ANALYSIS OF SKEW ANISOTROPIC PLATES BY THE SMALL PARAMETER METHOD

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1. Introduction

Bending problems of parallelogram anisotropic plates (to be referred to as skew anisotropic plates) are of importance first of all for skew-ribbed plates and skew grillages [4, 5, 6]. There are rather few studies on the analysis of skew anisotropic plates. Among them let us mention that by Suchar presenting a method of determining influence surfaces by means of polynomials [8]; Narouka applied the method of finite differences for a static load [7]; just as Mele [5], involving also the variation calculus.

In what follows, skew anisotropic plates under arbitrary static loads will be analyzed by the Sarkisyan small parameter method.

2. Fundamental relationships. Differential equation of skew anisotropic plates

The problem will be analyzed in a left-hand co-ordinate system where the xy-plane is coincident with the middle surface of the plate, and the z-axis is normal to it (Fig. 1).

Relationship between stress and strain components defined in this skew co-ordinate system is expressed by the matrix equation [6, 5]:

$$\boldsymbol{\sigma} = \mathbf{B}\boldsymbol{\epsilon} \tag{1}$$

where $\boldsymbol{\sigma} = \begin{bmatrix}\sigma_x \\ \sigma_y \\ \tau_{xy}\end{bmatrix}$ $\quad \mathbf{B} = [B_{ik}] \quad (B_{ik} = B_{ki}) \quad i = 1, 2, 3 \quad k = 1, 2, 3$ $\quad \boldsymbol{\epsilon} = \begin{bmatrix}\varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}\end{bmatrix}$

These expressions are formally identical to the well-known relationships written for an orthogonal co-ordinate system but the elements of $\boldsymbol{\sigma}$, $\boldsymbol{\epsilon}$ and $\mathbf{B}$ have different physical meanings [1, 6].

Differential equation of the skew anisotropic plate [5]:

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 4D_{13}\frac{\partial^4 w}{\partial x^3\,\partial y} + 2(D_{12} + 2D_{33})\frac{\partial^4 w}{\partial x^2\,\partial y^2} + 4D_{23}\frac{\partial^4 w}{\partial x\,\partial y^3} + D_{22}\frac{\partial^4 w}{\partial y^4} = q(x, y)\sin\varphi \tag{2}$$

where $D_{ik} = B_{ik}\dfrac{h^3}{12}$ and h is the plate thickness,

$w(x,y)$ is the displacement function of the plate in direction of the z-axis

$q(x,y)$ is the load acting normally to the middle surface of the plate (in direction z).

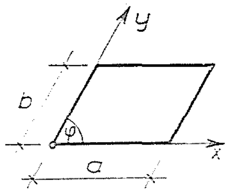

Fig. 1

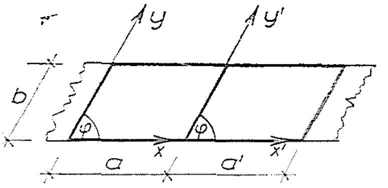

Fig. 2

In conformity with [5], boundary conditions can be written as:

a) If the edge $x =$ const. is simply supported, then along this edge

$$w = 0 \tag{3}$$

and

$$M_x = 0, \text{ that is } D_{11}\frac{\partial^2 w}{\partial x^2} + 2D_{13}\frac{\partial^2 w}{\partial x\,\partial y} = 0. \tag{4}$$

b) If the edge $x =$ const. is clamped, then along this edge

$$w = 0 \text{ and } \frac{\partial w}{\partial x} = 0. \tag{5}$$

c) If the edge $y =$ const. is simply supported by a beam of flexural rigidity EJ (to be referred to as elastically supported), then along this edge

$$M_y = 0 \text{ that is } D_{12}\frac{\partial^2 w}{\partial x^2} + D_{22}\frac{\partial^2 w}{\partial y^2} + 2D_{23}\frac{\partial^2 w}{\partial x\,\partial y} = 0 \tag{6}$$

and

$$Q_y + \frac{\partial M_{xy}}{\partial x} = EJ\frac{\partial^4 w}{\partial x^3} \quad \text{that is}$$

$$2D_{13}\frac{\partial^3 w}{\partial x^3} + (D_{12} + 4D_{33})\frac{\partial^3 w}{\partial x^2 \partial y} + 4D_{23}\frac{\partial^3 w}{\partial x\,\partial y^2} + D_{22}\frac{\partial^3 w}{\partial y^3} = -EJ\frac{\partial^4 w}{\partial x^4}. \tag{7}$$

Definitions of specific moments M and shears Q are the same as in [6] and [5]. The above boundary conditions were published by LEKHNITSKY for rectangular plates [3].

3. The small parameter method

3.1 Transformation of the differential equation of skew anisotropic plates

Sarkisyan developed the solution of the differential equation of form (2) by the small parameter method for simply supported rectangular plates [9]. Below, this method will shortly be described in the skew co-ordinate system presented in item 2. Let us apply co-ordinate transformation:

$$\left.\begin{aligned} x &= \alpha D_{11}^{1/4} & y &= \beta D_{22}^{1/4} \\ w(x, y) &= W[\alpha(x), \beta(y)] & q(x, y) \sin\varphi &= q_0(\alpha, \beta) \end{aligned}\right\}. \tag{8}$$

The differential equation (2) transformed according to (8) for the co-ordinate system α, β:

$$L_1[W] + \mu L_2[W] = q_0(\alpha, \beta) \tag{9}$$

where

$$\mu = \frac{D_{13}}{D_{11}^{1/2} D_{33}^{1/2}} \tag{10}$$

$L_1[W]$ and $L_2[W]$ are "products" of the function $W(\alpha, \beta)$ with operators:

$$\left.\begin{aligned} L_1 &= \frac{\partial^4}{\partial\alpha^4} - 2k\frac{\partial^4}{\partial\alpha^2\,\partial\beta^2} + \frac{\partial^4}{\partial\beta^4} \\ L_2 &= 4\left(k_1\frac{\partial^4}{\partial\alpha^3\,\partial\beta} + k_2\frac{\partial^4}{\partial\alpha\,\partial\beta^3}\right) \end{aligned}\right\} \tag{11}$$

respectively, where:

$$\left.\begin{aligned} k &= \frac{D_{12} + 2D_{33}}{D_{11}^{1/2} D_{22}^{1/2}}; & k_1 &= \frac{D_{33}^{1/2}}{D_{11}^{1/4} D_{22}^{1/4}}; \\ k_2 &= \lambda k_1; \quad \lambda = \frac{\mu'}{\mu}; & \mu' &= \frac{D_{23}}{D_{22}^{1/2} D_{33}^{1/2}} \end{aligned}\right\}. \tag{12}$$

Thereafter let us write function $W(\alpha, \beta)$ as power series of parameter μ.

$$W(\alpha, \beta) = W_0(\alpha, \beta) + \sum_{t=1}^{\infty} \mu^t W_t(\alpha, \beta). \tag{13}$$

Substituting (13) into (9) and separating this equation according to the powers of μ an infinite series of differential equations arise:

$$L_1[W_0] = q_0 \tag{14}$$

$$L_1[W_t] = -L_2[W_{t-1}] \quad (t = 1, 2, \ldots\ldots). \tag{15}$$

Thereby the solution of differential equation (2) of skew anisotropic plates has been reduced to that of Eqs (14), (15), similar in form to the differential equation of rectangular orthotropic plates. Boundary conditions for the former (see item 3.2) are, however, more complex than those of rectangular orthotropic plates with the same type of edges. Furthermore, the known function in the right-hand side of the t-th differential equation from (15) — "product" of the solution of the $(t-1)$-th equation by the operator L_2 as a "load" — is also more complex in form than usual for orthotropic plates.

Being written (14) and (15) in a skew co-ordinate system, their solution means physically the deflection function of a plate made of a material with two planes of symmetry including an angle φ — beside the xy-plane of symmetry of elasticity — viz. planes xz and yz, for which the directions of identical elasticity characteristics are obtained by skew reflection, i.e. parallelly to axes y and x, respectively. Such a material — resulting from mathematical abstraction — is termed a clinogonally anisotropic one, in short a clinotropic one.

The infinite series in (13) is rapidly converging, μ being much less than unity because of its physical meaning. Let us refer to favourable results of convergency analyses by SARKISYAN for anisotropic plates clamped along all edges [10].

For rectangular orthotropic plates $\mu = 0$, hence Eq. (14) yields the final solution, and (15) is superfluous. Of course, the described method lends itself also for skew isotropic or orthotropic plates, provided coefficients D_{ik} in (2) are replaced by those adjacent to the corresponding term of the differential equation of the skew isotropic or orthotropic plate [5, 6].

3.2 Transformation of boundary and continuity conditions

3.21 Simply supported edge at $\alpha =$ const. In conformity with boundary condition (3):

$$W_0 = 0 \qquad W_t = 0 \quad (t = 1, 2, \ldots). \tag{16}$$

Transforming relationship (4) according to (8), multiplying by $D_{11}^{-1/2}$, and separating according to powers of μ, Sarkisyan obtained the following relationships [9]:

$$\left.\begin{aligned} &\frac{\partial^2 W_0}{\partial \alpha^2} = 0 \\ &\frac{\partial^2 W_t}{\partial \alpha^2} = -2k_1 \frac{\partial^2 W_{t-1}}{\partial \alpha\, \partial \beta} \quad (t = 1, 2, \ldots) \end{aligned}\right\}. \tag{17}$$

3.22 Elastically supported edge at $\beta =$ const. Transforming relationship (6) according to (8), multiplying by $D_{22}^{-1/2}$, and separating according to powers of μ:

$$\left.\begin{aligned} k_3 \frac{\partial^2 W_0}{\partial \alpha^2} + \frac{\partial^2 W_0}{\partial \beta^2} &= 0 \\ k_3 \frac{\partial^2 W_t}{\partial \alpha^2} + \frac{\partial^2 W_t}{\partial \beta^2} &= - 2k_2 \frac{\partial^2 W_{t-1}}{\partial \alpha\, \partial \beta} \end{aligned}\right\} \tag{18}$$

where
$$k_3 = \frac{D_{12}}{D_{11}^{1/2} D_{22}^{1/2}} . \tag{19}$$

Transforming relationship (7) according to (8), multiplying by $D_{22}^{-1/4}$, and separating according to powers of μ:

$$(k + 2k_1^2) \frac{\partial^3 W_0}{\partial \alpha^2\, \partial \beta} + \frac{\partial^3 W_0}{\partial \beta^3} + k_4 \frac{\partial^4 W_0}{\partial \alpha^4} = 0$$

$$(k + 2k_1^2) \frac{\partial^3 W_t}{\partial \alpha^2\, \partial \beta} + \frac{\partial^3 W_t}{\partial \beta^3} + k_4 \frac{\partial^4 W_t}{\partial \alpha^4} = - 2k_1 \frac{\partial^3 W_{t-1}}{\partial \alpha^3} - 4k_2 \frac{\partial^3 W_{t-1}}{\partial \alpha\, \partial \beta^2} \tag{20}$$

$$(t = 1, 2, \ldots)$$

where
$$k_4 = \frac{EJ}{D_{11} D_{22}^{1/4}} . \tag{21}$$

3.23 Continuity conditions for continuous skew anisotropic plates with intermediate simple support. Fig. 2 is a detail of a continuous skew plate, with simple supports at $x = 0$, $x' = 0$ and $x' = a'$. Deflection function is sought for in form of functions defined between two intermediate supports. Plate deflections are given by functions $w = w(x,y)$ and $w' = w'(x',y')$ in domains $0 \leq x \leq a$, and $0 \leq x' \leq a'$, respectively. Along the boundaries of neighbouring domains i.e. at the intermediate supports a compatibility and an equilibrium condition can be written each.

According to the condition of compatibility $x = a$ i.e. $x' = 0$, functions w and w' have a common tangential plane. This condition applied to differential equations (14) and (15), respectively, yields the condition of continuity:

$$\frac{\partial W_0}{\partial \alpha} - \frac{\partial W_0'}{\partial \alpha'} = 0 \qquad \frac{\partial W_t}{\partial \alpha} - \frac{\partial W_t'}{\partial \alpha'} = 0 \quad (t = 1, 2, \ldots). \tag{22}$$

Equilibrium condition at the same support:

$$M_x - M_x' = 0. \tag{23}$$

Substituting in this condition the relationship for flexural moment M_x [5] and transforming according to (8), multiplying by $D_{11}^{-1\,2}$ and separating according to the powers of μ yields:

$$\left.\begin{aligned} &\frac{\partial^2 W_0}{\partial\alpha^2} + k_3\frac{\partial^2 W_0}{\partial\beta^2} - \frac{\partial^2 W_0'}{\partial\alpha'^2} - k_3\frac{\partial^2 W_0'}{\partial\beta'^2} = 0 \\ &\frac{\partial^2 W_t}{\partial\alpha^2} + k_3\frac{\partial^2 W_t}{\partial\beta^2} - \frac{\partial^2 W_t'}{\partial\alpha'^2} - k_3\frac{\partial^2 W_t'}{\partial\beta'^2} = 2k_1\left(\frac{\partial^2 W_{t-1}'}{\partial\alpha'\,\partial\beta'} - \frac{\partial^2 W_{t-1}'}{\partial\alpha\,\partial\beta}\right). \end{aligned}\right\} \qquad (24)$$

4. Solution of the differential equation of clinogonally anisotropic plates

4.1 General

Relationships (14) and (15) being similar in form to the differential equation of rectangular orthotropic plates with principal anisotropy directions coincident with the co-ordinate axes, the "clinotropic" plate problem in item 3 can in principle be solved by any method known from the bending theory of orthotropic plates, only that it has to be adapted to the more complex boundary conditions in item 3.2 and to the complex "load functions" in the right-hand side of Eqs (15). In what follows, application to the differential equation (14) of the double Fourier series method developed by KACZKOWSKI and WILDE for orthotropic plates [2, 11] will be presented.

4.2 The Wilde solution of orthotropic plates

WILDE expanded all terms of the differential equation of orthotropic plates into Fourier series, then, on the basis of equality between the mn-th Fourier coefficients, determined the coefficient

$$w_{mn} = \sum_{i=\mathrm{I.}}^{\mathrm{IV.}} w_{mn}^{(i)} \qquad (25)$$

of the solution of form

$$w(\alpha, \beta) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} w_{mn} \sin\alpha_m\alpha \sin\beta_n\beta \qquad (26)$$

The applied notations were:

$$\alpha_m = \frac{m\pi}{a_1} \qquad\qquad \beta_n = \frac{n\pi}{b_1} \qquad (27)$$

where a_1 and b_1 are the edge lengths of the skew plate in the co-ordinate system α, β, hence, from (8):

$$a_1 = a D_{11}^{-1/4} \qquad b_1 = b D_{22}^{-1/4}. \tag{28}$$

Terms of the coefficients w_{mn} according to (26) are:

$$w_{mn}^{(I)} = \frac{1}{\Delta_{mn}} q_{mn} \tag{29}$$

$$w_{mn}^{(II)} = -\frac{4}{a_1 b_1} \frac{1}{\Delta_{mn}} \left\{ \alpha_m \int_0^{b_1} \left[\frac{\partial^2 w}{\partial \alpha^2}(0, \beta) - (-1)^m \frac{\partial^2 w}{\partial \alpha^2}(a_1, \beta) \right] \sin \beta_n \beta \, d\beta + \right.$$
$$\left. + \beta_n \int_0^{a_1} \left[\frac{\partial^2 w}{\partial \beta^2}(\alpha, 0) - (-1)^n \frac{\partial^2 w}{\partial \beta^2}(\alpha, b_1) \right] \sin \alpha_m \alpha \, d\alpha \right\} \tag{30}$$

$$w_{mn}^{(III)} = \frac{4}{a_1 b_1} \frac{1}{\Delta_{mn}} \left\{ \Delta_x \int_0^{b_1} \left[w(0, \beta) - (-1)^m w(a_1, \beta) \right] \sin \beta_m \beta d\beta + \right.$$
$$\left. + \Delta_y \int_0^{a_1} \left[w(\alpha, 0) - (-1)^n w(\alpha, b_1) \right] \sin \alpha_m \alpha d\alpha \right\} \tag{31}$$

$$w_{mn}^{(IV)} = -\frac{4}{a_1 b_1} \frac{1}{\Delta_{mn}} 2k\alpha_m \beta_n \left[w(0, 0) - (-1)^n w(0, b_1) - \right.$$
$$\left. - (-1)^m w(a_1, 0) + (-1)^{m-n} w(a_1 b_1) \right] \tag{32}$$

where q_{mn} is the mn-th coefficient of the double Fourier sine series of the load $q_0(\alpha, \beta)$,

$$\left. \begin{array}{l} \Delta_{mn} = \alpha_m^4 + 2k\alpha_m^2 \beta_n^2 + \beta_n^4 \\ \Delta_x = \alpha_m^3 + 2k\alpha_m \beta_n^2 \qquad \Delta_y = \beta_n^3 + 2k\alpha_m^2 \beta_n \end{array} \right\} . \tag{33}$$

Boundary value functions in (30) and (31) can be written as simple Fourier series:

$$\left. \begin{array}{ll} \dfrac{\partial^2 w}{\partial \alpha^2}(0, \beta) = -\sum\limits_n A_n \sin \beta_n \beta & \dfrac{\partial^2 w}{\partial \alpha^2}(\alpha, b_1) = -\sum\limits_n C_n \sin \beta_n \beta \\ \dfrac{\partial^2 w}{\partial \beta^2}(\alpha, 0) = -\sum\limits_m B_m \sin \alpha_m \alpha & \dfrac{\partial^2 w}{\partial \beta^2}(\alpha, b_1) = -\sum\limits_m D_m \sin \alpha_m \alpha \end{array} \right\} \tag{34}$$

$$\left. \begin{array}{ll} w(0, \beta) = \sum\limits_n E_n \sin \beta_n \beta & w(a_1, \beta) = \sum\limits_n G_n \sin \beta_n \beta \\ w(\alpha, 0) = \sum\limits_m F_m \sin \alpha_m \alpha & w(\alpha, b_1) = \sum\limits_m K_m \sin \alpha_m \alpha \end{array} \right\} \tag{35}$$

which, substituted into (30) and (31) yield:

$$w_{mn}^{(II)} = \frac{4}{a_1 b_1} \frac{1}{\Delta_{mn}} \left\{ \frac{\alpha_m b_1}{2} \left[A_n - (-1)^m C_n\right] + \frac{\beta_n a_1}{2} \left[B_m - (-1)^n D_m\right] \right\} \quad (36)$$

$$w_{mn}^{(III)} = \frac{4}{a_1 b_1} \frac{1}{\Delta_{mn}} \left\{ \frac{\Delta_x b_1}{2} \left[E_n - (-1)^m G_n\right] + \frac{\Delta_y a_1}{2} \left[F_m - (-1)^n K_m\right] \right\}. \quad (37)$$

Thus, the Wilde analysis of orthotropic plates lends itself particularly for cases where boundary conditions are given in forms (34) and (35).

4.3 Satisfaction of arbitrary boundary conditions

The previous analysis in that form satisfies boundary conditions in the form of infinite series only. Therefore expressions (25), (26), (32), (36) and (37) will be transformed so that each mn-th term can in itself satisfy boundary conditions, consequently constants A_n, B_m etc., belonging to conditions in item 3.2, can also be determined (see item 5).

Replacing (26) into (25), the part of the double infinite series containing $w_{mn}^{(II)}$, $w_{mn}^{(III)}$, $w_{mn}^{(IV)}$ that is $w^{(II)}$, $w^{(III)}$, $w^{(IV)}$ are transformed in turn as follows.

Transformation of $w^{(II)}$

$$w^{(II)} = \overline{w}^{(II)} + \overline{\overline{w}}^{(II)} \quad (38)$$

where

$$\overline{w}^{(II)} = -2 \sum_n \sum_m \left[\frac{A_n - (-1)^m C_n}{a_1 \alpha_m^3 \Delta_{mn}} (\Delta_{mn} - \alpha^4 m) + \frac{B_m - (-1)^n D_m}{b_1 \beta_n^3 \Delta_{mn}} (\Delta_{mn} - \beta_n^4) \right] \cdot \sin \alpha_m \alpha \sin \beta_n \beta \quad (39)$$

$$\overline{\overline{w}}^{(II)} = 2 \sum_n \sum_m \left[\frac{A_n - (-1)^m C_n}{a_1 \alpha_m^3} + \frac{B_m - (-1)^n D_m}{b_1 \beta_n^3} \right] \sin \alpha_m \alpha \sin \beta_n \beta. \quad (40)$$

This latter can also be written as:

$$\overline{\overline{w}}^{(II)} = \left(\frac{a_1 \alpha}{3} - \frac{\alpha^2}{2} + \frac{\alpha^3}{6a_1} \right) \sum_n A_n \sin \beta_n \beta + \left(\frac{a_1 \alpha}{6} - \frac{\alpha^3}{6a_1} \right) \sum_n C_n \sin \beta_n \beta + \left(\frac{b_1 \beta}{3} - \frac{\beta^2}{2} + \frac{\beta^3}{6b_1} \right) \sum_m B_m \sin \alpha_m \alpha + \left(\frac{b_1 \beta}{6} - \frac{\beta^3}{6b_1} \right) \sum_m D_m \sin \alpha_m \alpha \quad (41)$$

the m-th coefficient of the Fourier sine series of functions

$$\left(\frac{a_1 \alpha}{3} - \frac{\alpha^2}{2} + \frac{\alpha^3}{6a_1}\right) \text{ and } \left(\frac{a_1 \alpha}{6} - \frac{\alpha^3}{6a_1}\right)$$

$$\text{being } \frac{2}{a_1 \alpha_m^3} \text{ and } \frac{-2(-1)^m}{a_1 \alpha_m^3}, \quad \text{respectively.}$$

Transformation of $w^{(III)}$:

$$w^{(III)} = \overline{w}^{(III)} + \overline{\overline{w}}^{(III)} \tag{42}$$

where

$$\overline{w}^{(III)} = -2 \sum_n \sum_m \left[\frac{E_n - (-1)^m G_n}{\alpha_m^2 \Delta_x + \alpha_m \beta_n^4} \cdot \frac{\beta_n^4}{a_1} + \frac{F_m - (-1)^n K^m}{\beta_n^2 \Delta_y + \beta_n \alpha_m^4} \cdot \frac{\alpha_m^4}{b_1}\right] \cdot$$
$$\cdot \sin \alpha_m \alpha \sin \beta_n \beta \tag{43}$$

$$\overline{\overline{w}}^{(III)} = 2 \sum_n \sum_m \left[\frac{E_n - (-1)^m G_n}{a_1 \alpha_m} + \frac{F_m - (-1)^n K_m}{b_1 \beta_n}\right] \sin \alpha_m \alpha \sin \beta_n \beta \tag{44}$$

This latter can also be written as:

$$\overline{\overline{w}}^{(III)} = \left(1 - \frac{\alpha}{a_1}\right) \sum_n E_n \sin \beta_n \beta + \frac{\alpha}{a_1} \sum_n G_n \sin \beta_n \beta +$$
$$+ \left(1 - \frac{\beta}{b_1}\right) \sum_m F_m \sin \alpha_m \alpha + \frac{\beta}{b_1} \sum_m K_m \sin \alpha_m \alpha \tag{45}$$

the m-th coefficients of the Fourier sine series of functions $\left(1 - \frac{\alpha}{a_1}\right)$ and $\frac{\alpha}{a_1}$ being $\frac{2}{a_1 \alpha_m}$ and $\frac{-2(-1)^m}{a_1 \alpha_m}$, respectively.

Similarly, for $w^{(IV)}$:

$$w^{(IV)} = \overline{w}^{(IV)} + \overline{\overline{w}}^{(IV)} \tag{46}$$

where

$$\overline{w}^{(IV)} = -\sum_n \sum_m \frac{8k\alpha_m^2 \beta_n^2 + 2\Delta_{mn}}{a_1 b_1 \alpha_m \beta_n \Delta_{mn}} \left[w(0,0) - (-1)^n w(0, b_1) - \right.$$
$$\left. - (-1)^m w(a_1 0) + (-1)^{m+n} w(a_1 b_1)\right] \cdot \sin \alpha_m \alpha \sin \beta_n \beta \tag{47}$$

$$\overline{\overline{w}}^{(IV)} = \frac{1}{2}\left[\left(1 - \frac{\alpha}{a_1}\right)\left(1 - \frac{\beta}{b_1}\right) w(0,0) + \left(1 - \frac{\alpha}{a_1}\right)\frac{\beta}{b_1} w(0, b_1) - \right.$$
$$\left. + \frac{\alpha}{a_1}\left(1 - \frac{\beta}{b_1}\right) w(a_1, 0) + \frac{\alpha\beta}{a_1 b_1} w(a_1 b_1)\right]. \tag{48}$$

Let us mention here that a transformation similar to (42) has also been applied by KACZKOWSKI [2].

Now, the suitable form to satisfy boundary conditions is obtained by substituting (26) then (29), (39), (41), (43), (45), (47) and (48) into (25).

4.4 *Consideration of an arbitrary load*

It is obvious from (29) that to solve differential equations (14) and (15), coefficients of the double Fourier sine series of the load function $q_0(\alpha, \beta)$ are needed. When solving, however, the t-th equation in (15), the right-hand side is obtained first as a double Fourier cosine series, it being the product of W_{t-1} of sine series form by operator L_2.

The double Fourier cosine series can be transformed in a double sine one in the manner to be described below.

Two respective forms of function $p(\alpha, \beta)$ defined in the ranges $0 \leq \alpha \leq a_1$, and $0 \leq \beta \leq b_1$ are

$$p(\alpha, \beta) = \sum_q \sum_p b_{pq} \cos \alpha_p \alpha \cos \beta_n \beta = \\ = \sum_n \sum_m d_{mn} \sin \alpha_m \alpha \sin \beta_n \beta. \tag{49}$$

If coefficients b_{pq} are known:

$$d_{mn} = \frac{8}{\pi^2} \sum_q \sum_p b_{pq} \frac{pq}{(p^2 - m^2)(q^2 - n^2)} \qquad \begin{array}{l} \text{for even } \begin{matrix} m \\ n \end{matrix} \\ \left.\begin{matrix} p \\ q \end{matrix}\right\} = 1,3,5 \ldots \\ \text{for odd } \begin{matrix} m \\ n \end{matrix} \\ \left.\begin{matrix} p \\ q \end{matrix}\right\} = 2,4,6 \ldots \end{array} \tag{50}$$

Thus, in this expression terms are identically zero where either both p and m or both q and n are even or odd.

5. Applications

5.1 *Skew anisotropic plates simply supported along all edges*

Sarkisyan developed the small parameter method for the analysis of rectangular anisotropic plates simply supported along all edges [9]. As first application of the method presented in items 3 and 4, determination of the deflection function of skew anisotropic plates simply supported along all edges subject to an arbitrary static load will be presented.

The solution satisfying differential equation (14) and boundary conditions (16) and (17):

$$W_0(\alpha, \beta) = \sum_n \sum_m W_{o,mn} \sin \alpha_m \alpha \sin \beta_n \beta \tag{51}$$

where

$$W_{o,mn} = \frac{q_{mn}}{\Delta_{mn}} .$$

Right-hand side of t-th differential equation in (15) transformed according to (49):

$$\begin{aligned} - L_2[W_{t-1}] &= \sum_q \sum_p b_{t-1,pq} \cos \alpha_p \alpha \cos \beta_q \beta = \\ &= \sum_n \sum_m d_{t-1,mn} \sin \alpha_m \alpha \sin \beta_n \beta \end{aligned} \tag{52}$$

where

$$b_{t-1,pq} = W_{t-1,pq} \frac{4\pi^4 pq}{a_1 b_1} \left(\frac{k_1 p^2}{a_1^2} + \frac{k_2 q^2}{b_1^2} \right);$$

$W_{t-1,pq}$ is the pq-th Fourier coefficient of the solution of the (t—1)-th differential equation;

coefficients $d_{t-1,mn}$ are obtained from $b_{t-1\,pq}$ according to (50).

Boundary conditions for the t-th equation are, according to (16) and (17):

For $\alpha = 0$, $W_t = 0$ and $$\frac{\partial^2 W_t}{\partial \alpha^2} = - \frac{2\pi^2 k_1}{a_1 b_1} \sum_n \sum_m mn W_{t-1,mn} \cos \beta_n \beta \tag{53}$$

For $\alpha = a_1$, $W_t = 0$ and $$\frac{\partial^2 W_t}{\partial \alpha^2} = - \frac{2\pi^2 k_1}{a_1 b_1} \sum_n \sum_m (-1)^m mn W_{t-1,mn} \cos \beta_n \beta \tag{54}$$

For $\beta = 0$, $W_t = 0$ and $$\frac{\partial^2 W_t}{\partial \beta^2} = - \frac{2\pi^2 k_2}{a_1 b_1} \sum_n \sum_m mn W_{t-1,mn} \cos \alpha_m \alpha \tag{55}$$

For $\beta = b_1$, $W_t = 0$ and $$\frac{\partial^2 W_t}{\partial \beta^2} = - \frac{2\pi^2 k_2}{a_1 b_1} \sum_n \sum_m (-1)^n mn W_{t-1,mn} \cos \alpha_m \alpha \tag{56}$$

Terms according to (26) of the coefficients of function W_t of the form (25) are:

$$W_{t,mn}^{(1)} = \frac{d_{t-1,mn}}{\Delta_{mn}} . \tag{57}$$

The other terms are obtained by replacing boundary conditions (53) through (56) into Eqs (29) through (31). Integration yields:

$$w_{t,mn}^{(II)} = \frac{4m\,\pi^2 k_1}{\Delta_{mn}\,a_1^3\,b_1} \sum_q \sum_p pq W_{t-1,pq} \left[1 - (-1)^{p-m}\right] \left[\frac{1-(-1)^{q-n}}{q+n} + \right.$$

$$\left. + \frac{1-(-1)^{q-n}}{q-n}\right] + \frac{4n\,\pi^2 k_2}{\Delta_{mn}\,a_1\,b_1^2} \sum_q \sum_p pq W_{t-1,pq} \cdot$$

$$\cdot \left[1-(-1)^{q+n}\right]\left[\frac{1-(-1)^{p+m}}{p+m} + \frac{1-(-1)^{p-m}}{p-m}\right] \tag{58}$$

$$W_{t,mn}^{(III)} = W_{t,mn}^{(IV)} = 0. \tag{59}$$

Accordingly, constants A_n and C_n in (36) will be written separately, since they will be needed in item 5.2.

$$A_n = \frac{2\pi k_1}{a_1 b_1} \sum_q \sum_p pq W_{t-1,pq} \left[\frac{1-(-1)^{q+n}}{q+n} + \frac{1-(-1)^{q-n}}{q-n}\right] \tag{60}$$

$$C_n = \frac{2\pi k_1}{a_1 b_1} \sum_q \sum_p pq W_{t-1,pq} (-1)^p \left[\frac{1-(-1)^{q-n}}{q+n} + \frac{1-(-1)^{q-n}}{q-n}\right]. \tag{61}$$

The problem is solved by substituting Eqs (51) and (57) through (59) into (26), (25) and (13), and transforming into the xy co-ordinate system according to Eqs (8).

5.2 "Bridge-like" skew anisotropic plate

A usual problem for bridge structures is that of a skew anisotropic plate with two opposite edges simply supported and the other two edges elastically supported. Let us consider now a plate simply supported on edges $x = 0$ and $x = a$ — or $\alpha = 0$ and $\alpha = a_1$ after transformation according to (8) — and supported elastically along edges $y = 0$ and $y = b$ i.e. $\beta = 0$, $\beta = b_1$.

Solution of the t-th differential equation in (15) will be sought for in forms (25) and (26). Coefficients $W_{t,mn}^{(i)}$ will be written according to (29), (36), (37) and (32). Conditions for edges $\alpha = 0$ and $\alpha = a_1$ equal (53) and (54), respectively, while those for edges $\beta = 0$ and $\beta = b_1$ can be written according to (18) and (20), respectively. Determination of function W_0 as solution of differential equation (14) will not be treated here separately, only referred to as a special case when analyzing W_t.

Also here, $W_{t,mn}^{(I)}$ will be determined according to (57), but $W_{0,mn}^{(I)}$ has to be computed with the Fourier coefficients q_{mn} of the effective load in the numerator.

For the determination of functions W_t and W_0, boundary conditions imply terms in (37) to be:

$$E_n = G_n = 0 \tag{62}$$

and

$$W^{(\mathrm{IV})}_{o,mn} = W^{(\mathrm{IV})}_{t,mn} = 0 \,. \tag{63}$$

Similarly as for the plate simply supported along all edges, when determining W_0, $A_n = C_n = 0$, while for W_t, the A_n and C_n values are obtained from (60) and (61), respectively.

Find now coefficients B_m, D_m, F_m, K_m in (36) and (37) by means of terms $W_t^{(\mathrm{II})}$ and $W_t^{(\mathrm{III})}$ transformed according to item 4.3.

According to boundary condition (18) written for edge $\beta = 0$, B_m and F_m are related by the expression:

$$B_m = \frac{2\pi^2 k_2}{a_1 b_1} g_m - k_3 \alpha_m^2 F_m \tag{64}$$

where $g_m = 0$ for W_0

$$\left. g_m = \frac{4}{\pi} \sum_q \sum_p W_{t-1,pq} \frac{p^2 q}{p^2 - q^2} \text{ for } W_t \right\} . \tag{65}$$

Condition (18) written for edge $\beta = b_1$ relates D_m and K_m as follows:

$$D_m = \frac{2\pi^2 k_2}{a_1 b_1} g'_m - k_3 \alpha_m^2 K_m \tag{66}$$

where $g_m = 0$ for W_0

$$\left. g'_m = \frac{4}{\pi} \sum_q \sum_p (-1)^q W_{t-1,pq} \frac{p^2 q}{p^2 - q^2} \text{ for } W_t \right\} . \tag{67}$$

According to boundary conditions (20) written for edges $\beta = 0$, and $\beta = b_1$ — involving (64) and (66) — to determine F_m and K_m, a series of linear equation systems with two unknows are obtained, in the following form:

$$\left. \begin{aligned} e_{1F} F_m + e_{1K} K_m = e_{10} \\ e_{2F} F_m + e_{2K} K_m = e_{20} \end{aligned} \right\} \tag{68}$$

where

$$\left. \begin{matrix} e_{1F} \\ e_{2F} \end{matrix} \right\} = (k + 2k_1^2) \left\{ \mp \frac{2k_3 \alpha_m^4}{b_1} \sum_n \frac{\Delta_{mn} - \beta_n^4}{\beta_n^2 \Delta_{mn}} \pm \right.$$

$$\left. \pm \frac{2\alpha_m^6}{b_1} \sum_n \frac{1}{\beta_n \Delta_y + \alpha_m^4} \pm \frac{b_1}{3} \alpha_m^4 k_3 \pm \frac{\alpha_m^2}{b_1} \right\} \mp$$

$$\mp \frac{2k_3 \alpha_m^2}{b_1} \sum_n \frac{\Delta_{mn} - \beta_n^4}{\Delta_{mn}} \pm \frac{2\beta_m^4}{b_1} \sum_n \frac{\beta_n^3}{\beta_n^2 \Delta_y - \beta_n \alpha_m^4} \mp$$

$$\mp \frac{k_3 \alpha_m^2}{b_1} + k_4 \alpha_m^4 . \tag{69}$$

Relationships for e_{2F} and e_{1K} are obtained from (69) by multiplying terms containing $\sum_n$ by $(-1)^n$, replacing term $\pm \frac{b_1}{3} \alpha_m^4 k_3$ in clamped brackets by $\mp \frac{b_1}{6} \alpha_m^4 k_3$, and omitting last term $(k_4 \alpha_m^4)$.

$$e_{10} = (k + 2k_1^2) \Bigg\{ \alpha_m^2 \sum_n \frac{d_{i-1,mn} \beta_n}{\Delta_{mn}} + \frac{2}{a_1 \alpha_m} \sum_n [A_n - (-1)^m C_n] \cdot$$

$$\cdot \left(1 - \frac{\Delta_{mn} - \alpha_m^4}{\Delta_{mn}}\right) \beta_n - \frac{4\pi^2 k_2 \alpha_m^2}{a_1 b_1^2} \sum_n [g_m - (-1)^n g'_m] \frac{\Delta_{mn} - \beta_n^4}{\beta_n^2 \Delta_{mn}} \Bigg\} +$$

$$+ \frac{2\pi^2 k_2 \alpha_m^2}{a_1} \left(\frac{g_m}{3} + \frac{g'_m}{6}\right) + \sum_n \frac{d_{i-1,mn} \beta_n^3}{\Delta_{mn}} + \frac{2}{a_1 \alpha_m^3} \sum_n [A_n -$$

$$- (-1)^m C_n] \left(1 - \frac{\Delta_{mn} - \alpha_m^4}{\Delta_{mn}}\right) \beta_n^3 - \frac{4\pi^2 k_2}{a_1 b_1^2} \sum_n [g_m - (-1)^n g'_m] \frac{\Delta_{mn} - \beta_n^4}{\Delta_{mn}} -$$

$$- \frac{2\pi^2 k_2}{a_1 b_1^2} (g_m - g'_m) + \frac{32\pi k_2^2}{a_1 b_1} \sum_r \alpha_r g_r \frac{r}{r^2 - m^2} +$$

$$+ \left(\frac{8k_1}{\pi} - \frac{16 k_2 k_3}{\pi}\right) \sum_r \alpha_r^3 F_r \frac{r}{r^2 - m^2}. \qquad (70)$$

Relationship for e_{20} is obtained from (70) by multiplying terms containing $\sum_n$ by $(-1)^n$, replacing $\left(\frac{g_m}{3} + \frac{g'_m}{6}\right)$ in the second term by $\left(-\frac{g_m}{6} - \frac{g'_m}{3}\right)$, coefficients g_r in the last but one term by g'_r, and F_r in the last term by K_r.

Since the last terms of e_{10} and e_{20} include all the F and K, respectively, in the first section of the solution this term will be zeroed, the computation repeated with the obtained F and K values, and iterated to the desired accuracy.

In course of the determination of coefficients F_m and K_m for $\overline{W}_0$, in the coefficients of the equation system (68) $d_{i-1,mn} = q_{mn}$, and $A_n = C_n = g_m = g'_m = 0$, furthermore, last two terms of e_{10} and e_{20} are zeroed.

Thereby, coefficients A_n, B_m, C_n, D_m, E_n, F_m, G_n, K_m have been determined. Substituting them into (36) and (37), and making use of (57), (63), (25) and (26), the solution is available in the form of Eq. (13).

Summary

Application of the small parameter method for the analysis of skew anisotropic plates under arbitrary loads has been treated. Solution of the differential equation of the skew anisotropic plate is reduced to that of equation series similar in form to the differential equation of orthotropic plates, only that here the boundary conditions are more complex. Arising partial problems — differential equation of clinogonally anisotropic (clinotropic) plates — have been solved by means of double Fourier series.

Application of this method has been demonstrated on skew anisotropic plates either simply supported along all edges or simply supported along two opposite edges and supported elastically along the other two ones. This method lends itself to satisfy other boundary or continuity conditions, e. g. for continuous plates, provided edges are parallel to the skew coordinate axes.

This method can be applied on a digital computer, since it contains several repetitive operations. As a consequence of the nature of both the power series of small parameters and of the double Fourier series, accuracy can be increased by the use of longer running time, rather than by greater storage capacity. Convergence of the applied infinite series is slower for stresses and near the edges than for deflections and farther from the edges.

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