AN ELECTRIC ANALOGUE TO CALCULATE VELOCITY DISTRIBUTION OF UNIFORM LAMINAR FLOW

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Introduction

If the laminar flow is uniform and velocity vectors are parallel to the x-axis, then velocity distribution in the plane (y, z) can be described by a simplified form of the Navier—Stokes equation:

$$\frac{1}{\mu} \frac{dp}{dx} = \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \tag{1}$$

with μ being the dynamic viscosity and p the dynamic pressure. If the pipe or open channel is a prismatic one and v_x is equal to the full velocity v then one obtains:

$$\frac{gS}{\nu} = \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = \nabla^2 v = \Delta v \tag{2}$$

where g is the gravity constant, $S = \frac{1}{\omega} \frac{dp}{dx}$ the hydraulic gradient, ν the kinematic viscosity, ω the specific weight of the fluid and $\Delta = \nabla^2$ the Laplace operator.

However simple Eq. (2) may appear, its exact solution by integration is known only for a few cases, as circular, elliptical or equilateral-triangular pipes. Solutions, where the integral consists of the sum of an infinite series have been developed for rectangular pipes and for those having an isosceles-triangular cross section.

The only way to be resorted to in cases of cross sections other than the above ones can be found in the application of various numerical methods.

All numerical methods, whether to be classified as relaxation methods or a direct one that will be described presently, are based upon the same principle of substituting the Poisson-type partial differential equation of (2) by as many linear equations as there are points with a velocity value to be determined. This set of linear algebraic equations may then be solved either by "manual relaxation", by relaxation or matrix inversion performed on a

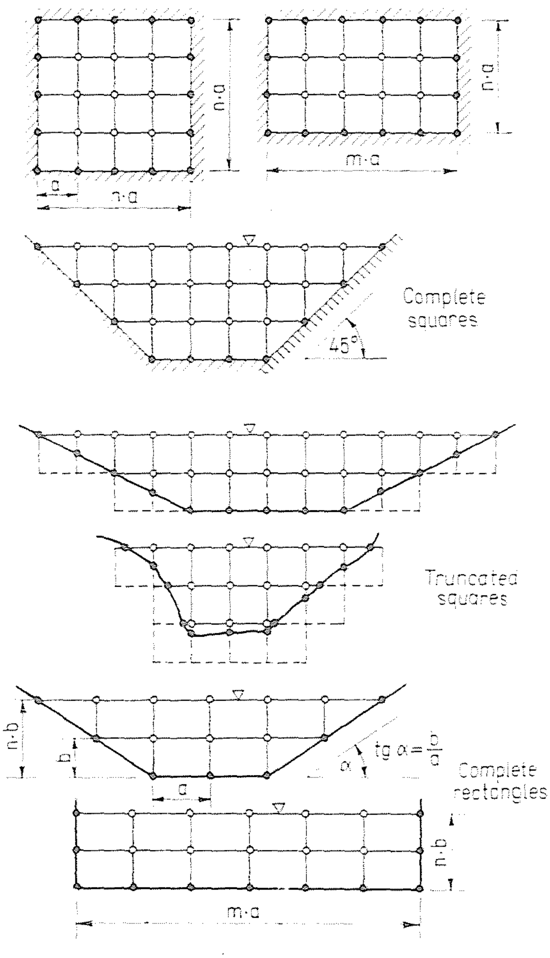


Fig. 1

digital computer, or, by using an electric analogue of the hydraulic system to be outlined below.

Again, the considerations underlying the establishment of a set of equations above mentioned will vary according to the geometry of the points representing the linear equations.

The geometric patterns to cover the velocity field can be the following ones:

1. Square
   a) complete
   b) truncated
2. Rectangular
   a) complete
   b) truncated

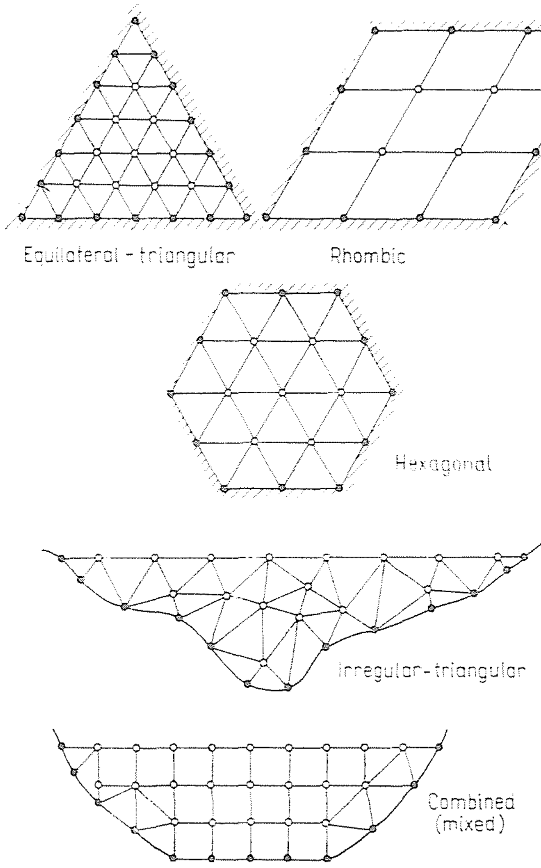


Fig. 2

3. Equilateral-triangular
4. Hexagonal
5. Polar
6. Rhombic
7. Irregular triangular
8. Combined.

Examples of these patterns, also pointing out some of the most favourable fields of their application are shown in Figs 1 and 2.

Difference equations

In the following, the difference equations obtained in each investigated point by expanding Eq. (2) into a Taylor series in the vicinity of the point, will be discussed.

If the cross section to be examined is covered by a *rectangular mesh* (Fig. 3), with elements measuring Δy by Δz, then the difference equation, by neglecting higher-order terms of the Taylor series, will be:

$$\frac{v_{m,n-1} + v_{m,n+1} - 2v_{m,n}}{(\Delta y)^2} + \frac{v_{m+1,n} + v_{m-1,n} - 2v_{m,n}}{(\Delta z)^2} = -\frac{gS}{\nu} \tag{3}$$

If the mesh is a square one then obviously $\Delta y = \Delta z = a$ and hence

$$\frac{v_{m,n-1} + v_{m,n+1} + v_{m+1,n} + v_{m-1,n} - 4v_{m,n}}{a^2} = -\frac{gS}{\nu} \tag{4}$$

Eqs (3) and (4) hold only if the network is a complete one, that is, the solid boundary does not intersect any of the mesh lines in points other than the nodes. Otherwise, instead of the *regular star* shown in Fig. 3 one may have one or more *truncated stars* along the boundary (Fig. 4) satisfying the equation

$$\frac{v_1}{a_1(a_1 + a_2)} + \frac{v_2}{a_2(a_1 + a_2)} + \frac{v_3}{a_3(a_3 + a_3)} + \frac{v_4}{a_4(a_3 + a_4)} - \\ - \left(\frac{1}{a_1 a_2} + \frac{1}{a_3 a_4}\right) v_0 = -\frac{gS}{\nu} \tag{5}$$

One can prove that Eqs (3) and (4) are but special cases of Eq. (5).

If the mesh used is a rhombic one with side lengths h inclined at an angle α, then the Navier—Stokes equation will be:

$$\nabla^2 v = \frac{1}{\sin^2 \alpha}\left(\frac{\partial^2 v}{\partial Y^2} - 2\frac{\partial^2 v}{\partial Y\,\partial Z} + \frac{\partial^2 v}{\partial Z^2}\right) = -\frac{gS}{\nu} \tag{6}$$

with Y and Z being the oblique co-ordinates parallel to the mesh lines. A difference equation for $\alpha = 60°$ can be written as:

$$\frac{1}{3h^2}\left[4(v_{m+1,n} + v_{m-1,n} + v_{m,n+1} + v_{m,n-1}) + v_{m+1,n-1} + v_{m-1,n+1} - \right. \\ \left. - v_{m-1,n-1} - v_{m+1,n+1} - 16v_{m,n}\right] = -\frac{gS}{\nu} \tag{7}$$

with notations in Fig. 5.

If the cross section to be investigated is either a circle, a circular sector, annulus or an annular segment then it may be appropriate to apply *polar co-ordinates* by using the equation

$$\frac{v_{m+1,n} - 2v_{m,n} + v_{m-1,n}}{(\Delta r)^2} + \frac{v_{m+1,n} - v_{m-1,n}}{2r_m \Delta r} +$$
$$+ \frac{v_{m,n+1} - 2v_{m,n} + v_{m,n-1}}{r_m^2 \Theta^2} = - \frac{gS}{\nu} \tag{8}$$

For notations see Fig. 6.

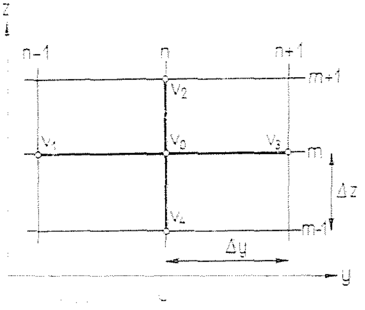


Fig. 3

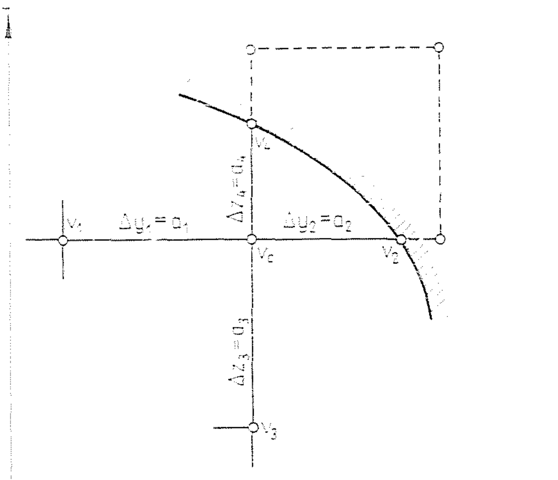


Fig. 4

In some cases, the use of an *equilateral-triangular mesh* may seem justified. An approximate solution of this case is the equation

$$\frac{4v_0}{a^2} + \frac{2}{3a^2} \sum_{k=1}^{6} v_k = - \frac{gS}{\nu} \tag{9}$$

when considering only six points around the investigated one. If however, the Taylor expansion is continued up to the 4th derivatives and six more points are considered, a more accurate result can be obtained from the equation

$$\frac{48v_0 - 9(v_1 + \dots + v_6) + (v_7 + \dots + v_{12})}{9a^2} = -\frac{gS}{\nu} \tag{10}$$

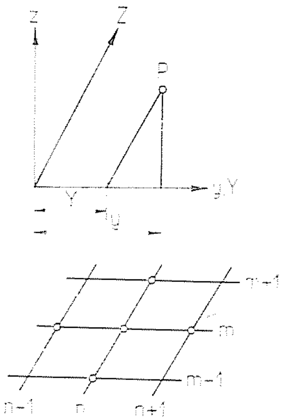


Fig. 5

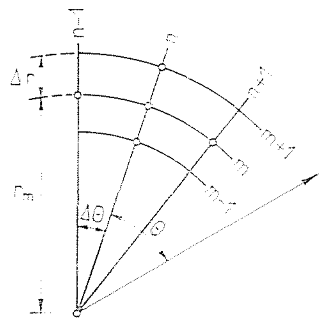


Fig. 6

The explanation of subscripts used in Eqs (9) and (10) is given in Fig. 7.

A quick but less accurate method lies in the application of a *hexagonal mesh* with only three points considered around the investigated one (Fig. 8). Its difference equation is

$$v_0 = \frac{4}{3a^2}(v_1 + v_2 + v_3 - 3v_0) = -\frac{gS}{\nu} \tag{11}$$

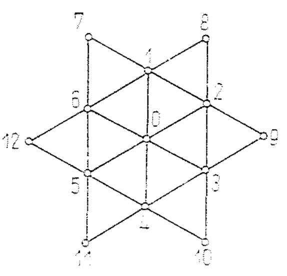


Fig. 7

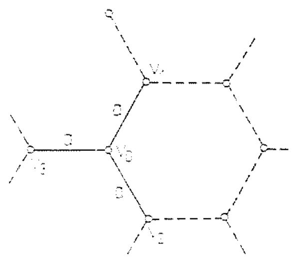


Fig. 8

The most flexible network of points is the one consisting of *triangles of arbitrary size* but containing no obtuse angles. This network, however, requires a different approach than the previous ones and it will thus be discussed only after the description of the main subject of this paper, the electric analogue of laminar-flow velocity distribution.

The electric simulation of a laminar-flow velocity field

It is a commonplace fact that two-dimensional potential flow in the (y, z) plane satisfies the Laplace differential equation

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \tag{12}$$

It is also known that when applying a square mesh on such a potential field, the following relationship will hold for the nodes:

$$\frac{\varphi_{m,n-1} + \varphi_{m,n+1} + \varphi_{m+1,n} + \varphi_{m-1,n} - \varphi_{m,n}}{a^2} = 0 \tag{13}$$

This difference equation shows a conspicuous similarity to Eq. (4), the only difference being in the fact that the underlying Laplace equation is a homogeneous one, whilst the Navier—Stokes equation of (2) is non-homogeneous.

Likewise it is known that the basic equation of two-dimensional potential theory in electrodynamics

$$\nabla^2 u = \frac{\partial^2 y}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{14}$$

can be modelled by aid of a resistor network shown in Fig. 9 where

$$\frac{u_1 + u_2 + u_3 + u_4 - 4u_0}{R} = 0 \tag{15}$$

or, in other terms:

$$i_1 + i_2 + i_3 + i_4 = 0 \tag{16}$$

i.e. the algebraic sum of currents flowing in or off a node must be zero.

But Eqs (15) and (16) are analogous to (13) and therefore the arrangement shown in Fig. 9 is frequently used to determine pointwise values of flow potential by measuring voltages u in the analogous network adjusted to prevailing boundary conditions.

Now, it is but an obvious step to extend this type of analogy to laminar flow as well, and hence, to replace the constant term gS/ν of the Navier—Stokes equation through a constant current i_B introduced from without in each node of the analogue network. If this condition, together with boundary conditions of laminar flow, is satisfied then the electric network of Fig. 10 will represent an analogue to the velocity field of uniform laminar flow.

Thus, for each node of the "square" electric network, the following equation will hold:

$$\frac{u_1 + u_2 + u_3 + u_4 - 4u_0}{R} + i_B = 0 \qquad (17)$$

for, by introducing the conductance $c = 1/R$,

$$c(u_1 + u_2 + u_3 + u_4 - 4u_0) + i_B = 0 \qquad (17a)$$

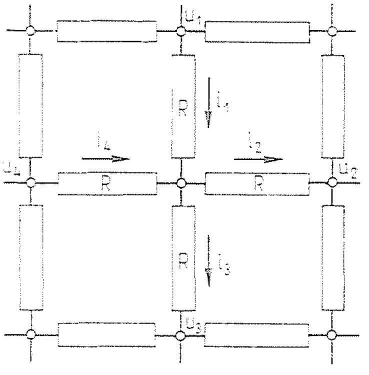


Fig. 9

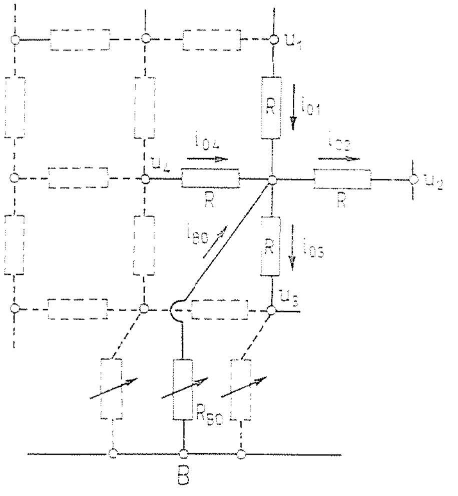


Fig. 10

By comparing Eqs (17) and (4) it can be seen that the linear relationship between voltage u_i and velocity v_i of the i-th node can only be maintained if

$$\frac{u_i}{v_i} = \frac{R}{a^2},$$

a condition to be satisfied easily in case of a square mesh consisting of equal side lengths a and resistors R, respectively, but meaningless in any other case. Therefore it is required to find a principle of general validity, applicable to the design of analogues of arbitrary geometrical pattern, by aid of which the resistors connecting two adjacent nodes can be dimensioned. This goal has already been achieved by TASNY—TSCHIASSNY who derived the relationship

$$c_i = k \cot \alpha_i \qquad (18)$$

for triangles of a network not containing obtuse angles, with c_i being the conductance of a triangle side, k a proportionality factor being constant all over an electrically isotropic continuum (by varying k it becomes possible to simulate stratified currents or other variations of viscosity), and α_i the angle opposite to the side in question (Fig. 11). Obtuse angles should be avoided because of their negative cotangent.

From Eq. (18) also it follows that conductances depend only upon the shape but not upon the size of the triangles.

Now, as said before, sides (except the solid boundary), are generally common with two adjacent mesh units (triangles or quadrangles) involved. Thus, the resistance value of the resistor R connecting two nodes is obtained as the reciprocal value of the sum of conductances $(c' + c'')$ pertaining to the adjacent two mesh units.

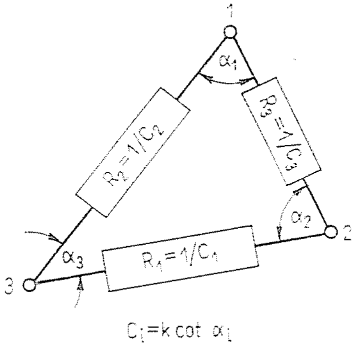


Fig. 11

Opposite to the diagonal of a square or a rectangle, there are two right angles resulting in zero conductance or infinite resistance, thus it is purposeless to subdivide quadrangles into triangles. However, such subdivision carried out theoretically may prove helpful in answering other problems. Thus for instance, this way it can be proved that for a rectangular mesh with a ratio of side lengths $\Delta y/\Delta z = a/b$, the corresponding ratio of resistors should be

$$\frac{R_a}{R_b} = \left(\frac{a}{b}\right)^2 . \tag{19}$$

Influence area

So far, the way of calculating resistor elements of an arbitrary network has been discussed but still the question of currents to be introduced in each node in order to simulate the hydraulic term gS/ν remained open.

In case of a rectangular network, one has the Navier—Stokes equation

$$\frac{v_1 + v_3 - 2v_0}{a^2} + \frac{v_2 + v_4 - 2v_0}{b^2} = -\frac{gS}{\nu} \tag{20a}$$

and the analogous electric equation

$$\frac{u_1 + u_3 - 2u_0}{R_a} + \frac{v_2 + v_4 - 2v_0}{R_b} = i_B\,. \tag{20b}$$

If one introduces the velocity-to-voltage ratio $\beta = v_i/u_i$ expressing the "model scale" between velocity v_i in a point and the voltage u_i in the corresponding node, then, after performing various transformations and rearrangements on Eqs (20), one arrives to

$$i_B = -\frac{gS}{\nu}\,\frac{ab}{\beta\sqrt{R_a R_b}}\,. \tag{21}$$

But in accordance with Eq. (19), there is a relationship between the resistances and the corresponding geometrical lengths:

$$\frac{a_2}{R_a} = \frac{b^2}{R_b} = \gamma$$

and hence (21) will go over into

$$i_B = -\frac{gS}{\nu}\,\frac{\gamma}{\beta} = -\frac{gS}{\nu}\,\delta \tag{22}$$

with δ appropriately called the analogue factor:

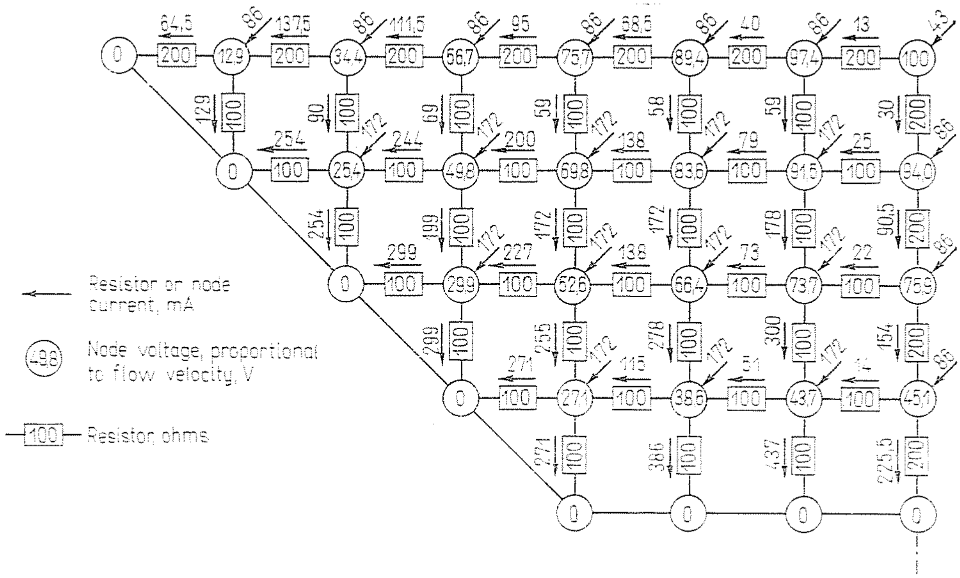
$$\delta = \frac{l^2 u}{Rv}\ [\text{amp. cm. sec.}] \tag{23}$$

But from Eq. (21) it becomes apparent that, apart from certain constants, i_B is proportional to the area ab of the mesh unit. If all these units are of the same size, as it is the case with rectangular, square or equilateral-triangular networks, then the same currents i_B should be introduced into each mode.

On the other hand, if the network is an irregular or mixed one, then each node has a particular *influence area* and currents i_B to be introduced must be linearly proportional to their particular influence area.

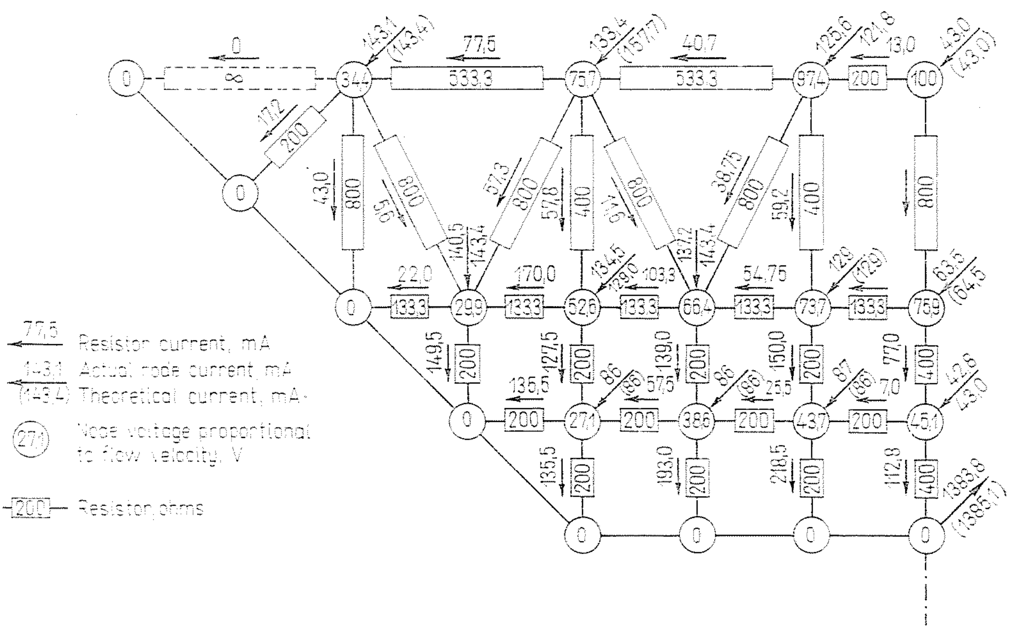
The boundary of influence areas can be drawn easily by connecting alternately the mid-points of mesh sides and the centres of gravity of the mesh field elements (rectangles, triangles, etc.).

Nodes lying along the solid boundary have influence areas of their own, with $i_B = 0$.



Resistor or node current, mA
Node voltage, proportional to flow velocity, V
Resistor, ohms

Fig. 12



Resistor current, mA
Actual node current, mA
Theoretical current, mA
Node voltage proportional to flow velocity, V
Resistor, ohms

Fig. 13

An application of the above principles will be shown by means of the following example:

The velocity distribution of laminar flow in a trapezoidal open channel will be investigated. Due to symmetry it will suffice to consider one half of the cross section only.

At first, the network to be applied should be a square one with mesh side lengths $a = 1$. Fig. 12 shows the resistor values representing the sides, and also the currents flowing through these resistors. Electric potentials measured in each node are indicated as well as currents i_B introduced into the nodes.

The compound network of Fig. 13 has been deliberately designed so as to utilize nodes that also exist in the square mesh. Thus, results can be compared or checked. The agreement between the two results is apparently a close one. The experiment has been carried out in the inverse way, that is, currents i_B to each node have been adjusted so as to obtain the same voltages as with the square network, and on Fig. 13, these actual currents i_B have been indicated together with those calculated from the influence areas. Besides, voltages have been chosen in both cases to be also the percentages of the maximum velocity appearing at the intersection of water surface and profile centre line.

Calculation of discharge

The discharge flowing through a square $a \times a$ can be obtained in the first approximation as

$$\Delta Q = \frac{a^2}{4}(v_1 + v_2 + v_3 + v_4) \tag{24}$$

based upon velocity values obtained in the four corners. It can be proved that by repeated subdividing this square into smaller and smaller ones, the above term will tend towards

$$\lim_{a \to 0} \Delta Q = \frac{a^2}{4}(v_1 + v_2 + v_3 + v_4) + \frac{a^4 gS}{12\nu} \tag{25}$$

As for the trapezoidal profile, calculation resulted in $Q = \Sigma \Delta Q_i = 20.172\ v_{max}$ which amounts to a mean velocity $v_m = 0.5\ v_{max}$ for the 40 area units of Fig. 12. This value agrees with that one well known of a circular profile.

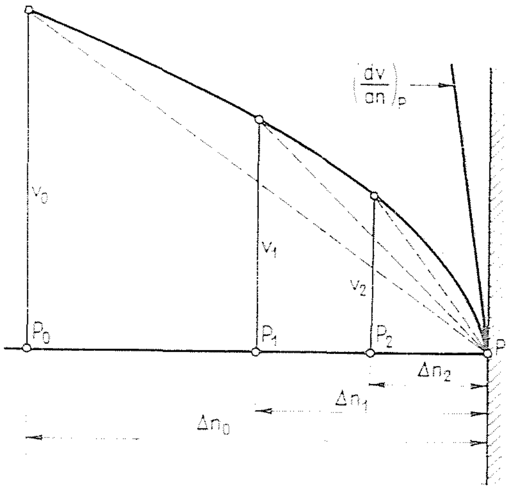


Fig. 14

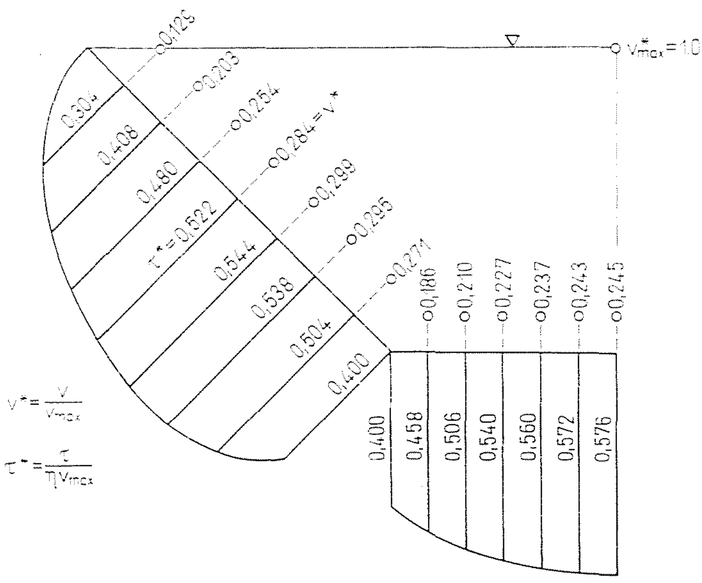


Fig. 15

Wall shear stress

If there is a point P_0 in the vicinity of a solid boundary, with a known velocity v_0, then, by assuming isotach lines to be parallel to the boundary in this environment, (by omitting the details) one can prove that wall shear stress at point P will be

$$\tau_0 = \mu \left|\frac{dv}{dn}\right|_P = \mu \left[\frac{\Delta v_0}{\Delta n_0} + \frac{\Delta n_0}{2} \frac{gS}{\nu}\right]. \tag{26}$$

This result has been obtained by repeated halving of the distance n_0 between P_0 and P, as demonstrated in Fig. 14. Fig. 15 shows wall shear stress distribution of the trapezoidal profile.

Apparatus

The simple electric analogue can be built up on a panel by wiring the resistors of previously calculated values. The adjustment of the node currents i_B can be a manual one, resembling to manual relaxation but much quicker than the numerical method, using variable resistors checked by an ammeter. But, in co-operation with the Department of Theoretical Electricity, the author succeeded in developing an automatic transistorized current stabilizer. If connecting to each node a stabilized one adjusted previously to the calculated value of i_B, immediate results can be obtained through measuring node voltages.

Acknowledgement

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Summary

The exact analytical solution of the Navier-Stokes equation is only known for a few profiles of a regular geometrical shape. The author has developed an electric analogue by aid of which the velocity distribution of laminar flow for any arbitrary profile can easily be determined. The homogeneous part of the differential equation is simulated by a network of resistors being much the same as the one used for modelling potential flow, whilst the inhomogeneous term gS/ν is simulated by currents introduced into the nodes of the network. Beyond the general theory and its conclusions, some practical applications are also described.

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