MATRIX EQUATION ANALYSIS IN THE FINITE ELEMENT METHOD

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1. Introduction

Since World War II, the event of digital computers, together with problems raised by the airplane and rocket industry, stimulated the development of appropriate up-to-date structural analysis methods suiting actual requirements and the available computer technique. Far from applying the methods already known, making use of the possibilities presented by the speed of computer methods to solve ever greater problems, they follow instead entirely new ways.

The new methods apply the matrix calculus in a wide range, not only to simplify the writing and programming of algorithms as the natural language of computation methods, but also to present an elegant and concise mathematical treatment.

The most widely extended of them is the finite element method, called by some authors the matrix displacement method, advantageous by its versatility. Though initially it had been applied in structural engineering, just as will be here, essentially it suits to any boundary value problem that can be described by partial (or ordinary) differential equations, for arbitrary domains, boundary conditions and loads. It is widely applied for vibration, heat transfer and hydraulic problems.

The disadvantage of the finite element method is that rather small problems require operations with quite large matrices, exceeding the capacity of comparatively up-to-date computers, at an important computer time demand.

In what follows, the finite element method will be briefly surveyed and a method will be presented, likely to cut computer time and storage capacity for some frequent but special cases.

2. The finite element method

2.1 General

A well-known fundamental principle of the analysis of hyperstatic structures is to consider the structure an entity of members connected at a finite number of nodes. If force-displacement relationships for each member as well as statical and geometrical boundary conditions are known, the behaviour of the whole structure can be cleared up. This is the basis of the analysis of hyperstatic trusses, frameworks, lattices etc.

For surface structures or continua there is in fact an infinite number of nodes. The method of finite elements eliminates this difficulty by considering the surface (for simplicity's sake, continua will not be treated below) to be divided into elements connected at a finite number of nodes, acted upon by nodal forces between elements (replacing boundary stresses of elements), and of course, external forces are also considered as acting only at these nodes. Relationship between nodal forces acting at the elements and nodal displacements are represented by the stiffness matrix of the element (not to be determined here because of space shortage). From the stiffness matrices of all the elements, that of the entire structure can be determined, delivering the relationship between forces and displacements of the structure as a whole.

Thereby force-induced displacements and from them the stresses can be determined.

Division of the structure into elements, as well as replacement of the continuous internal stress system by nodal forces is an approximation of real conditions. Another usual approximation is related to the establishment of force-displacement relationships for the element. In spite of these approximations, the method is a useful one, not only by permitting the analysis of till now (in closed form) untreatable problems, but by increasing accuracy upon making divisions finer. The fundamentals of this method are due to Turner, described in detail by ARGYRIS [4] and ZIENKIEWICZ [5], or in Hungarian by BERÉNYI [7].

2.2 *Finite element analysis of discs*

An in-plane loaded plate (disc) is known to be in a plane stress state. A conventional method of analysis is that by the Airy stress function, with the differential equation:

$$\frac{\partial^4 F}{\partial x^4} + 2\,\frac{\partial^4 F}{\partial x^2\,\partial y^2} + \frac{\partial^4 F}{\partial y^4} = 0 \tag{2.1}$$

where F is the Airy stress function, interpreted as:

$$\sigma_y = \frac{\partial^2 F}{\partial x^2}\,; \qquad \sigma_x = \frac{\partial^2 F}{\partial y^2}\,; \qquad \tau_{xy} = -\,\frac{\partial^2 F}{\partial x \partial y}\,. \tag{2.2}$$

Validity of this method is restricted to edge-loaded discs.

In the finite element method, the disc is usually divided into triangular elements (Fig. 1). Nodal forces and displacements can be divided into components of x and y direction. Nodal forces and displacements will be expressed by vectors $\mathbf{p}$ and $\mathbf{d}$, respectively:

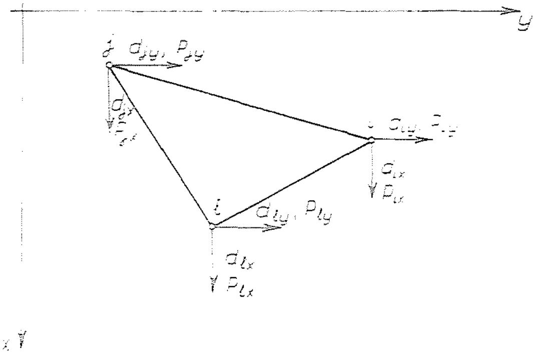


Fig. 1

$$\mathbf{p} = \begin{bmatrix} \mathbf{p}_i \\ \mathbf{p}_j \\ \mathbf{p}_l \end{bmatrix} = \begin{bmatrix} p_{ix} \\ p_{iy} \\ p_{jx} \\ p_{jy} \\ p_{lx} \\ p_{ly} \end{bmatrix}; \qquad \mathbf{d} = \begin{bmatrix} \mathbf{d}_i \\ \mathbf{d}_j \\ \mathbf{d}_l \end{bmatrix} = \begin{bmatrix} d_{ix} \\ d_{iy} \\ d_{jx} \\ d_{jy} \\ d_{lx} \\ d_{ly} \end{bmatrix}. \tag{2.3}$$

Displacements and loads are related by the stiffness matrix $\mathbf{k}$:

$$\mathbf{k}\,\mathbf{d} = \mathbf{p};$$

or, in particular

$$\begin{bmatrix} \mathbf{k}_{ii} & \mathbf{k}_{ij} & \mathbf{k}_{il} \\ \mathbf{k}_{ji} & \mathbf{k}_{jj} & \mathbf{k}_{jl} \\ \mathbf{k}_{li} & \mathbf{k}_{lj} & \mathbf{k}_{ll} \end{bmatrix} \begin{bmatrix} \mathbf{d}_i \\ \mathbf{d}_j \\ \mathbf{d}_l \end{bmatrix} = \begin{bmatrix} \mathbf{p}_i \\ \mathbf{p}_j \\ \mathbf{p}_l \end{bmatrix} \tag{2.4}$$

Blocks $\mathbf{k}$ are now size $2 \cdot 2$ and represent the force $\mathbf{p}$ at the node with the first subscript produced by the displacement $\mathbf{d}$ of the node with the second subscript. In case of isotropy problems according to Maxwell's reciprocal theorem the matrix $\mathbf{k}$ is always symmetric, hence $\mathbf{k}_{ij} = \mathbf{k}_{ji}$.

A rectangular field is conveniently treated by rectangular elements. Then $\mathbf{d}$ and $\mathbf{p}$ will contain $4 \cdot 2$ elements, and $\mathbf{k}$ $4 \cdot 4$ blocks, 64 elements.

Stiffness matrices of all elements being determined, matrix equation of the entire structure can be composed. Vectors $\mathbf{p}$ and $\mathbf{d}$ will include forces and displacements of all nodes consecutively, and the stiffness matrix $\mathbb{K}$ of the whole structure will contain as many rows and columns of blocks, as many nodes there are in the structure. Each block $\mathbf{k}_{ij}$ contains the sum of corresponding blocks of the stiffness matrices of all elements involving i and j nodes. Thus, the equation of the structure is of the form:

$$\mathbb{K}\,\mathbf{d} = \mathbf{p}. \tag{2.5}$$

Note that any block $\mathbf{k}_{ij}$ differs from zero only if there exists at least one element which involves both i and j nodes. Thereby most blocks of matrix $\mathbb{K}$ will be zero blocks, and the stiffness matrix $\mathbb{K}$ is invariably symmetrical.

2.3 Finite element analysis of bending plates

In the case of bending plates, the nodes have three degrees of freedom (neglecting other displacement possibilities), thus, an element in the xy plane has all nodes acted upon by displacements w_i, φ_{ix}, φ_{iy} and force components P_i, M_{ix}, M_{iy}. For instance, for a triangular element (Fig. 2):

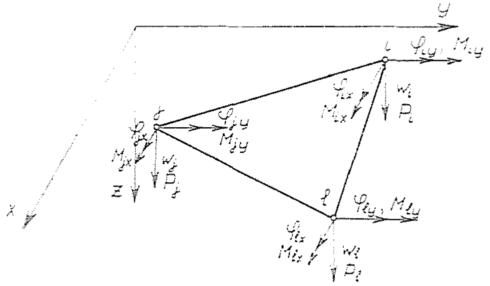


Fig. 2

$$\mathbf{d} = \begin{bmatrix} \mathbf{d}_i \\ \mathbf{d}_j \\ \mathbf{d}_l \end{bmatrix} = \begin{bmatrix} w_i \\ \varphi_{ix} \\ \varphi_{iy} \\ w_j \\ \varphi_{jx} \\ \varphi_{jy} \\ w_l \\ \varphi_{lx} \\ \varphi_{ly} \end{bmatrix} \quad \text{and} \quad \mathbf{p} = \begin{bmatrix} \mathbf{p}_i \\ \mathbf{p}_j \\ \mathbf{p}_l \end{bmatrix} = \begin{bmatrix} P_i \\ M_{ix} \\ M_{iy} \\ P_j \\ M_{jx} \\ M_{jy} \\ P_l \\ M_{lx} \\ M_{ly} \end{bmatrix}. \tag{2.6}$$

Accordingly, size of the block $\mathbf{k}_{ij}$ will be $3 \cdot 3$. Thereafter the procedure will be as before.

3. Proposed method of treatment for the hypermatrix equation

3.1 The hypermatrix equation

The finite element method was seen to lead to the matrix equation

$$\mathbf{K}\,\mathbf{d} = \mathbf{p} \tag{3.1}$$

with the solution

$$\mathbf{d} = \mathbf{K}^{-1}\,\mathbf{p}. \tag{3.2}$$

Stiffness matrix $\mathbf{K}$ in the equation can be composed of stiffness matrices of the elements. If both the domain and the elements are rectangular, matrix $\mathbf{K}$ is a hypermatrix of special structure, with hypermatrix blocks:

$$\mathbf{K} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & & & & \\ \mathbf{B}^* & \mathbf{A} & \mathbf{B} & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & \mathbf{B}^* & \mathbf{A} & \mathbf{B} \\ & & & & \mathbf{B}^* & \mathbf{A} \end{bmatrix} \tag{3.3}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{a} & \mathbf{b} & & & & \\ \mathbf{b}^* & \mathbf{a} & \mathbf{b} & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & \mathbf{b}^* & \mathbf{a} & \mathbf{b} \\ & & & & \mathbf{b}^* & \mathbf{a} \end{bmatrix}$$

and (3.4)

$$\mathbf{B} = \begin{bmatrix} \mathbf{c} & \mathbf{d}_2 & & & & & \\ \mathbf{d}_1 & \mathbf{c} & \mathbf{d}_2 & & & & \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & \mathbf{d}_1 & \mathbf{c} & \mathbf{d}_2 \\ & & & & & \mathbf{d}_1 & \mathbf{c} \end{bmatrix}$$

Blocks **a**, **b**, **c**, $\mathbf{d}_1$ and $\mathbf{d}_2$ are linear combinations of elementary stiffness matrix blocks, with sizes equalling the numbers of freedom of the nodes.

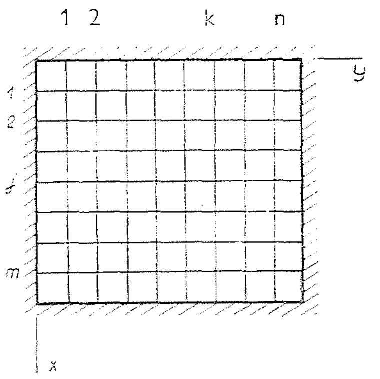


Fig. 3

The stiffness matrix of the system was seen to be a continuant hypermatrix, with non-zero blocks being also continuant hypermatrices. Apparently, most blocks of this hypermatrix are zero. In case of a problem of $m \cdot n$ nodes (Fig. 3) the matrix of order $m \cdot n \cdot s$ contains $m^2 \cdot n^2 \cdot s^2$ elements, among them at most $(3m—2)\,(3n—2)s^2$ non-zero ones, a minor part of all elements, thus it is uneconomical to store and handle the entire matrix both from storage capacity and running time aspects. Often but the upper band is being stored, with $(m+2) \cdot m \cdot n \cdot s^2$ elements, but also here the non-zero elements are a minority, at most $4mns^2$.

Fig. 4 shows the logarithmic plot of the above values for the range $s = 2$ to $m = n = 4 \sim 22$. For $m = n = 22$ the entire matrix contains nearly one million elements, with less than 2% non-zero ones, there being about 50,000 elements in the half-band, with a mere 15% non-zero ones.

Because of the high number of elements, the main store of the computer is insufficient even for rather modest problems, hence external store (magnetic

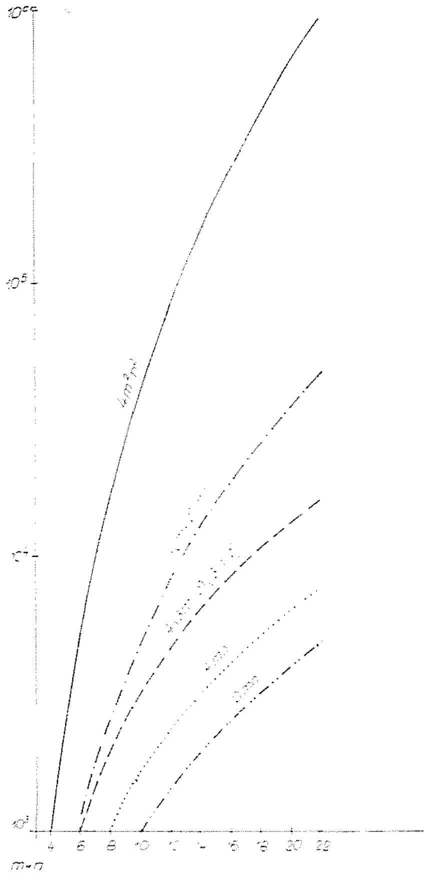


Fig. 4

drum, magnetic tape) has to be applied, inconvenient because for external store the frequent input exchange much increases running time.

The proposed method demands a mere $\sim 5mns$, thus it is accessible to rather small computers.

3.2 *Some matrix relationships*

Without entering into details, some less known matrix relationships will briefly be presented, described with all particulars by e.g. MacDuffee [3].

Direct product of two matrices is defined by the identities:

$$\mathbf{A} \cdot \times \mathbf{B} \equiv [\mathbf{A}\, b_{ij}] \qquad \text{and} \qquad \mathbf{A} \times \cdot\, \mathbf{B} \equiv [a_{ij}\, \mathbf{B}]\,. \tag{3.5}$$

From definition (3.5) is is easy to verify the following identity:

$$(\mathbf{A}_1 \cdot \times \mathbf{B}_1)(\mathbf{A}_2 \cdot \times \mathbf{B}_2) \equiv (\mathbf{A}_1 \mathbf{A}_2) \cdot \times (\mathbf{B}_1 \mathbf{B}_2).$$

Introducing symbol Π of the direct product defined by the identity

$$\prod_{i=1}^{n} \mathbf{A}_i \equiv \mathbf{A}_1 \cdot\times \mathbf{A}_2 \cdot\times \ \ldots \ \cdot\times \mathbf{A}_i \cdot\times \ \ldots \ \cdot\times \mathbf{A}_n$$

the identity

$$\prod_{i=1}^{n} (\mathbf{A}_i \mathbf{B}_i \mathbf{C}_i) \equiv \left(\prod_{i=1}^{n} \mathbf{A}_i \right) \left(\prod_{i=1}^{n} \mathbf{B}_i \right) \left(\prod_{i=1}^{n} \mathbf{C}_i \right) \tag{3.7}$$

that will be later of importance, can be proved by mathematical induction.

Applying direct multiplication, the hypermatrix $\widetilde{\mathbb{K}}$, resembling to the hypermatrix $\mathbb{K}$, and subject to stipulations

$$\begin{aligned} \mathbf{a} &= \mathbf{a}_{00} \\ \mathbf{b} &= \mathbf{b}^* = \mathbf{a}_{10} \\ \mathbf{c} &= \mathbf{c}^* = \mathbf{a}_{01} \\ \mathbf{d} &= \mathbf{d}^* = \mathbf{a}_{11} \end{aligned} \tag{3.8}$$

can be written as a direct polynomial:

$$\begin{aligned} \widetilde{\mathbb{K}} = \mathbf{a}_{00} \cdot\times \mathbb{E}_m \cdot\times \mathbb{E}_n + \mathbf{a}_{10} \cdot\times \mathbb{B}_m \cdot\times \mathbb{E}_n + \mathbf{a}_{01} \cdot\times \mathbb{E}_m \cdot\times \mathbb{B}_n + \\ + \mathbf{a}_{11} \cdot\times \mathbb{B}_m \cdot\times \mathbb{B}_n , \end{aligned} \tag{3.9}$$

where $\mathbb{E}_m$ and $\mathbb{E}_n$ are unit matrices of m and n order, respectively, and $\mathbb{B}_m$ denotes the uniform continuant matrix of order m.

$$\mathbb{B}_m = \begin{array}{c} \\ \\ \\ \\ \\ m \end{array} \begin{bmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & 1 \\ & & & & 1 & 0 \end{bmatrix} \begin{array}{c} m \\ \\ \\ \\ \\ \\ \end{array} . \tag{3.10}$$

In view of the fact that the zero-th power of any (square) matrix is the unit matrix of corresponding order:

$$\mathbb{K} = \sum_{j=0}^{1} \sum_{i=0}^{1} \mathbf{a}_{ij} \cdot\times \mathbb{B}_m^i \cdot\times \mathbb{B}_n^j . \tag{3.11}$$

3.3 Quasi-spectral decomposition of a direct polynomial

Let us solve the problem for the general case first. Let the spectral decompositions of matrices $\mathbf{A}_1, \mathbf{A}_2 \ldots \mathbf{A}_i \ldots \mathbf{A}_\varrho$ of simple structure and of $n_1, n_2 \ldots n_i \ldots n_\varrho$ order in the form:

$$\mathbf{A}_i = \mathbf{U}_i \Lambda_i \mathbf{U}_i^{-1}. \tag{3.12}$$

Theorem: The so-called quasi-spectral decomposition of the direct polynomial

$$\mathbf{N} = \sum_{\mu_\varrho=0}^{m_\varrho} \cdots \sum_{\mu_2=0}^{m_2} \sum_{\mu_1=0}^{m_1} \mathbf{C}_{\mu_1 \mu_2 \ldots \mu_\varrho} \cdot \times \mathbf{A}_1^{\mu_1} \cdot \times \mathbf{A}_2^{\mu_2} \cdot \times \ldots \cdot \times \mathbf{A}_\varrho^{\mu_\varrho} \tag{3.13}$$

of the order

$$n = \prod_{i=0}^{\varrho} n_i$$

is given by the formula:

$$\mathbf{N} = \left(\mathbf{E}_{n_0} \cdot \times \prod_{i=1}^{\varrho} \mathbf{U}_i\right) \left\{\sum_{\mu_\varrho=0}^{m} \cdots \sum_{\mu_2=0}^{m_2} \sum_{\mu_1=0}^{m_1} \mathbf{C}_{\mu_1 \mu^2 \ldots \mu_\varrho} \cdot \times \prod_{i=1}^{\varrho} \Lambda_i^{\mu_\varrho}\right\}. \tag{3.14}$$

$$\left(\mathbf{E}_{n_0} \cdot \times \prod_{i=1}^{\varrho} \mathbf{U}_i^{-1}\right) = \mathbf{V}\mathbf{\Gamma}\mathbf{V}^{-1}. \tag{3.15}$$

In the following the matrices

$$\left.\begin{aligned} \mathbf{V} &= \mathbf{E}_{n_0} \cdot \times \prod_{i=1}^{\varrho} \mathbf{U}_i \\ \text{and} \qquad \mathbf{\Gamma} &= \sum_{\mu_\varrho=0}^{m_\varrho} \cdots \sum_{\mu_2=0}^{m_2} \sum_{\mu_1=0}^{m_1} \mathbf{C}_{\mu_1\mu_2 \ldots M_\varrho} \cdot \times \prod_{i=1}^{\varrho} \Lambda_i^{\mu} \end{aligned}\right\} \tag{3.16}$$

will be called quasi-modal and quasi-spectrum. respectively.

Proof: Consider a term $\mathbf{T}_{\mu_1\mu_2\cdots\mu_\varrho}$ of the direct polynomial (3.13) belonging to settled $\mu_1, \mu_2, \ldots \mu_\varrho$ values and substitute the spectral form of $\mathbf{A}_i$ matrices as well as the identity

$$\mathbf{C}_{\mu_1\mu_2 \ldots \mu_\varrho} = \mathbf{E}_{n_0} \mathbf{C}_{\mu_1\mu_2 \ldots \mu_\varrho} \mathbf{E}_{n_0} \tag{3.17}$$

to yield:

$$\mathbf{T}_{\mu_1\mu_2 \ldots \mu_\varrho} = (\mathbf{E}_{n_0} \mathbf{C}_{\mu_1\mu_2 \ldots \mu_\varrho} \mathbf{E}_{n_0}) \cdot \times \prod_{i=1}^{\varrho} \mathbf{U}_i \Lambda_i^{\mu_i} \mathbf{U}_i^{-1}. \tag{3.18}$$

From the identity (3.7):

$$\mathbf{T}_{\mu_1\mu_2\ldots\mu_\varrho} = \left(\mathbf{E}_{n_0}\cdot\times\prod_{i=\mu}^{\varrho}\mathbf{U}_i\right)\left(\mathbf{C}_{\mu_1\mu_2\ldots\mu_\varrho}\cdot\times\prod_{i=1}^{\varrho}\mathbf{\Lambda}_i^{\mu_i}\right)\left(\mathbf{E}_{n_0}\cdot\times\prod_{i=1}^{\varrho}\mathbf{U}_i^{-1}\right). \quad (3.19)$$

Summing up and factoring out the first and the last term in brackets (occurring in all terms of the sum):

$$\mathbf{N} = \left(\mathbf{E}_{n_0}\cdot\times\prod_{i=1}^{\varrho}\mathbf{U}_i\right)\left\{\sum_{\mu_\varrho=0}^{m_\varrho}\cdots\sum_{\mu_2=0}^{m_2}\sum_{\mu_1=0}^{m_1}\mathbf{C}_{\mu_1\mu_2\ldots\mu_\varrho}\cdot\times\prod_{i=1}^{\varrho}\mathbf{\Lambda}_i^{\mu}\right\}\left(\mathbf{E}_{n_0}\cdot\times\prod_{i=1}^{\varrho}\mathbf{U}_i^{-1}\right).$$

Q.E.D.

Note that the proved theorem can be considered a generalization of a theorem by EGERVÁRY [1]. It should be stressed that proof of the theorem had the only restriction for the coefficient matrices $\mathbf{C}_{\mu_1\mu_2\cdots\mu_\varrho}$ to be regular and the blocks were not required to be commutable.

In the special case of the general theorem above where the direct polynomial has scalars $c_{\mu_1\mu_2\cdots\mu_\varrho}$ as coefficients, the spectral decomposition of the hypermatrix of $n = \prod_{i=1}^{\varrho} n_i$ order

$$\mathbf{M} = \sum_{\mu_\varrho=0}^{m_\varrho}\cdots\sum_{\mu_2=0}^{m_2}\sum_{\mu_1=0}^{m_1} c_{\mu_1\mu_2\ldots\mu_\varrho}\prod_{i=1}^{\varrho}\mathbf{A}_i^{\mu_i} \quad (3.20)$$

is delivered by the identity

$$\mathbf{M} = \left(\prod_{i=1}^{\varrho}\mathbf{U}_i\right)\left\{\sum_{\mu_\varrho=0}^{m_\varrho}\cdots\sum_{\mu_2=0}^{m_2}\sum_{\mu_1=0}^{m_1} c_{\mu_1\mu_2\ldots\mu_\varrho}\prod_{i=1}^{\varrho}\mathbf{\Lambda}_i^{\mu_i}\right\}\cdot\left(\prod_{i=1}^{\varrho}\mathbf{U}_i^{-1}\right)\mathbf{V}\mathbf{\Gamma}\mathbf{V}^{-1} \quad (3.21)$$

where modal matrix $\mathbf{V}$ is the direct product of modal matrices $\mathbf{U}_i$, and spectrum $\mathbf{\Gamma}$ is a diagonal matrix with polynomials of matrices $\mathbf{A}_i$ as elements.

3.4 Solution of a matrix equation with a direct polynomial coefficient

Let us consider the equation

$$\mathbf{N}\mathbf{x} = \mathbf{y} \quad (3.22)$$

where $\mathbf{N}$ is a direct polynomial according to (3.13), $\mathbf{x}$ and $\mathbf{y}$ are vectors of n order.

Also vectors $\mathbf{x}$ and $\mathbf{y}$ can be obtained as direct polynomials:

$$\left.\begin{aligned} \mathbf{x} &= \sum_{\nu_\varrho=1}^{n_\varrho} \cdots \sum_{\nu_2=1}^{n_2} \sum_{\nu_1=1}^{n_1} \mathbf{x}_{\nu_1\nu_2\ldots\nu_\varrho} \cdot\times \prod_{i=1}^{\varrho} \mathbf{e}_{\nu_i} \\ \mathbf{y} &= \sum_{\nu_\varrho=1}^{n_\varrho} \cdots \sum_{\nu_2=1}^{n_2} \sum_{\nu_1=1}^{n_1} \mathbf{y}_{\nu_1\nu_2\ldots\nu_\varrho} \cdot\times \prod_{i=1}^{\varrho} \mathbf{e}_{\nu_i} \end{aligned}\right\} \tag{3.23}$$

where $\mathbf{x}_{\nu_1\nu_2\ldots\nu_\varrho}$ and $\mathbf{y}_{\nu_1\nu_2\ldots\nu_\varrho}$ are vectors of n_0 order;
$\mathbf{e}_{\nu_i}$ is the ν_i-th unit vector of n_i order;
$\prod_{i=1}^{\varrho} \mathbf{e}_{\nu_i}$ are unit vectors of n/n_0 order.

Let us introduce the following subscript convention:

for
$$\mathbf{A} = \prod_{i=1}^{\varrho} \mathbf{A}_i ,$$
where
$$\mathbf{A}_i = [a_{i\,r_i,s_i}] ,$$

an arbitrary element of $\mathbf{A}$ is:

$$a_{r,s} = a_{r_1,r_2\ldots r_\varrho;\, s_1s_2\ldots s_\varrho} = \prod_{i=1}^{\varrho} a_{i,r_i,s_i} \tag{3.24}$$

and an arbitrary element of $\mathbf{B} = \mathbf{C} \cdot\times \mathbf{A}$ is:

$$b_{r_0r_1r_2\ldots r_\varrho;\, s_0s_1s_2\ldots s_\varrho} = c_{r_0s_0}\, a_{1r_1s_1}\, a_{2r_2s_2} \ldots a_{\varrho r_\varrho s_\varrho} = c_{r_0s_0} \prod_{i=1}^{\varrho} a_{i,r_i,s_i} \tag{3.25}$$

subscripts are related by:

$$\left.\begin{aligned} r - 1 &= \sum_{i=0}^{\varrho} \left\{ \left(\prod_{k=0}^{i-1} n_{k-1} \right) \left(r_i - 1 \right) \right\} \\ s - 1 &= \sum_{i=0}^{\varrho} \left\{ \left(\prod_{k=0}^{i-1} n_{k-1} \right) \left(s_i - 1 \right) \right\} \end{aligned}\right\} \tag{3.26}$$

where $n_{-1} = 1$ by definition.
Introducing a similar subscript convention for vectors $\mathbf{x}$ and $\mathbf{y}$

$$x_\nu = x_{\nu_0\nu_1\nu_2\ldots\nu_\varrho} \tag{3.27}$$

where

$$\nu - 1 = \sum_{i=0}^{\varrho} \left\{ \left(\prod_{k=0}^{i-1} n_{k-1} \right) \left(\nu_i - 1 \right) \right\}. \tag{3.28}$$

Replacing expressions for N, x and y into (3.22) we obtain

$$\left[\mathbf{E}_{n_0} \cdot \times \prod_{i=1}^{\varrho} \mathbf{U}_i\right] \left\{\sum_{\mu_\varrho=0}^{m_\varrho} \cdots \sum_{\mu_2=0}^{m_2} \sum_{\mu_1=0}^{m_1} \mathbf{C}_{\mu_1\mu_2\ldots\mu_\varrho} \cdot \times \prod_{i=1}^{\varrho} \Lambda_i^{\mu_i}\right\} \cdot$$

$$\cdot \left[\mathbf{E}_{n_0} \cdot \times \prod_{i=1}^{\varrho} \mathbf{U}_i^{-1}\right] \left\{\sum_{\nu_\varrho=1}^{n_\varrho} \cdots \sum_{\nu_2=1}^{n_2} \sum_{\nu_1=1}^{n_1} \mathbf{x}_{\nu_1\nu_2\ldots\nu_\varrho} \cdot \times \prod_{i=1}^{\varrho} \mathbf{e}_{\nu_i}\right\} =$$

$$= \sum_{\nu_\varrho=1}^{n_\varrho} \cdots \sum_{\nu_2=1}^{n_2} \sum_{\mu_1=1}^{n_1} \mathbf{y}_{\nu_1\nu_2\ldots\nu_\varrho} \cdot \times \prod_{i=1}^{\varrho} \mathbf{e}_{\nu_i} . \tag{3.29}$$

Let us examine first matrix product:

$$\left[\mathbf{E}_{n_0} \cdot \times \prod_{i=1}^{\varrho} \mathbf{U}_i\right] \left[\mathbf{E}_{n_0} \cdot \times \prod_{i=1}^{\varrho} \mathbf{U}_i^{-1}\right] .$$

In view of identity (3.7), it is obvious that:

$$\left[\mathbf{E}_{n_0} \cdot \times \prod_{i=1}^{\varrho} \mathbf{U}_i\right] \left[\mathbf{E}_{n_0} \cdot \times \prod_{i=1}^{\varrho} \mathbf{U}_i^{-1}\right] = \mathbf{E}_{n_0} \mathbf{E}_{n_0} \cdot \times \prod_{i=1}^{\varrho} (\mathbf{U}_i \mathbf{U}_i^{-1}) = \prod_{i=0}^{\varrho} \mathbf{E}_{n_i} = \mathbf{E}_n . \tag{3.30}$$

According to this relationship the matrix equation has the solution:

$$\mathbf{x} = \mathbf{N}^{-1} \mathbf{y} .$$

or, in detail

$$\sum_{\nu_\varrho=1}^{n_\varrho} \cdots \sum_{\nu_2=1}^{n_2} \sum_{\nu_1=1}^{n_1} \mathbf{x}_{\nu_1\nu_2\ldots\nu_\varrho} \cdot \times \prod_{i=1}^{\varrho} \mathbf{e}_{\nu_i} =$$

$$= \underbrace{\left[\mathbf{E}_{n_0} \cdot \times \prod_{i=1}^{\varrho} \mathbf{U}_i\right]}_{1} \underbrace{\left\{\sum_{\mu_\varrho=0}^{m_\varrho} \cdots \sum_{\mu_2=0}^{m_2} \sum_{\nu_1=0}^{m_1} \mathbf{c}_{\mu_1\mu_2\ldots\mu_\varrho} \cdot \times \prod_{i=1}^{\varrho} \Lambda_i^{\mu_i}\right\}^{-1}}_{2}$$

$$\underbrace{\left[\mathbf{E}_{n_0} \cdot \times \prod_{i=1}^{\varrho} \mathbf{U}_i^{-1}\right]}_{3} \underbrace{\left(\sum_{\nu_\varrho=1}^{n_\varrho} \cdots \sum_{\nu_2=1}^{n_2} \sum_{\nu_1=1}^{n_2} \mathbf{y}_{\nu_1\nu_2\ldots\nu_\varrho} \cdot \times \prod_{\nu_2=1}^{\varrho} \mathbf{e}_{\nu_i}\right)}_{4} . \tag{3.31}$$

Let us examine the product of the four factors. Let us consider first the product of factors 3 and 4 of n order, denoted by $\mathbf{g}$:

$$\mathbf{g} = \left[\mathbf{E}_{n_0} \cdot \times \prod_{i=1}^{\varrho} \mathbf{U}_i^{-1}\right] \left\{\sum_{\nu_\varrho=1}^{n_\varrho} \cdots \sum_{\nu_2=1}^{n_2} \sum_{\nu_1=1}^{n_1} \mathbf{y}_{\nu_1\nu_2\ldots\nu_\varrho} \cdot \times \prod_{i=1}^{\varrho} \mathbf{e}_{\nu_i}\right\} . \tag{3.32}$$

Putting factor ③ under the sign of summation, and taking relationship (3.7) into account, we may write:

$$\mathbf{g} = \sum_{\nu_\varrho=1}^{n_\varrho} \cdots \sum_{\nu_2=1}^{n_2} \sum_{\nu_1=1}^{n_1} \mathbf{y}_{\nu_1\nu_2\ldots\nu_\varrho} \cdot \times \prod_{i=1}^{\varrho} \mathbf{U}_i^{-1} \mathbf{e}_{\nu_i} \tag{3.33}$$

Denoting the ν_i-th column vector of the inverse of the i-th modal matrix by $\mathbf{U}_i^{-1}$ i.e.:

$$\mathbf{U}_i^{-1} \mathbf{e}_{\nu_i} = \mathbf{u}_{i,\nu_i}^{(-1)} \tag{3.34}$$

we obtain

$$\mathbf{g} = \sum_{\nu_\varrho=1}^{n_\varrho} \cdots \sum_{\nu_2=1}^{n_2} \sum_{\nu_1=1}^{n_1} \mathbf{y}_{\nu_1\nu_2\ldots\nu_\varrho} \cdot \times \prod_{i=1}^{\varrho} \mathbf{u}_{i,\nu_i}^{(-1)}. \tag{3.35}$$

A term $\mathbf{t}^{\nu_1\nu_2\ldots\nu_\varrho}$ of the above summation, belonging to fixed $\nu_1, \nu_2 \ldots \nu_\varrho$ values:

$$\mathbf{t}^{\nu_1\nu_2\ldots\nu_\varrho} = \mathbf{y}_{\nu_1\nu_2\ldots\nu_\varrho} \cdot \times \prod_{i=1}^{\varrho} \mathbf{u}_{i,\nu_i}^{(-1)}. \tag{3.36}$$

A vector block of $\mathbf{t}^{\nu_1\nu_2\ldots\nu_\varrho}$ is as follows

$$\mathbf{t}_r^{\nu_1\nu_2\ldots\nu_\varrho} = \mathbf{t}_{rr_1r_2\ldots r_\varrho}^{\nu_1\nu_2\ldots\nu_\varrho} = \mathbf{y}_{\nu_1\nu_2\ldots\nu_\varrho} \prod_{i=1} \mathbf{u}_{i,r_i,\nu_i}^{(-1)}. \tag{3.37}$$

After summation, an elementary vector $\mathbf{g}_r$ of $\mathbf{g}$ is:

$$\mathbf{g}_r = \mathbf{g}_{r_1r_2\ldots r} = \sum_{\nu_\varrho=1}^{n_\varrho} \cdots \sum_{\nu_2=1}^{n_2} \sum_{\nu_1=1}^{n_1} \mathbf{y}_{\nu_1\nu_2\ldots\nu_\varrho} \prod_{i=1}^{\varrho} \mathbf{u}_{i,r_i,\nu_i}^{(-1)} \tag{3.38}$$

Considering vectors $\mathbf{g}$ and $\mathbf{y}$ of dimension n as tensors $\mathbb{G}$ and $\mathbb{Y}$ of $\varrho + 1$ order in a Cartesian system, matrices $\mathbf{U}_i^{-1}$ as tensors of second order $\mathbb{U}_i^{-1}$, and their direct product as tensors of order 2ϱ, relationship (3.38) can also be interpreted so that tensor $\mathbb{G} = [g_r]$ is the transformation of tensor $\mathbb{Y} = [y_r]$ with respect to tensors $\mathbb{U}_i^{-1}$ or, by other words, the contraction of 4 tensors with tensors $\mathbf{U}_i^{-1}$, with respect to the second subscript. With tensorial notation:

$$\mathbf{g}_{r_i} = \mathbf{y}_{\nu_i} \, u_{i,r_i\nu_i}^{(-1)} \tag{3.39}$$

or

$$\mathbb{G} = \mathbb{Y} \times \mathbf{U}^{-1} \tag{3.40}$$

where $\times$ is the symbol of contraction.

Returning to Eq. (3.31), let us consider factor ②:

$$\mathbf{\Gamma}^{-1} = \qquad \ldots \sum_{\mu_2=0}^{m_2} \sum_{\mu_1=0}^{m_1} \mathbf{C}_{\mu_1\mu_2\ldots\mu_\varrho} \prod_{i=1}^{\varrho} \mathbf{\Lambda}_i^{\mu_i} \Bigg\}^{-1}. \tag{3.41}$$

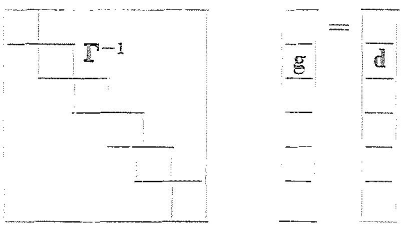
In this expression all factors of the direct product are diagonal matrices, thus, the whole term in brackets will be a hyperdiagonal matrix, with blocks of order n:

$$\gamma_{\beta_1\beta_2\ldots\beta_\varrho} = \sum_{\mu_\varrho=0}^{m_\varrho}\cdots\sum_{\mu_2=0}^{m_2}\sum_{\mu_1=0}^{m_1}\mathbf{C}_{\mu_1\mu_2\ldots\mu_\varrho}\prod_{i=0}^{\varrho}\lambda_{i,\beta_i}^{\mu_i}. \tag{3.42}$$

Since the inverse of a hyperdiagonal matrix is also a hyperdiagonal one and blocks of the inverse are the inverses of its blocks:

$$\gamma_{\beta_1\beta_2\ldots\beta_\varrho}^{-1} = \left(\sum_{\mu_\varrho=0}^{m_\varrho}\cdots\sum_{\mu_2=0}^{m_2}\sum_{\mu_1=0}^{m_1}\mathbf{C}_{\mu_1\mu_2\ldots\mu_\varrho}\prod_{i=1}^{\varrho}\lambda_{i,\beta_i}^{\mu_i}\right)^{-1} \tag{3.43}$$

From this expression it is obvious that this procedure is only valid if of the polynomials with matrix coefficients $\mathbf{C}_{\mu_1\mu_2\ldots\mu_\varrho}$, eigenvalues of matrices $\mathbf{A}_i$ are regular. The product of hyperdiagonal matrix $\mathbf{\Gamma}^{-1}$ by vector $\mathbf{g}$ can be illustrated schematically as:



Apparently:

$$\mathbf{d}_{\beta_i} = \gamma_{\beta_i}^{-1}\,\mathbf{g}_{\beta_i} = \left(\sum_{\mu_\varrho=0}^{m_\varrho}\cdots\sum_{\mu_2=0}^{m_2}\sum_{\mu_1=0}^{m_1}\mathbf{C}_{\mu_1\mu_2\ldots\mu_\varrho}\prod_{i=1}^{\varrho}\lambda_{i,\beta_i}^{\mu_i}\right)^{-1}\mathbf{g}_{\beta_1\beta_2\ldots\beta_\varrho} \tag{3.44}$$

where $\mathbf{d}_{\beta_i}$ and $\mathbf{g}_{\beta_i}$ are vectors of dimension n_0 and $\gamma_{\beta_i}^{-1}$ is a matrix of order n_0.

With tensorial notation, $\mathbf{\Gamma}$ and $\mathbf{\Gamma}^{-1}$ are tensors of order $\varrho + 2$:

$$\mathbf{\Gamma}^{-1}\left[\gamma^{(-1)}_{s_0r_0\beta_1\beta_2\ldots\beta_\varrho}\right],$$

$\mathbb{D}$ and $\mathbb{G}$ are tensors of order $\varrho + 1$:

$$\mathbb{D} = [d_{s_0\beta_1\beta_2\ldots\beta_\varrho}]\,, \qquad \mathbb{G} = [g_{r_0s_1s_2\ldots\beta_\varrho}]\,,$$

relationship (3.44) is by tensorial notation:

$$\mathbb{D} = \mathbf{\Gamma}^{-1} \odot \mathbb{G} \tag{3.45}$$

where $\odot$ is symbol of the so-called logical multiplication defined as:

$$\mathbf{d}_{s_0\beta_1\beta_2\ldots\beta_\varrho} = \sum_{r_0=1}^{n_0}\gamma^{-1}_{s_0r_0\beta_1\beta_2\ldots\beta_\varrho}\,\mathbf{g}_{r_0\beta_1\beta_2\ldots\beta_\varrho}. \tag{3.46}$$

Nothing but the previously discussed operation of multiplying by a quasi-modal matrix is left to solve Eq. (3.22). By tensorial notation:

$$\mathbb{X} = \mathbb{U} \times \{\mathbb{\Gamma}^{-1} \odot (\mathbb{U}^{-1} \times \mathbb{Y})\}. \tag{3.47}$$

3.5 Special cases

3.51 *Solution of the Poisson differential equation by the method of finite differences.* Both for biharmonical and Poisson equations the method of finite differences leads to a matrix equation with a direct polynomial coefficient of scalar coefficient, e.g. to the Poisson equation of the form:

$$\mathbb{M}\,\mathbf{w} = \left(\sum_{\mu_2=1}^{1} \sum_{\mu_1=1}^{1} a_{\mu_1\mu_2}\, \mathbb{B}_m^{\mu_1} \cdot \times \mathbb{B}_n^{\mu_2}\right) \mathbf{w} = \mathbf{p} \tag{3.48}$$

where: $a_{00} = +4$
$a_{10} = a_{01} = -1$
$a_{11} = 0$

Solution with tensorial notation will be:

$$\mathbb{W} = \mathbb{U} \times \{\mathbb{\Gamma}^{-1} \odot (\mathbb{U}^{-1} \times \mathbb{P})\}. \tag{3.49}$$

Innermost contraction:

$$g_{r_1,r_2} = \sum_{\nu_1=1}^{m} \sum_{\nu_2=1}^{n} \mathbb{U}^{(-1)}_{m,r_1\beta_1}\, \mathbb{U}^{(-1)}_{n,r_2\beta_2}\, p_{\nu_1\nu_2} \tag{3.50}$$

considering vectors $\mathbf{p}$ and $\mathbf{g}$ as matrices $\mathbb{G} = [g_{r_1r_2}]$ and $\mathbb{P} = [p_{\nu_1\nu_2}]$, taking into consideration that since $\mathbb{B}$ is a symmetrical matrix, its spectral form is $\mathbb{B} = \mathbb{U}\ \mathbb{L}\ \mathbb{U}^*$

$$\mathbb{G} = \mathbb{U}_m \mathbb{P}\, \mathbb{U}_n. \tag{3.51}$$

Now $\mathbb{\Gamma}^{-1}$ is a diagonal matrix:

$$\mathbb{\Gamma}^{-1} = \left\langle \frac{1}{a_{00} + a_{10}\,\lambda_{mi} + a_{01}\,\lambda_{nj}} \right\rangle = \left\langle \frac{1}{-\lambda_{mi} - \lambda_{nj} + 4} \right\rangle,$$

this again can be considered as a two-dimensional matrix:

$$\tilde{\mathbb{\Gamma}}^{-1} = [\tilde{\gamma}^{(-1)}_{i,j}] = \left[\frac{1}{-\lambda_{mi} - \lambda_{nj} + 4} \right].$$

and now the logical multiplication will consist in multiplying elements with appropriate subscripts by each other:

$$\mathbf{D} = \widetilde{\mathbf{\Gamma}}^{-1} \odot (\mathbf{U}_m \mathbf{P} \mathbf{U}_n). \tag{3.52}$$

Contracting again yields:

$$\mathbf{W} = \mathbf{U}_m \{\widetilde{\mathbf{\Gamma}}^{-1} \odot (\mathbf{U}_m \mathbf{P} \mathbf{U}_n)\} \mathbf{U}_n \tag{3.53}$$

result analogous to the matrix equation method developed by SZABÓ [2] for the difference method for the solution of partial differential equations of even order.

This justifies the statement that this method can be considered a generalization of the matrix equation method.

3.52. *The finite element method, the disc problem.* Analysis of rectangular discs with rigidly clamped edges by the finite element method leads to matrix equation (3.1), where the structure of matrix $\mathbb{K}$ is found in (3.3) and (3.4). Matrix $\mathbb{K}$ differs but slightly from matrix $\widetilde{\mathbb{K}}$ defined by (3.11), therefore now only the hypermatrix equation

$$\widetilde{\mathbb{K}}\, \mathbf{w} = \mathbf{f} \tag{3.54}$$

will be discussed. Iteration can be applied to take into account the deviation and the deviation excess due to accidental variations of the boundary conditions, to be reconsidered in item 3.6.

Remind that $\widetilde{\mathbb{K}}$ is a hypermatrix of $m \cdot n$ block rows and block columns, with a structure expressed by the relationship:

$$\widetilde{\mathbb{K}} \equiv \sum_{\mu_2=0}^{1} \sum_{\mu_1=0}^{1} \mathbf{a}_{\mu_1\mu_2} \cdot \times \mathbb{B}_m^{\mu_1} \cdot \times \mathbb{B}_n^{\mu_2} \tag{3.55}$$

$\mathbf{a}_{\mu_1\mu_2}$ is a block of second order, while relationships $\mathbb{B}_m$ and $\mathbb{B}_n$ are simple continuous matrices of m and n order, respectively.

Vectors $\qquad \mathbf{w} = [w_{i,j,k}]$ and $[\mathbf{f} = f_{i,j,k}]$

contain displacement and external force components of disc nodes, subscripts indicating rows, columns of the point and the direction x or y in this order. Thus, vectors $\mathbf{w}$ and $\mathbf{f}$ can be considered three-dimensional matrices (blocks) (of the type $m \cdot n \cdot 2$). Now, the procedure is the same as in item 3.51, to yield:

$$\mathbf{w} = \mathbf{U}_m \{\mathbf{\Gamma}^{-1} \odot (\mathbf{U}_m \mathbf{F} \mathbf{U}_n)\} \mathbf{U}_n. \tag{3.56}$$

When interpreting Eq. (3.56), remind that:

— matrix multiplication of a three-dimensional block from left and right is defined by respective expressions

$$\mathbf{G} = \mathbf{U}\ \mathbf{F}; \qquad g_{ijk} = \sum_{l=1}^{m} u_{il} f_{ljk} \tag{3.57}$$

$$\mathbf{G} = \mathbf{F}\mathbf{U}; \qquad g_{ijk} = \sum_{l=1}^{m} f_{ilk}\, u_{lj} \tag{3.58}$$

that is, any layer of matrix $\mathbf{F}$ is to be multiplied by U:

— $\mathbf{\Gamma}^{-1}$ is defined as: $$\gamma_{i,j}^{-1} = \left(\sum_{\mu_2=0}^{1} \sum_{\mu_1=0}^{1} \mathbf{a}_{\mu_1\mu_2}\, \lambda_{mi}^{\mu_1}\, \lambda_{ni}^{\mu_2}\right)^{-1} \tag{3.59}$$

where γ_{ij}^{-1} is a two-dimensional matrix. If all matrices $\mathbf{a}_{\mu_1\mu_2}$ are diagonal matrices, then also $\lambda_{i,j}^{-1}$ will be a diagonal matrix.

— the logical multiplication $\mathbf{D} = \mathbf{\Gamma}^{-1} \odot \mathbf{G}$:

a) if γ^{-1} is a matrix

$$d_{ijk} = \sum_{l=1}^{2} \gamma_{i,j,k,l}^{(-1)}\, g_{i,j,l} \tag{3.60}$$

b) if γ^{-1} is a diagonal matrix, then also $\mathbf{\Gamma}^{-1}$ can be considered a three-dimensional block, and the logical multiplication can be interpreted as the product of elements of both blocks with the same subscripts.

3.6 Solution of the hypermatrix equation by iteration

As it was seen in 3.1 and 3.2 in case of rectangular domain and rigidly clamped edge, the stiffness matrix of the finite element method is a matrix $\mathbf{K}$ close to the direct polynomial $\tilde{\mathbf{K}}$. If boundary conditions or eventually the shape of domain vary, the stiffness matrix will differ by more from the direct polynomial $\mathbf{K}$. Therefore the equation system of the finite element method lends itself to iteration. Let us see now the convergence condition of the iteration

$$\mathbf{K}\,\mathbf{x} = \mathbf{y} \tag{3.61}$$

where $\mathbf{K}$ differs from the known (quasi) spectral-decomposed matrix $\mathbf{N}$ only by a matrix $\mathbf{F}$, so that:

$$\mathbf{K} = \mathbf{N} - \mathbf{F}. \tag{3.62}$$

Substituting and solving for $\mathbf{x}$:

$$\mathbf{x} = \mathbb{N}^{-1}(\mathbf{F}\mathbf{x} + \mathbf{y}), \tag{3.63}$$

expression readily iterated in form:

$$\mathbf{x}_{n+1} = \mathbb{N}^{-1}(\mathbf{F}\,\mathbf{x}_n + \mathbf{y}) \tag{3.64}$$

Obviously, since two subsequent iterations are related by the constant matrix

$$\mathbf{H} = \mathbb{N}^{-1}\,\mathbf{F} \tag{3.65}$$

convergence of the iteration has as condition:

$$\|\mathbf{F}\| < \|\mathbb{N}\|, \tag{3.66}$$

with $\|\mathbf{F}\|$ norm of matrix $\mathbf{F}$.

Since the proposed method has the advantage of not to establish the large-size coefficient matrix but only some factors of the direct polynomial, and considering that blocks of the coefficient matrix are combinations of the blocks of the elementary stiffness matrix, two rather rigorous criteria have been proved for the convergence, which we can, however, easily handle in our case.

Provided blocks of matrices $\mathbb{N}$ and $\mathbb{F}$ are known, a sufficient condition of the convergence is the inequality

$$\|\mathbf{F}_{ij}\| < \|\mathbb{N}_{ij}\| \tag{3.67}$$

to be valid for each block of identical subscript.

Provided hypermatrices $\mathbb{N}$ and $\mathbb{F}$ are direct polynomials of the same structure, i.e. they only differ by the coefficients $\mathbf{a}_{ij}$ and $\mathbf{a}'_{ij}$, a sufficient condition of the convergence is the inequality

$$\|\mathbf{a}'_{ij}\| < \|\mathbf{a}_{ij}\| \tag{3.68}$$

to be valid for each pair of coefficient blocks (where $\mathbf{a}_{ij}$ and $\mathbf{a}'_{ij}$ are coefficients of direct polynomials $\mathbb{N}$ and $\mathbb{F}$, respectively).

4. Conclusions

Last but not least, one may wonder why to apply spectral decomposition, a complex and tedious procedure, and besides iteration, instead of directly solving the matrix equation?

Stiffness matrices for the finite elements were seen in item 3 to be rather large-size ones. Among their elements and blocks, however, there is an obvious majority of zero blocks and zero elements, a percentage further growing with increasing sizes (and refined divisions). As a conclusion, storage of the entire matrix, and conventional solution of the equation system, is almost impossible but at least very lengthy a procedure even for the most up-to-date computers. Let us consider a disc problem of 20 by 20 divisions. The coefficient matrix measures $2 \cdot 20 \cdot 20 = 800$, its elements amounting to 640 000. A single solution of the equation system by Gaussian algorithm requires $\sim \frac{n^3}{3} = 17 \cdot 10^7$ operations of multiplication and division, without mentioning the external storage, needed because of the matrix size, much increasing the running time.

Current methods requiring to store but the upper non-zero band still mean in our case to store $800 \cdot (2+20) \cdot 2 = 35\,200$ elements, and according to BERÉNYI [8], there will be 167 000 operations for the first, and 67 000 for any subsequent solution.

The method proposed here has two advantages:

1. Reduction of occupied storage capacity, storage involving:

— matrices $\mathbf{U}_i$: occupied storage place: $\sum_{i=1}^{\varrho} n_i^2$

— diagonal matrices $\mathbf{\Lambda}_i$: $\sum_{i=1}^{\varrho} n_i$

— vectors $\mathfrak{d}$ and $\mathfrak{p}$: $2s \,\Pi\, n_i$

— coefficient matrices a_{ij} (or $\mathbf{C}_{\mu_1\mu_2\ldots\mu_\varrho}$).

For the presented case this amounts to $800 + 40 + 1600 + 16 = 2456$. A few vector places are still needed for iteration, so not more than 5000 words are needed, available even in the main store of a small computer of MINSK—22 or GIER type.

2. Reduced running time. One step of iteration requiring in fact $4s$ multiplications between m by n matrices: this means in our case $8 \cdot 20^3 \sim$ $\sim 64\,000$ simple operations. For a rapid convergence, the process is equivalent or but slightly slower than the band matrix system.

One may ask why the time for the spectral decomposition is not accounted with the running time? It is because there exist simple trigonometric formulae for the spectral decomposition of the uniformly continuant matrix $\mathbf{B}$, appropriate to establish both modal and spectrum elements in some seconds (or fractions thereof). And here another significant advantage appears: modal matrix $\mathbf{U}$ of matrix $\mathbf{B}$ needs not be stored in full, since in knowledge of the first vector, the others can be obtained by simply changing the sign and the element.

If, however, the spectral decomposition of the factors of the direct polynomial is not available in closed form, the economy of the method needs a previous analysis. Anyhow, the method seems to be economical in cases where similar structures are to be designed for different loads, since then the work of spectral decomposition emerges only once.

Summary

After a short presentation of the finite element method, its use for discs and bending plates will be described. The stiffness matrix can often be written as a direct polynomial or in a rather similar form. So-called quasi-spectral decomposition of the direct polynomial is suggested for the matrix equation, correcting the deviation from the direct polynomial by iteration. The method is advantageous in that it suffices to produce and store a mere of 4—5 vectors rather than to produce the entire stiffness matrix so that it lends itself to the use of a computer of medium size.

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