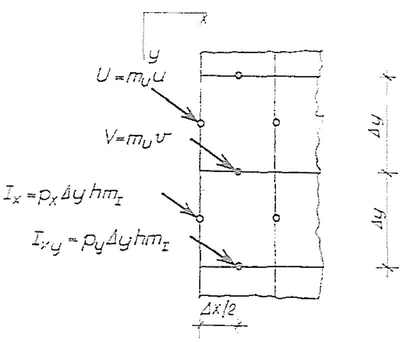

Fig. 4

The voltage representing displacement of the boundary point normally to the edge will be conducted to the edge point of the network belonging to the given direction, and the voltage representing the displacement parallel to the edge, to a point near to the edge of the network in that direction (at a distance $\Delta x/2$ and $\Delta y/2$, respectively). Fig. 4 shows a case of an edge parallel to the y axis.

The most frequent case is that of zero edge displacements. In this case the corresponding network points will be connected to the earth point.

2.7.2 Edge loads. Edge displacements due to edge loads have to satisfy relationships (3). This is automatically so in the electric network if, by analogy to the equilibrium equations (3), external loads are modelled by corresponding current supply to peripherial network nodal points, applying the Kirchhoff law.

For specific p_x and p_y values of load components in directions x and y, respectively, the corresponding currents for an edge of e.g. y direction are:

$$I_x = p_x \Delta y h\, m_I$$
$$I_{xy} = p_y \Delta y h\, m_I .$$

Notation I_{xy} refers to the shear force character. Of course, this current has to be fed to the corresponding network nodal points by a half division inward, these being quantities in direction y. Also this case is illustrated in Fig. 4. The former are valid to the edge in direction x, to the sense.

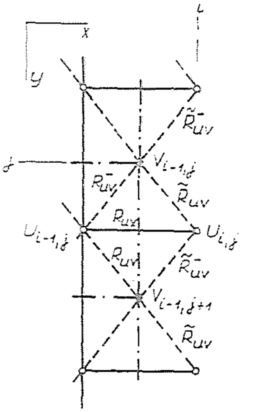

Fig. 5

A frequent case is that of a free edge, along which: $\sigma_x = 0$, $\tau_{xy} = 0$ or $\sigma_y = 0$, $\tau_{yx} = 0$. Zero normal stress is represented by no external supply to the edge point. Zero shear stresses have, however, other consequences too. Fig. 5 shows a part of the networks along an edge in direction y. There is no supply to the points near the edge of network V, since $\tau_{xy} = 0$. At the same time in each point $\tau_{yx} = 0$ or between these points the shear stress varies by $\dfrac{\partial \tau_{yx}}{\partial y} = 0$. Equation in direction x of expressions (6) can be written for edge point i, j as:

$$\Delta x \Delta y\, h \left(\frac{\partial \sigma_x}{\partial x} \right)_{i,j} = 0$$

or, the Lamé equation:

$$(\lambda' + 2G) \frac{\Delta y}{\Delta x} h(u_{i-1,j} - u_{i,j}) + \mu(\lambda' + 2G)\, h(v_{i-1,j} - v_{i-1,j+1}) = 0$$

boundary conditions involving $u_{i+1,j} - u_{i,j} = 0$ and $v_{i,j+1} - v_{i,j} = 0$.

Accordingly, no current can flow between edge points of network U. ($R_{uy} = \infty$), and resistance of conductors connecting both networks is:

$$\tilde{R}_{uv} = \frac{1}{\mu(\lambda' + 2G)h} \quad m_R = \frac{1 - \mu^2}{\mu E h} m_R . \tag{12}$$

Thus, in the edge point the Kirchhoff law can be written as:

$$-\frac{1}{R_{ux}} (U_{i,j} - U_{i-1,j}) \quad \frac{1}{\tilde{R}_{uv}} (U_{i,j} - V_{i-1,j}) + \frac{1}{\tilde{R}_{uv}} (U_{i,j} - V_{i-1,j+1}) = 0$$

2.7.3 General conditions. Boundary conditions may vary periodically, corresponding to the division, and the two types can be realized mixed. One great advantage of the applied model is exactly to ease designation of the domain and specification of either inner or outer boundary conditions of any type, varying even along one edge.

The potential represents the displacement values true to sign. Therefore the origin of the co-ordinate system has to be indicated unambiguously. For most problems this is done automatically by designating points of zero displacement. Otherwise, however, specially to this aim, at least three nodal points of zero potential have to be designated.

3. Test equipment

Scheme of the constructed model is seen in Fig. 6. Boundary conditions are given by current and voltage supplies. The two networks are balanced by means of negative resistances.

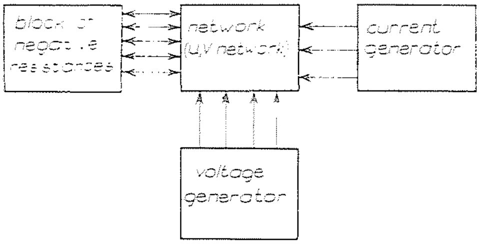

Fig. 6

The equipment models $6 \times 8 = 48$ elementary fields ($m = 8$, $n = 6$). Divisions are equal in both directions ($\Delta x = \Delta y$). Incorporated resistances R_{ux}, R_{uy}, R_{vx}, R_{vy} arranged as seen in Fig. 3 are of constant value. If the domain to be modelled cannot be covered by a square net, among the constant resistances one of the required value has specially to be inserted. To this aim 52 resistances of variable value are available on a special board. In addition to the resistances R_{uv} connecting the two networks, those of $\tilde{R}_{uv}$ value have also been inserted to permit realization of the free edge anywhere by simple plugging. Any resistance can simply be disconnected to ease formation of the domain.

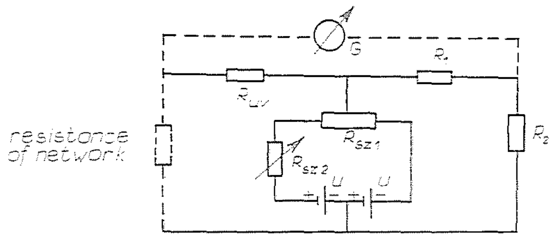

Fig. 7

Constant resistance values permit to determine proportionality factors from the resistance values in conformity with the starting data. For instance, from the R_{ux} value, applying (11):

$$m_R = R_{ux} \frac{Eh}{1-\mu^2} \frac{\Delta y}{\Delta x}.$$

For convenience of reading off, it is advisable to choose a rounded up value for the proportionality factor m_U of the potential.

Them m_I can be computed from (9).

The "negative resistances" have been realized by the Golovko method [3] by supplying I_{UV} between two corresponding points of both networks, in conformity with the given resistance values $R_{uv} = -\frac{U-V}{I_{UV}}$. Circuit diagram is shown in Fig. 7. Voltage is adjusted by means of variable resistances R_{sz_1} and R_{sz_2} to zero the galvanometer G. Since resistances R_1 and R_2 are equal, current I_{UV} corresponding to R_{UV} is fed into the network.

Presented voltage supplies U are 24 V each, setting the maximum voltages in the network at about $+10 V$. The equipment contains 96 "negative resistance" units.

The problem is solved by adjusting these currents I_V in the model. Because of the network reaction, the adjustment is done by successive approximations.

Fig. 8 shows a circuit diagram representing the supply needed for the boundary conditions. As compared to the resistance R, the resistance r representing the network can be neglected (0.5—1.0 $K\Omega$ as against 50—100 $K\Omega$).

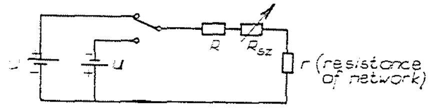

Fig. 8

The exact current intensity can be adjusted by means of the variable resistance R_{sz}. Voltage supply U is of 200 V.

Voltage supply was made by transforming AC of 24 V 50 Hz. Its low inner resistance provided for a convenient operation.

All units (Fig. 6) of the equipment in Fig. 9 were housed in a common closet together with meters and accessories.

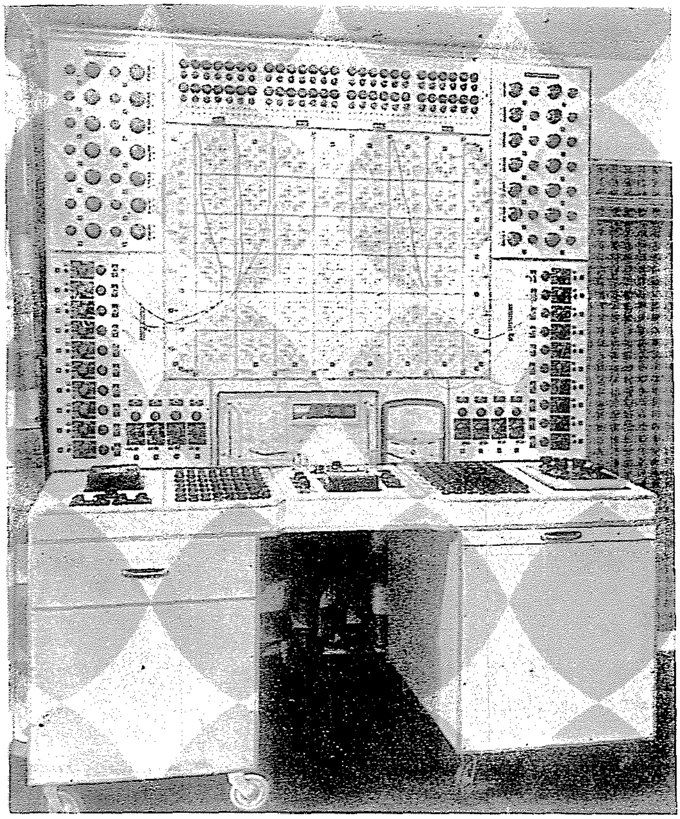

Fig. 9

Before constructing the final system, measurements were made on a little, temporary network model to collect observations applicable to the design of the final equipment. Recently, model tests were begun with. Test results, outcomes and accuracy data will be described in a subsequent paper.

Summary

An electric analogue model has been built at the Department of Civil Engineering Mechanics of the Budapest Technical University to analyze plates loaded in plane (discs). G. Kron (USA) published in 1944 the fundamentals of the applied network analogy, then M. D. Golovko (USSR) practically developed it. The constructed equipment applied the displacement method to produce displacement function values in a finite number of points for rectangular discs outlined by rectilinear edges, with multiply connected domains, for arbitrary boundary conditions. Plates may be braced by ribs parallel to the rectangular co-ordinates, or may be orthotropic otherwise.

References

1. KRON, G.: Journ. of Appl. Mech. II, 149 (1944).
2. CARTER, G .K.: Journ. of Appl. Mech. II, 162 (1944).
3. Гагарина, А. А. — Голоеко, М. Д.: Исследование напряхенного состояния крупноразмерних стеновых панелей методом электрических аналогий.

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