CONTRIBUTION TO THE ANALYSIS OF STRESSES IN PRESTRESSED CONCRETE BEAMS

By

T. Klatsmányi and G. Tassi

Department of Reinforced Concrete Structures, Technical University, Budapest

(Received April 8, 1970)

Presented by Prof. Dr. E. Bölcskei

1. Introduction

Design specifications for prestressed concrete beams prescribe the determination of stresses due to any load in the concrete and prestressing steel. The calculation should, according to certain specifications, confirm the restriction of the residual deformations, and according to other ones, assure the load capacity by limiting the stresses. Calculations may be carried out in order to check the cracklessness, to restrict cracking and to determine the quantity of ordinary reinforcement required to withstand tensile stresses.

Provided the materials are linearly elastic, the stresses in a homogeneous cross-section may readily be calculated by applying the elementary methods of the strength of materials. Determination of a bending moment associated with the deformation of a concrete fibre — let only by the method of successive approximations — is rather simple even in the case of a cracked beam. However, the evaluation of stresses caused by a moment higher than that producing cracks by taking into account the cracking, requires tedious calculation work [1]. Therefore, the approximation to consider a cracked beam to be homogeneous is a current practice in stress analysis. This approximation permits to check particularly easily the stress induced in the extreme fibre. For the application of approximate procedure [2] it is required to justify this basic assumption. The approximation may be justified by the limitation of the calculated (so-called fictitious) tensile stress or by the application of ordinary reinforcement to withstand the resultant of the calculated concrete tensile stresses [3].

The problem becomes especially difficult in the case where the restriction of the compressive stresses in the concrete and steel stresses should be pointed out for a moment significantly greater than the cracking moment. This is the case also with the Hungarian Code for Highway Bridges of 1968 [4, 5] which, nevertheless, permits to calculate the stresses even in the cracked beam by making use of the relationships valid to crackless beams.

Here, it is intended to investigate on what conditions the assumption of a homogeneous cross-section in analysing the stresses is permissible without making a gross error, *i.e.*, what a correction should be applied in the case of approximate evaluation to obtain a more exact result.

Symbols

b [cm] — width of flange of T or I beams;
b_1 [cm] — web width of rectangular, T or I beams;
b_2 [cm] — width of bottom flange of I beam;
v [cm] — thickness of top flange of T or I beam;
u [cm] — thickness of bottom flange of I beam (without concrete cover);
h [cm] — effective (approximately total) depth of beam;
x [cm] — distance from neutral axis to top fibre;
F_s [cm^2] — cross-sectional area of prestressing steel;
N_c [kp] — resultant of the compressive stresses in concrete due to prestressing force and external loads;
T_s [kp] — resulting tensile force due to prestress and external loads (for cracked beams only in the prestressing steel; for a crackless beam both in prestressing steel and concrete);
P_p [kp] — effective prestressing force;
M [kp/cm] — calculated bending moment due to external load;
σ_c [kp/cm^2] — stress calculated in top fibre;
σ_s^{ζ} [kp/cm^2] — surplus part of stress in prestressing steel above prestress;
σ_c^s [kp/cm^2] — stress in concrete at level of the centre of gravity of prestressing steel;
σ_r [kp/cm^2] $= \dfrac{P_p}{b_1 h}$;
n [ ] — ratio of moduli of elasticity of prestressing steel to concrete;
ξ [ ] $= \dfrac{x}{h}$;
β_1 [ ] $= \dfrac{b_1}{b}$;
β_2 [ ] $= \dfrac{b_2}{b}$;
$\varkappa$ [ ] $= \dfrac{v}{h}$;
$\varkappa_2$ [ ] $= \dfrac{u}{h}$;
μ [ ] $= \dfrac{F_s}{b_1 h}$;
γ [ ] $= \dfrac{M}{P_p\, h}$.

2. Basic assumptions

The analysis of stresses is based on the following assumptions:

a) Plane cross sections remain plane in bending (validity of the Bernoulli—Navier hypothesis).

b) The stress-strain diagrams for both concrete and steel are linear (validity of Hooke's law).

c) There is a perfect bond between prestressing bars and concrete.

d) The cross-section of the beam is symmetric, with the axis of symmetry in the plane of bending.

e) There is no or negligible ordinary reinforcement.

f) Analysis of the cracked cross-section is done with the tensile strength of concrete omitted.

g) Analysis of the non-cracked cross-section is done with the concrete cover omitted or with the assumption of all prestressing wires aligned in the extreme fibre subject to tension by an external load.

It should be noted that in our investigations the problem of losses of prestress is not treated of, therefore the variation of the number n derivable according to our assumptions will not be discussed either.

3. Analysis of the cracked cross-section

For analyzing the stresses in the cracked cross-section, the depth of the neutral axis should be determined. In stress state II following cracking, this is achieved for the general case by finding a straight line normal to the symmetry axis in the plane of bending and bordering a concrete area, straight line along which the value of stresses is zero as calculated with the areas of concrete in compression and prestressing steel [6].

Stresses written for simple cross-sections lead to relationships involving the distance of the neutral axis from the top fibre but this is given in an implicit form only (*e.g.* in [7]). In the following we shall proceed this way.

3.1 Choice of the cross-section to be analyzed

For evaluation of the stresses it is practicable to write the relationships for the T section shown in Fig. 1. Thus, in the case of $b = b_1$ or $v = 0$ they are valid for a rectangular section. The T section is equivalent to the I or box section in the case if the neutral axis intersects the web.

3.2 Relationships for the determination of stresses

Stresses will be determined according to the basic assumptions in Section 2. The stresses are represented in Fig. 1.

The stress excess in the prestressing steel above the prestress — corresponding to the joint deformation of steel and concrete — is:

$$\sigma_s^c = n\sigma_c^s = n\sigma_c^t \frac{h-x}{x}, \tag{1}$$

the compressive force:

$$N_c = \frac{\sigma_c^t}{2}\left[bx - \frac{b-b_1}{x}(x-v)^2\right], \tag{2}$$

and the tensile force:

$$T_s = P_p + F_s \sigma_s^c . \tag{3}$$

From the projection equilibrium condition $N_c = T_s$, the stress in the extreme fibre of the concrete is given by

$$\sigma_c^t = \frac{2P_p x}{bx^2 - (b - b_1)(x - v)^2 - 2nF_s(h - x)} . \tag{4}$$

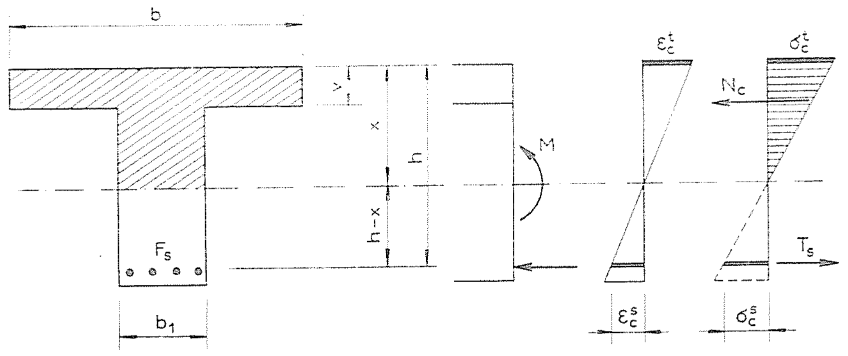


Fig. 1

From the condition of the equilibrium of moments about the neutral axis, by substitution of the expression (4) we obtain

$$2P_p \frac{\dfrac{bx^3}{3} - \dfrac{b - b_1}{3}(x - v)^3 + nF_s(h - x)^2}{bx^2 - (b - b_1)(x - v)^2 - 2nF_s(h - x)} = M - P_p(h - x) .$$

By introducing the parameters using symbols in Section 1, the stress in the extreme fibre of the concrete will be given in the form

$$\sigma_c^t = \frac{2\sigma_r \xi}{\dfrac{\xi^2}{\beta_1} - \dfrac{1 - \beta_1}{\beta_1}(\xi - \varkappa)^2 - 2n\mu(1 - \xi)} .$$

The value of ξ in this equation may be determined by the implicit expression

$$\frac{\dfrac{2}{3}\dfrac{\xi^3}{\beta_1} - \dfrac{2}{3}\dfrac{1 - \beta_1}{\beta_1}(\xi - \varkappa)^3 + 2n\mu(1 - \xi)^2}{\dfrac{\xi^2}{\beta_1} - \dfrac{1 - \beta_1}{\beta_1}(\xi - \varkappa)^2 - 2n\mu(1 - \xi)} = \gamma - (1 - \xi) .$$

In the particular case of a rectangular cross-section, for $\beta_1 = 1$ and $\varkappa = 0$, the relationships become simplified to:

$$\sigma_c^t = \frac{2\sigma_r \xi}{\xi^2 - 2n\mu(1-\xi)}$$

and

$$\frac{\frac{2}{3}\xi^3 + 2n\mu(1-\xi)^2}{\xi^2 - 2n\mu(1-\xi)} = \gamma - (1-\xi)\,.$$

4. Stresses in the crackless cross-section

In the case of a crackless cross-section, the place of the neutral axis has a significance different from that of a cracked beam. The stresses may be evaluated by the elementary methods of the strength of materials, making allowance for the basic assumptions in Section 2. Naturally, the same values are obtained by introducing ideal cross-sectional characteristics usual for the calculation of crackless prestressed beams.

4.1 Assumption of the cross-section

Since in the case of a crackless cross-section also the bottom flange is of importance for the development of the stresses, the investigations will concern the general case of I sections symmetrical about a single axis (Fig. 2). For $b = b_1 = b_2$ or $v = u = 0$, this cross-section is a rectangular one. For $b_2 = b_1$, or $u = 0$, it is a T section. A box girder subject to symmetric bending may be calculated in the same way as an I beam.

4.2 Relationships for the calculation of the stresses

If the beam is subject to an external bending moment M, and to a prestressing force P_p (Figs. 1 and 2) by taking into account the stress σ_s^c acting in the prestressing steel [1], and provided the neutral axis intersects the web, the force N_c may be given by Eq. (2) where, as a matter of course, x is still unknown. The tensile force is:

$$T_s = \frac{\sigma_c^t}{2}\frac{h-x}{x}\left[b_2(h-x) - (b_2 - b_1)\frac{(h-x-u)^2}{h-x}\right] +$$
$$+ P_p + nF_s\sigma_c^t\frac{h-x}{x}\,.$$

The concrete top stress may be obtained from the condition of projection equilibrium:

$$\sigma_c^t = \frac{2P_p x}{bx^2 - (b-b_1)(x-v)^2 - b_2(h-x)^2 + (b_2-b_1)(h-x-u)^2 - 2nF_s(h-x)} .$$

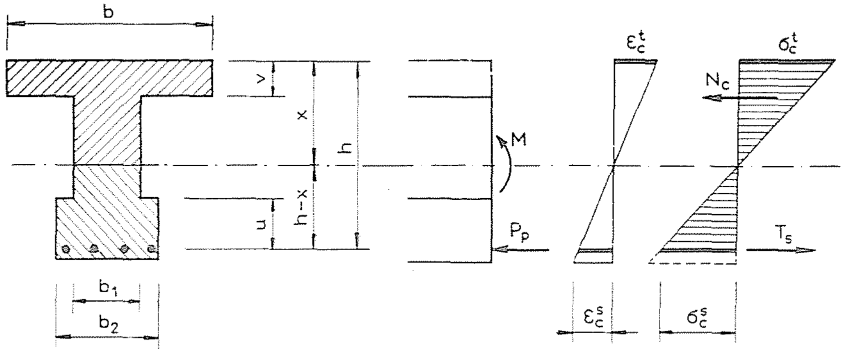


Fig. 2

From the equilibrium of the moments about the neutral axis:

$$\frac{2P_p\left[\frac{bx^3}{3} - \frac{(b-b_1)(x-v)^3}{3} + \frac{b_2(h-x)^3}{3} - (b_2-b_1)(h-x-u)^3 + nF_s(h-x)^2\right]}{bx^2 - (b-b_1)(x-v)^2 - b_2(h-x)^2 + (b_2-b_1)(h-x-u)^2 - 2nF_s(h-x)} =$$

$$= M - P_p(h-x) .$$

Introducing the parameters using symbols in Section 1:

$$\sigma_c^t =$$

$$= \frac{2\sigma_r \xi}{\frac{\xi^2}{\beta_1} - \frac{1-\beta_1}{\beta_1}(\xi - \varkappa)^2 - \frac{\beta_2}{\beta_1}(1-\xi)^2 + \left(\frac{\beta_2}{\beta_1} - 1\right)(1-\xi-\varkappa_2)^2 - 2n\mu(1-\xi)} , \tag{5}$$

and the relationship implicit for ξ

$$\frac{\frac{2}{3}\frac{\xi^3}{\beta_1} - \frac{2}{3}\frac{1-\beta_1}{\beta_1}(\xi-\varkappa)^3 + \frac{2}{3}\frac{\beta_2}{\beta_1}(1-\xi)^3 - \frac{2}{3}\left(\frac{\beta_2}{\beta_1}-1\right)(1-\xi-\varkappa_2)^3 + 2n\mu(1-\xi)^2}{\frac{\xi^2}{\beta_1} - \frac{1-\beta_1}{\beta_1}(\xi-\varkappa)^2 - \frac{\beta_2}{\beta_1}(1-\xi)^2 + \left(\frac{\beta_2}{\beta_1}-1\right)(1-\xi-\varkappa_2)^2 - 2n\mu(1-\xi)} =$$

$$= \gamma - (1-\xi) . \tag{6}$$

For an I section symmetric about two axes

$$\beta_2 = 1; \quad \varkappa = \varkappa_2$$

and thus

$$\sigma_c^t = \frac{2\sigma_r \xi}{\frac{1}{\beta_1}\left[\xi^2 - (1-\xi)^2\right] - \frac{1-\beta_1}{\beta_1}\left[(\xi-\varkappa)^2 - (1-\xi-\varkappa)^2\right] - 2n\mu(1-\xi)}$$

and

$$\frac{\frac{2}{3\beta_1}\left[\xi^3 + (1-\xi)^3 - \frac{2}{3}\frac{1-\beta_1}{\beta_1}\right]\left[(\xi-\varkappa)^3 + (1-\xi-\varkappa)^3\right] + 2n\mu(1-\xi)^2}{\frac{1}{\beta_1}\left[\xi^2 - (1-\xi)^2\right] - \frac{1-\beta_1}{\beta_1}\left[(\xi-\varkappa)^2 - (1-\xi-\varkappa)^2\right] - 2n\mu(1-\xi)} =$$

$$= \gamma - (1-\xi).$$

In the case of a T section

$$\frac{\beta_2}{\beta_1} = 1; \qquad \varkappa_2 = 0,$$

hence

$$\sigma_c^t = \frac{2\sigma_r \xi}{\frac{\xi^2}{\beta_1} - \frac{1-\beta_1}{\beta_1}(\xi-\varkappa)^2 - (1-\xi)^2 - 2n\mu(1-\xi)},$$

$$\frac{\frac{2}{3}\frac{\xi^3}{\beta_1} - \frac{2}{3}\frac{1-\beta_1}{\beta_1}(\xi-\varkappa)^3 + \frac{2}{3}(1-\xi)^3 + 2n\mu(1-\xi)^2}{\frac{\xi^2}{\beta_1} - \frac{1-\beta_1}{\beta_1}(\xi-\varkappa)^2 - (1-\xi)^2 - 2n\mu(1-\xi)} = \gamma - (1-\xi).$$

Substitution of

$$\beta_1 = \beta_2 = 1; \quad \varkappa = \varkappa_2 = 0$$

into Eqs (5) and (6) delivers for the stresses in the top fibre of the rectangular cross-section:

$$\sigma_c^t = \frac{2\sigma_r \xi}{\xi^2 - (1-\xi)^2 - 2n\mu(1-\xi)},$$

and

$$\frac{\frac{2}{3}\xi^3 + \frac{2}{3}(1-\xi)^3 + 2n\mu(1-\xi)^2}{\xi^2 - (1-\xi)^2 - 2n\mu(1-\xi)} = \gamma - (1-\xi).$$

The steel stresses may be determined in the same way.

5. Comparison of stresses obtained by considering the beams crackless and by taking cracks into account, respectively

The very aim of our investigation is to establish the differences in stresses when, according to the theory of elasticity, the cracking of the tensile zone is neglected and the whole cross-section is taken into account. Comparative calculations of a general validity being almost impossible by elementary means, our investigations are restricted to some particular, typical cases.

5.1 Cross-sections and parameters analyzed

The case of the rectangular section in Fig. 3 has been investigated for

$$n\mu = 0.05;\ 0.10;\ 0.20$$

and

$$\sigma_r = 50;\ 70;\ 100\ \mathrm{kp/cm^2}.$$

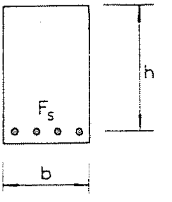


Fig. 3

The assumed values correspond to prestressing steel percentages of 0.5 to 1.5, commonly applied in practice.

The parameters of the T section shown in Fig. 1 are:

$$n\mu = 0.10;\ 0.20;\ 0.30$$
$$\beta_1 = 0.1$$
$$\varkappa = 0.1$$
$$\sigma_r = 100;\ 150;\ 200\ \mathrm{kp/cm^2}.$$

For the I section symmetric about two axes (Fig. 4):

$$n\mu = 0.10;\ 0.20;\ 0.30$$
$$\beta_1 = 0.1$$
$$\varkappa = 0.1$$
$$\sigma_r = 100;\ 200;\ 300\ \mathrm{kp/cm^2}.$$

5.2 Relative depth of the compression zone and relationship between the relative moments and concrete stresses for different prestresses and different percentages of prestressing steel

For the considered cross-sectional parameters and concrete stresses due to prestressing, the values of ξ have been determined in dependence of γ,

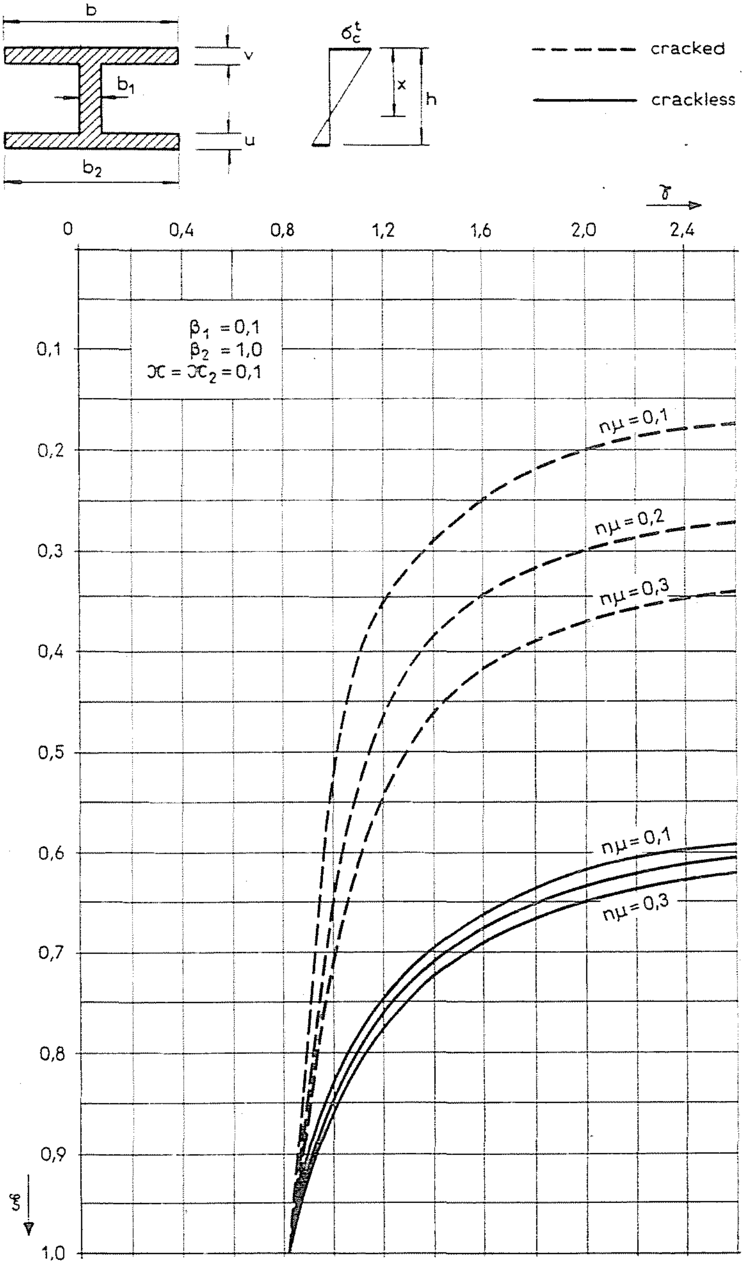

b
v
b1
u
b2
σct
x
h
cracked
crackless
γ
0
0,4
0,8
1,2
1,6
2,0
2,4
β1 = 0,1
β2 = 1,0
ϰ = ϰ2 = 0,1
0,1
0,2
0,3
0,4
0,5
0,6
0,7
0,8
0,9
ξ
1,0
nμ = 0,1
nμ = 0,2
nμ = 0,3
nμ = 0,1
nμ = 0,3

Fig. 4

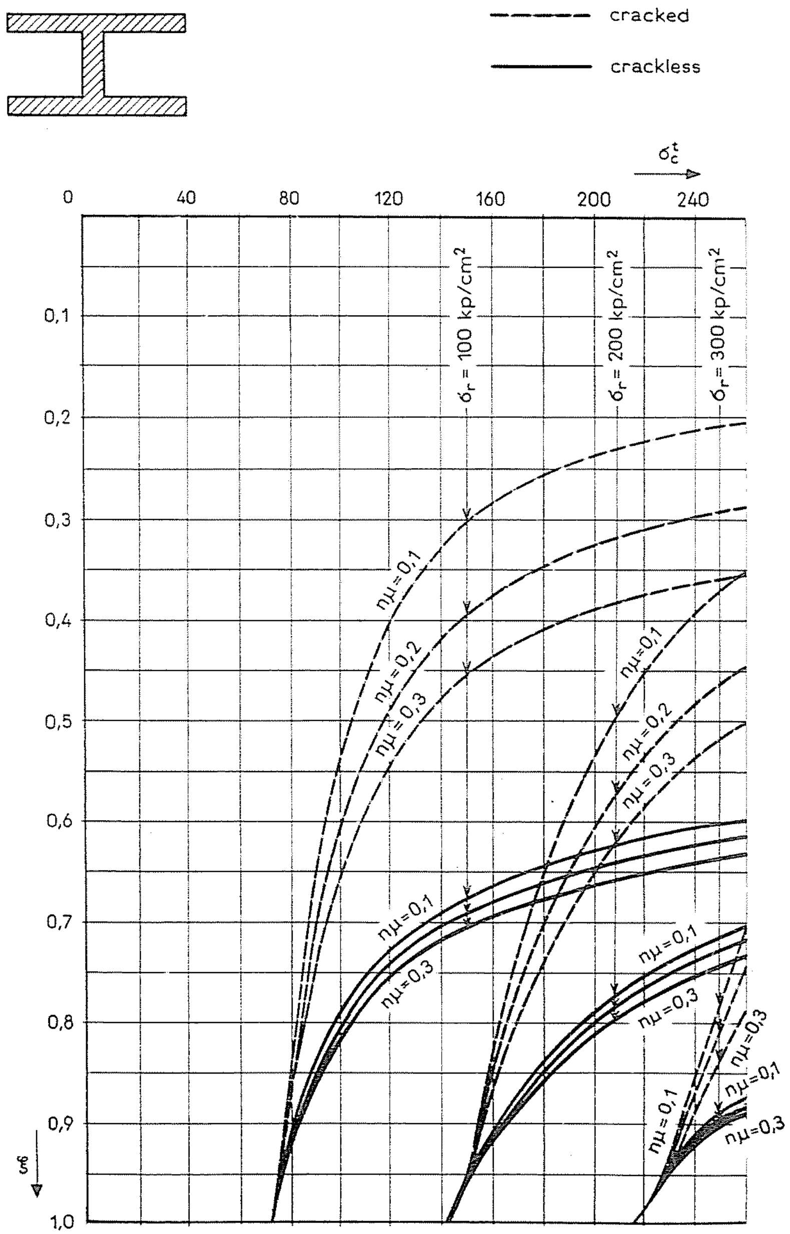

cracked
crackless
σ_c^t
0
40
80
120
160
200
240
0,1
0,2
0,3
0,4
0,5
0,6
0,7
0,8
0,9
1,0
ξ
σ_r = 100 kp/cm²
σ_r = 200 kp/cm²
σ_r = 300 kp/cm²
nμ=0,1
nμ=0,2
nμ=0,3
nμ=0,1
nμ=0,3
nμ=0,1
nμ=0,2
nμ=0,3
nμ=0,1
nμ=0,3
nμ=0,3
nμ=0,1
nμ=0,1
nμ=0,3

Fig. 5

for $0 \leq \xi \leq 1$ and $0 \leq \gamma \leq 2.6$, together with the relationships between ξ and σ_c^t in the ranges $0 \leq \xi \leq 1$ and $0 \leq \sigma_c^t \leq 260$ kp/cm^2 of practical interest.

The analysis has concerned a symmetric I section and the results are represented in Figs 4 and 5.

5.3 *Beam stress error due to neglecting the cracking*

The stresses acting in cracked or crackless cross-sections may be compared by reading the values belonging to the cracked and the crackless cross-sections, respectively, on the ξ, γ curves at the γ values corresponding to the given external moment and prestress, and applying the ξ, σ_c^t curves at the corresponding σ_r to determine the difference between the σ_c^t values for the two cases (Fig. 6).

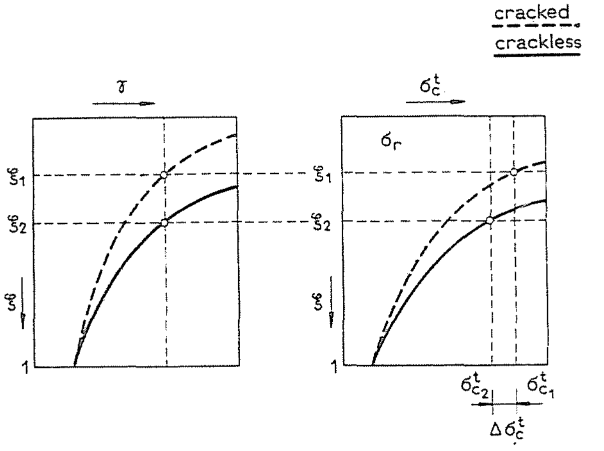
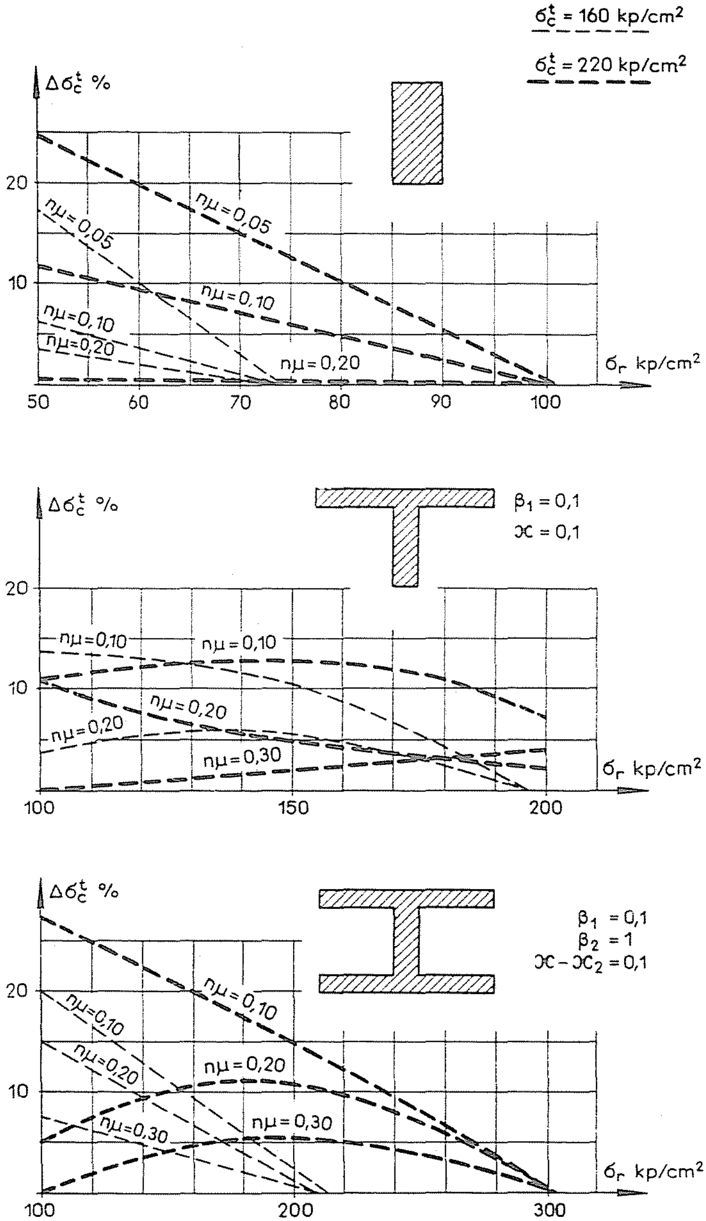


Fig. 6

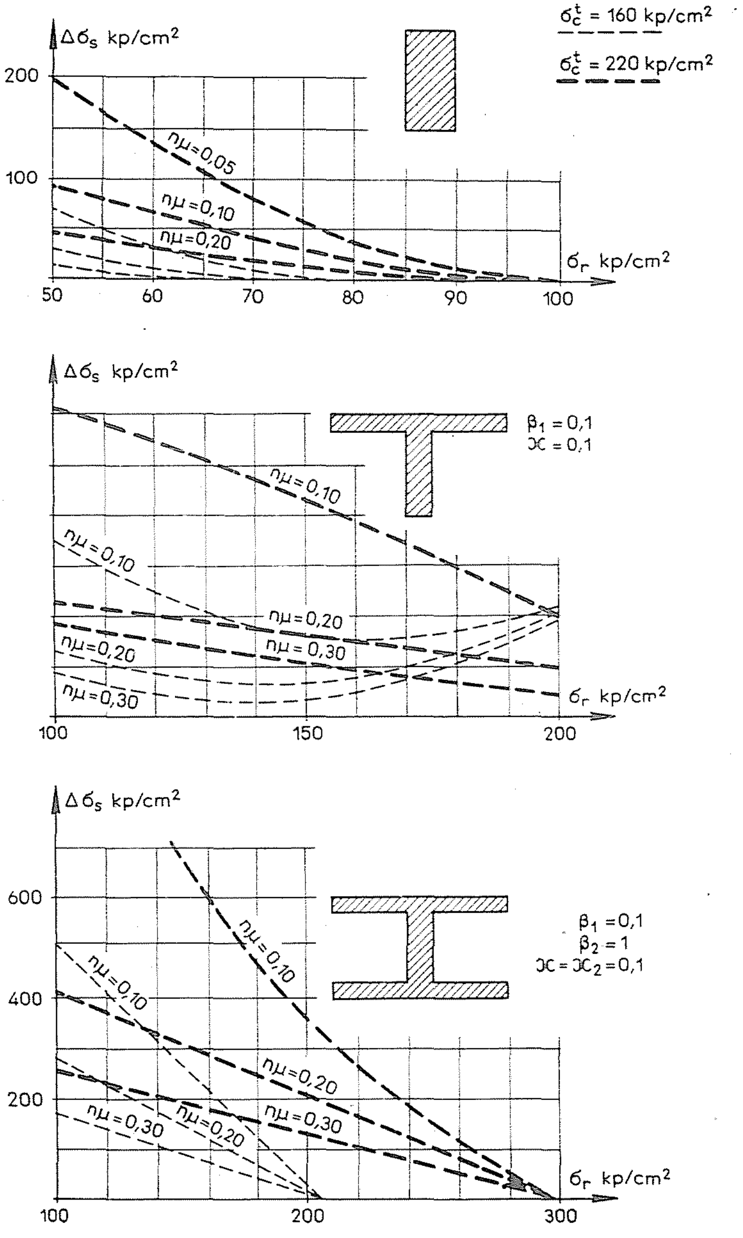
The error made by assuming cracklessness in the calculation of stresses in the prestressing steel is:

$$\Delta\sigma_s = n\left(\sigma_{c\ \mathrm{cracked}}^t \frac{1-\xi_{\mathrm{cracked}}}{\xi_{\mathrm{cracked}}} - \sigma_{c\ \mathrm{crackless}}^t \frac{1-\xi_{\mathrm{crackless}}}{\xi_{\mathrm{crackless}}}\right).$$



σ$_c^t$ = 160 kp/cm²
σ$_c^t$ = 220 kp/cm²
Δσ$_c^t$ %
20
10
nμ=0,05
nμ=0,05
nμ=0,10
nμ=0,10
nμ=0,20
nμ=0,20
σ$_r$ kp/cm²
50
60
70
80
90
100
Δσ$_c^t$ %
β$_1$ = 0,1
ϰ = 0,1
20
10
nμ=0,10
nμ=0,10
nμ=0,20
nμ=0,20
nμ=0,30
σ$_r$ kp/cm²
100
150
200
Δσ$_c^t$ %
β$_1$ = 0,1
β$_2$ = 1
ϰ - ϰ$_2$ = 0,1
20
10
nμ=0,10
nμ=0,10
nμ=0,20
nμ=0,20
nμ=0,30
nμ=0,30
σ$_r$ kp/cm²
100
200
300

Fig. 7



Δσs kp/cm²
200
100
σc^t = 160 kp/cm²
σc^t = 220 kp/cm²
nμ=0,05
nμ=0,10
nμ=0,20
σr kp/cm²
50
60
70
80
90
100
Δσs kp/cm²
β1 = 0,1
ϰ = 0,1
nμ=0,10
nμ=0,10
nμ=0,20
nμ=0,20
nμ=0,30
nμ=0,30
σr kp/cm²
100
150
200
Δσs kp/cm²
600
400
200
β1 = 0,1
β2 = 1
ϰ = ϰ2 = 0,1
nμ=0,10
nμ=0,10
nμ=0,20
nμ=0,20
nμ=0,30
nμ=0,30
σr kp/cm²
100
200
300

Fig. 8

5.4 *Evaluation of the comparative analysis*

The method in Section 5.3 has been applied to three typical cross-sections to determine the error made by considering the cracked cross-section to be crackless in the calculation of stresses in the compressed extreme fibre of concrete and in the prestressing steel.

Calculated stresses are plotted in Figs 7 and 8 for concrete and steel, respectively. The error due to the neglection of cracking is seen to decrease with the increase of the prestressing steel percentage and the prestress.

It should be noted that the mean concrete stress due to prestressing is σ_r for a rectangular section, $\frac{\sigma_r}{1{,}9}$ for a T section and $\frac{\sigma_r}{2{,}8}$ for an I section.

It can be concluded that at or over half of the design (permissible or limit) stress, the error in the mean value of the concrete stress due to prestressing calculated from a sound cross-section is less than 10 per cent. The error decreases by increasing the prestressing force using a higher percentage of prestressing steel, rather than by applying a higher prestress using a higher tensile prestressing steel.

When the mean concrete compressive stress calculated from the effective prestressing force reaches the half of design stress, the error made in calculating the stress in the prestressing steel will be less than $n \cdot 150$ to $n \cdot 200$ kp/cm^2. This is a special justification of setting a minimum for the prestressing force. It should be noted that other viewpoints, for example the elimination of the risk of brittle failure as upper limit leads to similar conclusions [8] and thus, requirements for the effective prestressing force or mean stress in concrete due to prestressing are likely to be advantageous not only with a view to simplified calculation.

Summary

From the analysis of certain typical, generalizable cases it is stated that stresses in bonded-wire prestressed concrete beams may closely be approximated by assuming cracklessness even at moments higher than that of cracking when about half value of the design (permissible or limit) concrete stress is reached. Otherwise, a more exact calculation is required for which the basic relationships also are presented in this paper.

References

1. TASSI, G.: Limit Analysis of Prestressed Concrete Beams. Periodica Polytechnica C. E. **13** (1969) 83–92.
2. BÖLCSKEI, E.—TASSI, G.—KLATSMÁNYI, T.: Reinforced Concrete Structures. Design of Prestressed Beams.* Tankönyvkiadó, Budapest, 1964.
3. Betonkalender 1967. W. Ernst. Berlin, 1967.

4. Ministry of Transport and Post. HI/1—68. Code for Highway Bridges.* Közlekedési Dokumentációs Vállalat, Budapest, 1968.
5. TASSI, G.: Specifications for Prestressed Concrete Structures of the new Code for Highway Bridges.* Mélyépítéstudományi Szemle XVIII 502—505 (1968).
6. BÖLCSKEI, E.—TASSI, G.: Reinforced Concrete Structures. Prestressed Beams.* Tankönyvkiadó, Budapest, 1970.
7. HAMPE, E.: Vorgespannte Konstruktionen. Band I. VEB Verlag für Bauwesen, Berlin 1964.
8. WINDISCH, A.: Safety to Brittle Failure in Prestressed Concrete Structures with Bond. Periodica Polytechnica C. E. **14** (1970)

* In Hungarian.

Associate Professor Dr. Géza TASSI
Sr. Assistant Professor Dr. Tibor KLATSMÁNYI
Budapest XI., Sztoczek u. 2, Hungary