POTENTIAL DEVELOPMENTS IN THE THEORY OF SUSPENDED SEDIMENT MOVEMENT

By

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1. Introduction

During the past few years several papers have been devoted to the description of the investigation results [13, 14, 15] which, together with a critical review, outlined a few potential approaches for the suggested development of the generalized theory on turbulent sediment transportation. Some of the more recent concepts will be considered below, and their detailed physical-analytical expansion will be described. For the sake of completeness it is deemed advisable to present a brief historical review of the problem.

During the development of the familiar *diffusion* theory, VELIKHANOV [24], and later ISMAIL [11], NING CHIEN [16], HUNT [10] and others, started essentially from a semi-empirical turbulent theory by determining the value of ε from the logarithmic distribution of velocity. The shortcomings of this approach were pointed out first in Hungary by BOGÁRDI [2].

Other authors, e.g. DOBBINS [4], and ROUSE [18], introduced a simplifying assumption by considering the turbulent mixing coefficient to be equal for both phases.

After a detailed analysis of the problem, based on the theoretical works of KOLMOGOROV and OBUHOV it was concluded by TEVEROVSKY and MINSKY (1952) that the turbulent mixing coefficient could be characterized by the ratio of the settling to the mean velocity. The above considerations have led to the conclusion that the diffusion theory is applicable with a fair approximation up to $d < 0.5$ mm.

Subsequent experiments of VELIKHANOV induced him to modify his earlier equations (1953) and to propound the advantages of the *gravitational* theory. The new approach was criticized, however, besides the advocates of the KOLMOGOROV theory, also by the protagonists of the diffusion theory. Earlier developments of this controversy have repeatedly been described [2, 13].

The theoretical development of the problem is due to BARENBLATT [1], who introduced into the earlier sets of equations a new energy equation, in which the work of suspension was already included in the sum of pulsation energy. The idea is perfectly acceptable, since in agreement with the classic experiments of VANONI, the intensity of pulsation decreases with increasing concentration. This phenomenon has been investigated in detail at the laboratory of the *Department for Water Resources* in the years 1960 to 1962, and the above theorem was fully corroborated by the experimental results obtained. However, the solution of the set of equations proposed by BARENBLATT leads eventually back to the diffusion theory, the only difference being that the KÁRMÁN coefficient $\varkappa$ is a function of concentration as well.

A substantially well founded and theoretically exact suggestion has been developed by G. TELETOV [22], who defined the tensor of turbulent stresses and the interaction between the two phases by semi-empirical relationships. Aside from the results of research concerned purely with sediment transportation, highly interesting information has been offered by the investigations of SLESKIN [20], who derived as a particular case of the seepage problem the differential equation of mixture transfer and developed the theorem of continuity for the laminar movement of both phases. The same problem has also been analysed by H. A. RAHMATULIN [17], who considered the movement of multi-phase mixtures by assuming them to be compressible.

Considering the problem as a whole, a substantially new approach to the problem is followed in the work of FRANKL [7, 8], who derived, besides the continuity and dynamical equations for the two phases, also the energy equations in an exact manner, and the early history of research on this problem is thus essentially concluded. Entirely similar conclusions have been arrived at by DEEMTER and LAAN [3], in deriving the energy, movement and continuity equations for both phases, assuming laminar motion.

More positive results have been achieved by SANOYAN and ANANYAN [19] in the solution of the basic equations, and although limited to the case of highly concentrated mixtures flowing under pressure, one of the possible fundamental alternatives of a solution is offered. Investigations by the author connected to this stage of theoretical development. For the case of turbulent flow of variable velocity and concentration in open channels the author succeeded in developing the generalized Reynolds equations [14], and relying on his experiments, in suggesting one of the possible solutions.

For obtaining a more comprehensive understanding of various aspects of the problem, the results of DZHRBASHYAN [5] should be considered of great interest. By analysing the relative velocities of the two phases he offered, on the basis of experimental evidence, a novel solution for the vector equations of HASKIND.

In fact, disregarding the early concepts of STOKES (1856) and MAYER (1871), no substantial advances have been made in this problem up to 1947 [2]. By assuming the validity of the linear and non-linear resistance law and introducing the variation of the relative velocity according to a specific, periodic relationship, the mutual influence between the particles of the two phases can be traced back to the phenomenon of dispersion. Still the analysis of turbulent diffusion, representing a closer approximation to the substance of the problem, is encountered first in the works of VI-CHENG-LIU [23] only, who applied analytical methods relating to random phenomena together with relationships describing the periodicity of turbulence.

The equation describing the movement of a solid particle moving alone is derived in an analytically exact form by HASKIND [9] for both the linear and non-linear ranges of resistance and considering — at least for the time being — an infinite field of motion and uniform movement. The theory is developed, however, far enough to define the relationships of relative velocity even for the cases characterized by pulsation of different frequency in turbulent flow. The methods of operator calculus are applied for solving the fundamental equations, assuming that turbulent pulsations are of a periodic character and that the extent of turbulence can be described by harmonic functions.

Similar equations have been introduced also by PANTSHEV [16], who investigated the movement of raindrops in the atmosphere. A solution is presented on the basis of considerations relating to probability theory of random phenomena concerning the distribution of pulsation velocity components of the water droplets and the air.

In the foregoing it has been attempted to present a sketchy, yet essentially complete description of the development that has occurred so far in the theory of suspended sediment transportation. Hereafter it is deemed advisable, and at the same time feasible, to summarize the theoretical foundations of the problem and to develop therefrom the solutions available at different boundary conditions. Subsequently the potential trends of future research can be outlined.

2. Theoretical foundations of the problem

To begin with, the concepts introduced will be defined and the rules of the necessary averaging operations will be described [7, 8].

The fluid and the sediment particles will be regarded as incompressible. The densities of water and sediment particles will be denoted by ϱ and ϱ_c, respectively. In the conventional system of x_1, x_2, x_3 co-ordinates the velocity components of the fluid and solid phases be u_1, u_2, u_3 and u_{c1}, u_{c2}, u_{c3}, respectively. The inertia forces related to unit mass will be accordingly X_i and X_{ci} $(i = 1, 2, 3)$. The tensor of transient stresses (which will be considered continuous) arising within the interior of the fluid, as well as of the solid particles will be p_{ik} $(i, k = 1, 2, 3)$.

A discontinuity function c will furthermore be introduced, which equals in the interior of solid particles unity, while assumes zero value in the fluid.

In the course of subsequent averaging operations this function will define the concentration c.

The rules of averaging will hereafter be reviewed. For this purpose a four-dimensional cylinder $Z(\bar{x}, \bar{t})$ is ascribed around each point of the four-dimensional space $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{t})$ i.e.,

$$\sum_{i=1}^{3} (x_i - \bar{x}_i)^2 \leqq r^2 ; \quad |t - \bar{t}| < \Delta t, \tag{1}$$

wherein r and Δt are fixed quantities.

The usual averaging form is

$$\bar{f}(\bar{x},\bar{t}) = \frac{\iiiint\limits_{Z(\bar{x},\bar{t})} f \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 \, \mathrm{d}t}{\iiiint\limits_{Z(\bar{x},\bar{t})} \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 \, \mathrm{d}t} . \tag{2}$$

Averaging according to the spaces occupied by the fluid and the particles will be performed subsequently according to the following relationships:

$$f^* = \frac{\overline{f(1-c)}}{1-\bar{c}} ; \quad f_c^* = \frac{\overline{fc}}{\bar{c}} . \tag{3}$$

The *continuity equation* obtained for the condition of incompressible solid particles is

$$\frac{\mathrm{d}}{\mathrm{d}t} = \iiiint\limits_{F(x_1,x_2,x_3)<0} c(x_1, x_2, x_3, t) \, \mathrm{d}\bar{x}_1 \, \mathrm{d}\bar{x}_2 \, \mathrm{d}\bar{x}_3 = - \iint\limits_{F(x_1,x_2,x_3)=0} c(x_1 x_2 x_3 t) \, v_{cn}(x_1, x_2, x_3, t) \, \mathrm{d}F \tag{4}$$

wherein v_{cn} — components of velocity v_c in the direction of the outward normal,

$\mathrm{d}F$ — elementary part of the surface $F(x_1, x_2, x_3) = 0$ defined arbitrarily,

$F(x_1, x_2, x_3) < 0$ — the internal area of the above surface.

Introducing under the integral sign the substitutions

$$x_i \to \bar{x}_i + \xi_i ; \quad t \to \bar{t} + \tau \tag{5}$$

where ξ_i, τ are constants, while $\bar{x}_i, \bar{t}$ variable values, from Eq. (4) we have

$$\frac{\mathrm{d}}{\mathrm{d}\bar{t}} \iiint\limits_{F(\bar{x}_1,\bar{x}_2,\bar{x}_3)<0} c(x_1+\xi_1, \ldots, \bar{t}+\tau)\,\mathrm{d}\bar{x}_1\,\mathrm{d}\bar{x}_2\,\mathrm{d}\bar{x}_3 = \\ = - \iint\limits_{F(\bar{x}_1,\bar{x}_2,\bar{x}_3)=0} c(\bar{x}_1+\xi_1, \ldots \bar{t}+\tau)\, v_{cn}(\bar{x}_1+\xi_1, \ldots \bar{t}+\tau)\,\mathrm{d}\bar{F} \tag{6}$$

where $\mathrm{d}\bar{F}$ is an elementary part of the surface $F(x_1, x_2, x_3) = 0$.

Eq. (3) is integrated with respect to the four-dimensional cylinder $Z(\bar{x}, \bar{t})$, i.e., taking into consideration that

$$\bar{\xi}_1^2 + \bar{\xi}_2^2 + \bar{\xi}_3^2 < r^2; \quad |\tau| < \Delta t. \tag{7}$$

In this case, owing to the constancy of the limits of integration defined by the relationship (7), the sequence of integration with respect to $\mathrm{d}\xi_1$, $\mathrm{d}\xi_2$, $\mathrm{d}\xi_3$, and differentiation with respect to t may be reversed. Logically, the sequence of integration with respect to $\mathrm{d}F$ (or $\mathrm{d}\bar{x}_1$, $\mathrm{d}\bar{x}_2$, $\mathrm{d}\bar{x}_3$) and $\mathrm{d}\xi_1$, $\mathrm{d}\xi_2$, $\mathrm{d}\xi_3$ can also be changed.

Dividing the volume of the cylinder by the expression

$$\frac{4}{3} 2\pi r^3 \Delta t$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint\limits_{F(\bar{x}_1,\ldots)<0} \bar{c}(\bar{x}_1, \ldots, \bar{t})\,\mathrm{d}\bar{x}_1\,\mathrm{d}\bar{x}_2\,\mathrm{d}\bar{x}_3 = - \iint\limits_{F(\bar{x}_1\ldots)=0} \bar{c}(\bar{x}_1, \ldots, \bar{t})\, v^*_{\bar{c}n}(\bar{x}_1, \ldots, \bar{t})\,\mathrm{d}F \tag{8}$$

wherein allowance has already been made to the fact that

$$\bar{c}\,\bar{v}_{cn} = \bar{c} v^*_{\bar{c}n}.$$

It should be mentioned here furthermore that all values of f are continuous and can be differentiated with respect to time and the co-ordinates alike.

The components along the co-ordinates are

$$\mathrm{grad}_{\bar{x}} \bar{f} = \frac{1}{\frac{4}{3} 2\pi r^3 \Delta t} \iiint\limits_{\substack{\Sigma\xi_i^2 < r^2 \\ |\tau| < \Delta t}} f(\bar{x}_1 + \xi_1, \ldots, \bar{t} + \tau)\,\mathrm{d}\xi\, d\tau\,, \tag{9}$$

wherein $\mathrm{d}F$ is the vector element of the surface $\Sigma\xi^2 = r^2$ along the outward normal. Furthermore

$$\frac{\partial \bar{f}}{\partial \bar{t}} = \frac{1}{\frac{4}{3} 2\pi r^3 \Delta t} \iiint_{\Sigma\xi^2<r^2} [f(\bar{x}_1+\xi_1, \ldots, \bar{t}+\Delta t) - f(\bar{x}_1+\xi_1, \ldots, \bar{t} - \Delta t)] \, d\xi_1 \, d\xi_2 \, d\xi_3. \tag{10}$$

The differentiation with respect to $\bar{t}$ on the left-hand side of Eq. (5) can be transferred obviously under the integral sign, and, in accordance with the *Gauss-Ostrogradsky* theorem

$$\iint_{F(\bar{x}_1,\ldots)=0} \bar{c}(\bar{x}_1, \ldots, \bar{t}) \, v^*_{cn} \cdot (\bar{x}_1, \ldots, \bar{t}) \, d\bar{F} = \iiint_{F(\bar{x}_1,\ldots)<0} \mathrm{div}_{\bar{x}}(\bar{c}v^*_c) \, d\bar{x}_1 \, d\bar{x}_2 \, d\bar{x}_3. \tag{11}$$

Consequently, for an arbitrary volume

$$\iiint_{F(\bar{x}_1,\ldots)<0} \left[\frac{\partial \bar{c}}{d\bar{t}} + \sum_{k=1}^{3} \frac{\partial(\bar{c}v^*_{ck})}{\partial \bar{x}_k} \right] d\bar{x}_1 \, d\bar{x}_2 \, d\bar{x}_3 = 0$$

and thus

$$\frac{\partial \bar{c}}{\partial \bar{t}} + \sum_{k=1}^{3} \frac{\partial(\bar{c}v^*_k)}{\partial \bar{x}_k} = 0 \tag{12}$$

and analogously

$$\frac{\partial(1-\bar{c})}{\partial \bar{t}} + \sum_{k=1}^{3} \frac{\partial[(1-\bar{c}) \, v^*_k]}{\partial x^*_k} = 0 \tag{13}$$

It should be remembered that the values v_c and v^* denote velocities related to the centers of gravity of solid and fluid phases contained in the sphere

$$\sum_{k=1}^{3} (x_i - \bar{x}_i)^2 < r^2$$

and averaged for the time interval

$$\bar{t} - \Delta t < t < \bar{t} + \Delta t.$$

The *equations of motion* may hereafter be written. In accordance with the momentum theorem

$$\begin{aligned} \frac{d}{dt} \iiint_{F(x_1,\ldots)<0} \varrho_c \, c(x_1, \ldots, t) \, v_{ci} \, dx_1 \, dx_2 \, dx_3 = \\ = - \iint_{F(x_1,\ldots)=0} \varrho_c \, c v_{ci} \, (x_1, \ldots, t) \, v_{cn}(x_1, \ldots, t) \, dF - \\ - \iint_{\Phi(x_1,\ldots)=0} p_{in} \, d\Phi + \iiint_{F(x_1,\ldots)<0} \varrho_c \, c X_{ci}(x_1, \ldots, t) \, dx_1 \, dx_2 \, dx_3 \end{aligned} \tag{14}$$

where $\Phi(x_1, x_2, x_3, t) < 0$ is the area occupied by the solid phase within the space $F(x_1, x_2, x_3) = 0$, whereas $\mathrm{d}\Phi$ is its elementary part. The vector p_{in} is the component along the outward normal of the tensor p_{ik} of the surface $\Phi(x_1, x_2, x_3, t) = 0$. Quite obviously,

$$\iint\limits_{\Phi(x_1,\ldots)=0} p_{in}\,\mathrm{d}\Phi = \iiint\limits_{\Phi(x_1,\ldots)<0} \sum_{k=1}^{3} \frac{\partial p_{ik}}{\partial x_k}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 = \iiint\limits_{F(x_1,\ldots)<0} c\,\frac{\partial p_{ik}}{\partial x_k}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 . \tag{15}$$

Substituting Eq. (5) and with the relationships expressed by Eq. (15),

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\bar{t}} \iiint\limits_{F(x_1,\ldots)<0} \varrho_c\, c(\bar{x}_1+\xi_1,\ldots,\bar{t}+\tau)\,v_{ci}(\bar{x}_1+\xi_1,\ldots\bar{t}+\tau)\,\mathrm{d}\bar{x}_1\,\mathrm{d}\bar{x}_2\,\mathrm{d}\bar{x}_3 = \\
&= -\iint\limits_{F(x_1,\ldots)=0} [\varrho_c c(\bar{x}_1+\xi_1,\ldots,t+\tau)]\,[v_{ci}(\bar{x}_1+\xi_1,\ldots,\bar{t}+\tau)\,v_{cn}(\bar{x}_1+\xi_1,\ldots,\bar{t}+\tau)\,\mathrm{d}F] - \\
&- \iiint\limits_{F(\bar{x}_1,\ldots)<0} c(\bar{x}_1+\xi_1,\ldots,\bar{t}+\tau) \sum_{k=1}^{3} \frac{\partial p_{ik}(\bar{x}_1+\xi_1,\ldots,\bar{t}+\tau)}{\partial \bar{x}_k}\,\mathrm{d}\bar{x}_1\,\mathrm{d}\bar{x}_2\,\mathrm{d}\bar{x}_3 + \\
&+ \iiint\limits_{F(x_1,\ldots)<0} \varrho_c\, c(\bar{x}_1+\xi_1,\ldots)\,X_{ci}(\bar{x}_1+\xi_1,\ldots)\,\mathrm{d}\bar{x}_1\,\mathrm{d}\bar{x}_2\,\mathrm{d}\bar{x}_3 .
\end{aligned} \tag{16}$$

Integrating with respect to the cylinder $Z(\bar{x}, \bar{t})$ and dividing by its volume:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\bar{t}} \iiint\limits_{F(\bar{x}_1,\ldots)<0} \varrho_c\,\overline{cv}_{ci}\,\mathrm{d}\bar{x}_1\,\mathrm{d}\bar{x}_2\,\mathrm{d}\bar{x}_3 = -\iint\limits_{F(\bar{x}_1,\ldots)=0} \varrho_c\,\overline{cv}_{ci}\,\bar{v}_{cn}\,\mathrm{d}F - \\
&- \iiint\limits_{F(\bar{x}_1,\ldots)<0} \sum_{k=1}^{3} c\,\overline{\frac{\partial p_{ik}}{\partial x_k}}\,\mathrm{d}\bar{x}_1\,\mathrm{d}\bar{x}_2\,\mathrm{d}\bar{x}_3 + \iiint\limits_{F(\bar{x}_1,\ldots)<0} \varrho_c\, cX_{ci}\,\mathrm{d}\bar{x}_1\,\mathrm{d}\bar{x}_2\,\mathrm{d}\bar{x}_3 .
\end{aligned} \tag{17}$$

Let us now write

$$\begin{aligned}
v_{ci}(\bar{x}_1+\xi_1,\ldots,\bar{t}+\tau) &= v_{ci}^{*}(\bar{x}_1,\ldots,\bar{t}) + v_{ci}'(\bar{x}_1,\ldots,\bar{t};\ \xi_1,\ldots,\tau), \\
p_{ik}(\bar{x}_1+\xi_1,\ldots,\bar{t}+\tau) &= \bar{p}_{ik}(\bar{x}_1,\ldots,\bar{t}) + p_{ik}'(\bar{x}_1,\ldots,\bar{t};\ \xi_1,\ldots,\tau), \\
c(\bar{x}_1+\xi_1,\ldots,\bar{t}+\tau) &= \bar{c}(\bar{x}_1,\ldots,\bar{t}) + c'(\bar{x}_1,\ldots,\bar{t};\ \xi_1,\ldots,\tau).
\end{aligned} \tag{18}$$

Introducing the relationships given by Eq. (18) into Eq. (17) and applying repeatedly the theorem of *Gauss-Ostrogradsky* we have

$$\varrho_c \frac{\partial(\bar{c}v_{ci}^*)}{\partial \bar{t}} + \varrho_c \sum_{s=1}^{3} \frac{\partial[\bar{c}v_{ci}^* v_{ck}^*]}{\partial \bar{x}_k} = - \sum_{k=1}^{3} \frac{\partial[\varrho_c \bar{c}(v_{ci}' v_{ck}')]^*}{\partial \bar{x}_k} - \\ - \bar{c} \sum_{k=1}^{3} \frac{\partial \bar{p}_{ik}}{\partial \bar{x}_k} - \sum_{k=1}^{3} c' \frac{\partial p_{ik}'}{\partial \bar{x}_k} + \varrho_c \bar{c} X_{ci}^* \tag{19}$$

and accordingly

$$\varrho \frac{\partial[(1-\bar{c})v_i^*}{\partial \bar{t}} + \varrho \sum_{k=1}^{3} \frac{\partial[(1-\bar{c})\, v_i^* v_k^*]}{\partial \bar{x}_k} = \sum_{k=1}^{3} \frac{\partial[\varrho(1-\bar{c})(v_i' v_k')]^*}{\partial \bar{x}_k} - \\ - (1-\bar{c}) \sum_{k=1}^{3} \frac{\partial \bar{p}_{ik}}{\partial \bar{x}_k} + \sum_{k=1}^{3} \overline{c' \frac{\partial p_{ik}}{\partial \bar{x}_k}} + \varrho(1-\bar{c})X_i^* . \tag{20}$$

The tensors

$$\Pi_{cik} = \varrho_c \bar{c}(v_{ci}' v_{ck}')^*; \\ \Pi_{ik} = \varrho(1-\bar{c})\,(v_i' v_k')^* \tag{21}$$

denote the secondary stresses caused by turbulent flow; thus e.g. the tensor Π_{ik} is analogous to the osmotic pressure developing in solutions.

The vector expressed by

$$\bar{c} \sum_{k=1}^{3} \frac{\partial \bar{p}_{ik}}{\partial \bar{x}_k}$$

represents essentially a generalized Archimedian force caused by the averaged microscopic stresses $\bar{p}_{ik}$ acting on the solid particles contained in the volume considered. The vector

$$R_i = \sum_{k=1}^{3} \overline{c' \frac{\partial p_{ik}'}{\partial \bar{x}_k}} \tag{22}$$

is the average fluid resistance to the movement of solid particles.

In the demonstration of the *energy equation* the following simplifying assumptions will be introduced:

$$v_i = v_{ic}; \; X_i = \text{const.}$$

and

$$X_i^* = X_{ic}^* = X_i ; \; X_i' = X_{ic}' = 0 .$$

In the interior of the fluid and solid phases the Eqs. (12), (13), further (19) and (20) may be assumed to be satisfied.

The energy equation relating to average movement can be developed directly from the foregoing equations, i.e.,

$$\left(\frac{\partial}{\partial \bar{t}} + \sum_{k=1}^{3} \frac{\partial}{\partial \bar{x}_k} v_{kc}^*\right)\left(\varrho_c \bar{c} \frac{w_c^{*2}}{2}\right) = \\ = \sum_{i=1} v_{ic}^* \left[- \sum_{k=1}^{3} \bar{c} \frac{\partial \bar{p}_{ik}}{\partial \bar{x}_k} - \sum_{k=1}^{3} \frac{\partial \Pi_{ikc}}{\partial \bar{x}_k} + R_i + \varrho_c \bar{c} X_i \right]; \tag{23}$$

$$\left(\frac{\partial}{\partial \bar{t}} + \sum_{k=1}^{3} \frac{\partial}{\partial \bar{x}_k} v_k^*\right)\left[\varrho (1-\bar{c}) \frac{w^{*2}}{2}\right] = \\ = \sum_{k=1}^{3} v_i^* \left[- \sum_{k=1}^{3} (1-\bar{c}) \frac{\partial \bar{p}_{ik}}{\partial \bar{x}_k} - \sum_{k=1}^{3} \frac{\partial \Pi_{ik}}{\partial \bar{x}_k} - R_i + \varrho (1-\bar{c}) X_i \right], \tag{24}$$

wherein

$$w_c^{*2} = \sum_{i=1}^{3} v_{ic}^{*2} \tag{25}$$

$$w^{*2} = \sum_{k=1}^{3} v_i^{*2} \tag{26}$$

It should be noted at this point that the sign of R_i has been reversed relative to that in the equations of movement, in order to obtain $R_i > 0$ in the case when this force acts in the direction of the positive x_i axis.

The equation of *pulsation energy* will be derived from Eqs. (12) and (13), rearranged in the form:

$$\varrho_c c \left(\frac{\partial v_i}{\partial t} + \sum_{k=1}^{3} v_k \frac{\partial v_i}{\partial x_k}\right) = - \sum_{k=1} c \frac{\partial p_{ik}}{\partial x_k} + \varrho_c c X_i ;$$

$$\varrho (1-c) \left(\frac{\partial v_i}{\partial t} + \sum_{k=1}^{3} v_k \frac{\partial v_i}{\partial x_k}\right) = - \sum_{k=1}^{3} (1-c) \frac{\partial p_{ik}}{\partial x_k} + \varrho (1-c) X_i .$$

After averaging and rewriting in differential form these yield

$$\varrho_c \left(\frac{\partial}{\partial \bar{t}} + \sum_{k=1}^{3} \frac{\partial}{\partial \bar{x}_k} v_{kc}^*\right)\left[\bar{c}\left(\frac{w_c^{*2}}{2} + \frac{(w_c'^2)_c^*}{2}\right)\right] = - \sum_{i,k=1}^{3} \frac{\partial (v_{ic}^* \Pi_{ikc})}{\partial \bar{x}_k} - \\ - \sum_{i=1}^{3} \frac{\partial K_{ic}}{\partial \bar{x}_i} - \bar{c} \sum_{i,k=1}^{3} v_{ic}^* \frac{\partial \bar{p}_{ik}}{\partial \bar{x}_k} + \sum_{i=1}^{3} v_{ic}^* R_i + A_c + \varrho_c \bar{c} \sum_{i=1}^{3} v_{ic}^* X_i \tag{27}$$

where

$$(w_c'^2)_c^2 = \sum_{i=1}^{3} (v_{ic}'^2)_c^*$$

and

$$A_c = - \sum_{i,k=1}^{3} c v_{ic}' \frac{\partial p_{ik}'}{\partial \bar{x}_k} \tag{28}$$

denote the work performed by the forces resulting from the irregular movement of the solid particles, while

$$K_{ic} = \varrho_c \overline{c v'_{ic} \frac{w_c'^2}{2}} \tag{29}$$

is the "conductivity of turbulent energy", in other words the average value of pulsation energy due to pulsation velocities.

Subtracting Eq. (23) from Eq. (27) we have

$$\begin{aligned} \varrho_c \left(\frac{\partial}{\partial \bar{t}} + \sum_{k=1}^{3} \frac{\partial}{\partial \bar{x}_k} v_{kc}^* \right) \left[\bar{c} \frac{(w_c'^2)_c^*}{2} \right] = \\ = - \frac{1}{2} \sum_{i,k=1}^{3} \left(\frac{\partial v_{ic}^*}{\partial \bar{x}_k} + \frac{\partial v_{kc}^*}{\partial \bar{x}_i} \right) \Pi_{ikc} + A_c - \sum_{k=1}^{3} \frac{\partial K_{ic}}{\partial \bar{x}_i} \end{aligned} \tag{30}$$

and for the fluid phase

$$\begin{aligned} \varrho \left(\frac{\partial}{\partial \bar{t}} + \sum_{k=1}^{3} \frac{\partial}{\partial \bar{x}_k} v_k^* \right) \left[(1 - \bar{c}) \frac{(w'^2)^*}{2} \right] = \\ = - \frac{1}{2} \sum_{i,k=1}^{3} \left(\frac{\partial v_i^*}{\partial \bar{x}_k} + \frac{\partial v_k^*}{\partial \bar{x}_i} \right) \Pi_{ik} - A - \sum_{i=1}^{3} \frac{\partial K_i}{\partial \bar{x}_i} \end{aligned} \tag{30'}$$

wherein

$$A = - \sum_{i,k=1}^{3} \overline{(1 - \bar{c}) v'_i \frac{\partial p'_i}{\partial \bar{x}_k}} \tag{28'}$$

is the average value of work performed by forces due to the irregular movement of fluid particles, while

$$K_i = \varrho \overline{(1 - \bar{c}) v'_i \frac{w'^2}{2}} \tag{29'}$$

denotes the "conductivity of turbulent energy" in the fluid phase.

For obtaining the *thermal equations* for the two phases consider the equations of total energy, expressed in integral form, without averaging. Thus

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \iiint_G \varrho_c c \left(\frac{w^2}{2} + e \right) \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 = - \iint_F \varrho_c c v_n \left(\frac{w^2}{2} + e \right) \mathrm{d}F - \\ - \iint_\Phi \left(\sum_{i=1}^{3} v_i p_{in} + q_n \right) \mathrm{d}\Phi + \iiint_G \varrho_c c \sum_{i=1}^{3} v_i X_i \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 \end{aligned} \tag{31}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_G \varrho(1-c)\left(\frac{w^2}{2}+e\right)\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 = -\iint\limits_F \varrho(1-c)\,v_n\left(\frac{w^2}{2}+e\right)\mathrm{d}F - \\ -\iint\limits_\psi\left(\sum_{i=1}^{3} v_i\, p_{in}+q_n\right)\mathrm{d}\,\psi + \iiint\limits_G \varrho(1-c)\cdot\sum_{i=1}^{3} v_i\,X_i\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 \tag{31'}$$

where

e is the internal (thermal) energy of unit mass;
q_i is the vector of molecular heat conductivity;
Φ is the part area of surface G occupied by the solid particles, and
ψ is the part area of surface G occupied by the fluid particles.

Having performed the averaging operations, the result can be written in a form entirely similar to the foregoing:

$$\varrho_c\left(\frac{\partial}{\partial \bar{t}}+\sum_{k=1}^{3}\frac{\partial}{\partial \bar{x}_k}v^*_{kc}\right)(\bar{c}e^*_c) = -\sum_{k=1}^{3}\frac{\partial Q_{kc}}{\partial \bar{x}_k} = \sum_{k=1}^{3}\overline{c\frac{\partial q_k}{\partial \bar{x}_k}} \tag{32}$$

$$\varrho\left(\frac{\partial}{\partial \bar{t}}+\sum_{k=1}^{3}\frac{\partial}{\partial \bar{x}_k}v^*_k\right)[(1-\bar{c})\,e^*] = -\sum_{k=1}^{3}\frac{\partial Q_k}{\partial \bar{x}_k} - \\ -\sum_{k=1}^{3}\overline{(1-\bar{c})\frac{\partial q_k}{\partial \bar{x}_k}} - \frac{1}{2}\sum_{i,k=1}^{3}\left(\frac{\partial v_i}{\partial \bar{x}_k}+\frac{\partial v_k}{\partial \bar{x}_i}\right)p_{ik}\,. \tag{32'}$$

The thermal energy is seen to increase because of turbulent and laminar heat conduction, while the energy of the fluid phase is increased also by the averaged work of micro-deformations. This term is not involved in the equation of the solid phase, the deformations thereof having been neglected.

No solution of the system of equations described above is possible unless the values

$$\Pi_{ikc},\,\Pi_{ik},\,R_i,\,K_{ic},\,K_i,\,A_c,\,A,\,Q_{ic},\,Q_i$$

and the term

$$-\frac{1}{2}\overline{\sum_{i,k=1}^{3}\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\right)p_{ik}}$$

are known. Owing primarily to imperfections of instrumentation and measuring technique, the above quantities can only be determined — at the present level of knowledge — with certain approximation. Some of the possibilities will be dealt with subsequently.

3. Possibilities for solving the system of equations in the case of steady flow

The system of equations introduced in the foregoing is suitable — as pointed out in several papers by FRANKL — for describing in principle any type of flow, provided the resistance forces R_i due to the presence of solid particles can be determined. In the case of steady flow one possibility thereof is offered, when — according to the familiar basic equation of *diffusion* theory —

$$cw + \alpha v' l \frac{dc}{dy} = 0 \tag{33}$$

where

w — fall velocity of particles;
y — vertical direction;
l — mixing length;
v' — average module of the vertical pulsational velocity component; and
α — an empirical coefficient.

The resistance to the movement of individual particles is, in the case of movement at a velocity w:

$$D(w) = (\varrho_c - \varrho) g V \tag{34}$$

where

V is the average volume of a particle.

On the other hand, according to the semi-empirical theory of turbulence,

$$\tau = \beta \varrho v' l \frac{dv}{dy} \tag{35}$$

where

v is the average velocity in the direction of travel and
β is an empirical coefficient.

According to VANONI [25], the values of α and β lie close enough to each other to be taken identical.

It is further generally accepted that

$$\tau = \tau_0 \left(1 - \frac{y}{h}\right)$$

and

$$l = \varkappa y,$$

further, that

$$\beta v' = l \frac{dv}{dy} \tag{36}$$

where

h — water depth; and
$\varkappa$ — the Kármán velocity coefficient (~ 0.4).

Since

$$\tau = \varrho \left(l \frac{\mathrm{d}v}{\mathrm{d}y} \right)^2 = \varrho (\beta v')^2 = \tau_0 \left(1 - \frac{y}{h} \right);$$

$$\frac{\mathrm{d}v}{\mathrm{d}y} = \frac{1}{\varkappa y} \sqrt{\frac{\tau_0}{\varrho} \left(1 - \frac{y}{h} \right)},$$

the mixing coefficient becomes

$$\varepsilon = \beta v' l = l^2 \frac{\mathrm{d}v}{\mathrm{d}y} = \varkappa \sqrt{\frac{\tau_0}{\varrho}}\, y \sqrt{1 - \frac{y}{h}}. \tag{37}$$

Replacing in Eq. (19) the notations and simplifications $y = x_2$; $u_{1c} \cong u_1$ (for fine particles); $p_{11} = p_{22} = p_{33} = p$; $p_{ik} = 0$ for $i \neq k$ (i.e. neglecting the effect of micro-stresses resulting from friction); and assuming that $X_{1c} = g \sin i \approx g_i$; $X_{2c} = -g \cos i \approx -g$; $X_{3c} = 0$, we have

$$-\frac{\mathrm{d}\pi_{yyc}}{\mathrm{d}y} = -c \frac{\mathrm{d}p}{\mathrm{d}y} + R_y - \varrho_c c g = 0. \tag{38}$$

Accepting further that the presence of sediment particles is of no influence on pressure distribution (for low concentrations), it can be written:

$$\frac{\mathrm{d}p}{\mathrm{d}y} = -\varrho g,$$

and thus

$$-\frac{\mathrm{d}\Pi_{yyc}}{\mathrm{d}y} + R = -\frac{\mathrm{d}[\varrho_c (v_c'^2)_c^* \bar{c}]}{\mathrm{d}y} + R_y = 0. \tag{39}$$

If the mean square of the vertical pulsation velocity components does not depend appreciably on y (except for layers in the vicinity of the bottom), then according to MINSKY it may be assumed that

$$(v'^2)^* = \alpha v_{\max}^2 = 0.02\, v_{\max}^2.$$

In this case

$$0.02 \frac{\varrho_c v_{\max}^2}{(\varrho_c - \varrho) g} \frac{1}{c} \frac{dc}{dy} = - 0.02 \frac{v_{\max}^2}{gh} \frac{\varrho_c}{\varrho_c - \varrho} \frac{d \, ln \, c}{d(y/h)} =$$

$$= - 0.02 \frac{\varrho_c}{\varrho_c - \varrho} \frac{v_{\max}^2}{gh} \frac{wh}{\varkappa v' l} = \tag{40}$$

$$= - 0.02 \frac{\varrho_c}{\varrho_c - \varrho} \frac{wh}{\varkappa y \sqrt{\tau_0/\varrho} \sqrt{1 - y/h}} .$$

Concerning the physical phenomenon the following picture is thus obtained: owing to the value

$$\frac{w}{\sqrt{\tau_0/\varrho}} \ll 0.1$$

this term appears to be negligible in practice (at least in the case of low concentrations); the *Froude* number is in general

$$\frac{v_{\max}}{\sqrt{gh}} \leq 1.0 ,$$

so that neither this influence is considered significant. It is to be inferred therefrom that — disregarding the surface and bottom layers — the direct weight of the particles is counterbalanced only by the resistance acting on them and not by the gradient of turbulent stresses (consequently the latter may be neglected).

The prerequisite for the validity of these statements is, naturally, that the specific weight (velocity) of the sediment particles equals the specific weight (velocity) of water. In this case, owing to the diffusion of the sediment particles, a closely uniform distribution may take place along the depth, and then the average value of the resistance R_i may actually become zero, and the sediment particles move relative to the fluid particles at the velocity of turbulent diffusion, i.e.

$$v_{id} = - \varepsilon \frac{\partial \, ln \, c}{\partial x_i} . \tag{41}$$

Consequently, if the values $D_i(v)$ are components of the resistance to movement of the particles moving in an immobile fluid at velocity v, then the average resistance of the particles contained in unit volume is

$$R_i = \frac{c}{v} D_i (v_c - v - v_d) , \tag{42}$$

wherein

v_c the average velocity of particles;
v the average velocity of fluid particles;
v_d the average velocity of turbulent diffusion; and
C/V the number of particles contained in unit volume.

Assuming further the validity of *Stokes'* law:

$$R_i = -\frac{c}{V} 6\,\pi\mu r(v_{ic} - v_i - v_{id}) = \frac{9}{2}\frac{c}{r^2}(v_{ic} - v_i - v_{id})\,\mu\,, \tag{42'}$$

finally, the equation describing the movement of suspended particles can be written for the case of steady movement:

$$\varrho_c\left(\frac{\partial v_{ic}}{\partial t} + \sum_{k=1}^{3} v_{kc}\frac{\partial v_{ic}}{\partial x_k}\right) = -\frac{\partial p}{\partial x_i} + \frac{1}{c}R_i + \varrho_c X_{ic}. \tag{43}$$

Since the equation

$$\frac{\partial_s}{\partial t} + \sum_{k=1}^{3}\frac{\partial(cv_{kc})}{\partial x_k} = 0 \tag{44}$$

is also available, the system of equations is closed (the only unkown quantities being u_{ic} and c).

As will be perceived, for the special case of steady movement the results of the diffusion theory have been obtained. In fact, from Eq. (43)

$$c\left(\varrho_c g + \frac{dp}{dy}\right) = c\,(\varrho_c - \varrho)\,g = R_y = \frac{c}{V}D(-v_d) \tag{45}$$

or

$$D(-v_d) = (\varrho_c - \varrho)\,gV \tag{46}$$

which leads, as indicated by Eq. (34), to the equality

$$w = -v_d \tag{47}$$

corresponding to Eq. (33).

Data obtained by extensive experimental checks conducted at the laboratory of the *Department for Water Management* lead, however, to the conclusion that the assumption concerning the identity between sediment and fluid particle velocities is, unfortunately, not satisfied in the majority of cases of practical interest. For this reason the above solution suggested by FRANKL is considered acceptable for the approximately laminar flow conditions in *settling basins* only. The point of theoretical significance is, consequently, that the term Π_{xyc} (the transverse gradient of turbulent shearing stresses) involved in the condition

$$-c\frac{\mathrm{d}p_{yx}}{\mathrm{d}y}-\frac{\mathrm{d}\Pi_{xyc}}{\mathrm{d}y}+R_x+c\varrho_c gi\approx-\frac{\mathrm{d}\Pi_{xyc}}{\mathrm{d}y}+R_x+c\varrho gi=0 \qquad (48)$$

defines essentially the lag of solid particles, and this is not eliminated unless $R_x = 0$, i.e., when (in an extreme case) there is no physical difference between the two phases, consequently a single phase (water) is only present.

4. Solution of the system of equations for the case of quasi-steady flow

Consider the potential solutions of the dynamical relationships expressed by Eqs. (19) and (20) for the case of quasi-steady, plane and uniform flow. For this case it is assumed, in agreement with SANOYAN and ANANYAN [19] that

— the pressure distribution is hydrostatic;

— the distribution of solid particles is statistically steady, the concentration prevailing in a particular elementary volume of space during a particular elementary interval of time remains unchanged both in time and along the coordinate axis in the direction of flow;

— concentration is a function of depth;

— the distribution of pulsational velocity components does not differ appreciably from that in clear water.

With these in mind, the fundamental expressions of Eqs. (19) and (20) assume the form

$$-\frac{\mathrm{d}}{\mathrm{d}x_2}[\varrho_c\bar{c}(v'_{c1}v'_{c2})^*]+g\varrho_c i\bar{c}-R_1=0 \qquad (49)$$

$$-\frac{\mathrm{d}}{\mathrm{d}x_2}[\varrho_c\bar{c}(v'^2_{c2})^*]-\bar{c}\frac{\mathrm{d}\bar{p}}{\mathrm{d}x_2}-g\varrho_c\bar{c}-R_2=0 \qquad (50)$$

$$-\frac{\mathrm{d}}{\mathrm{d}x_2}[\varrho(1-\bar{c})(v'_1v'_2)^*]+\varrho(1-\bar{c})gi+R_1=0 \qquad (51)$$

$$-\frac{\mathrm{d}}{\mathrm{d}x_2}[\varrho(1-\bar{c})(v'^2_2)^*]-(1-\bar{c})\frac{\mathrm{d}\bar{p}}{\mathrm{d}x_2}-\varrho(1-\bar{c})g+R_2=0. \qquad (52)$$

In the case under consideration the x_1 axis points into the direction of flow and coincides with the bottom; the x_2 axis is perpendicular to x_1 ; i is the bottom slope and, as previously, $\bar{c}$ is the average value of concentration at a particular point (elementary volume) of space.

One possibility for the solution is then according to SANONYAN and ANANYAN to introduce on the basis of experiments by MINSKY, the following conditions:

$$(u_2'^2)^* = \varkappa u_{\max}^2(1 - k_1 \bar{c}) \tag{53}$$

$$(u_{c2}'^2)^* = \varkappa u_{\max}^2(1 - k_2 \bar{c}) \tag{54}$$

$$(u_1' u_2')^* = - gi(h - x_2)(1 - k_3 \bar{c}). \tag{55}$$

The coefficients k_1, k_2, k_3 may then be regarded as parametric functions of the mean particle diameter d alone and their magnitudes are determined experimentally.

Eqs. (49) through (52) are preferably rewritten into dimensionless form using the following notations:

$$x_1 = xh\,;\; x_2 = yh\,;\; (u_{c1}' u_{c2}')^* = (u_c' v_c')^* ghi\,;\; (u_c'^2)^* = u_c'^2 u_{\max}^2 .$$

Thus

$$g\varrho_c\, i \frac{\mathrm{d}}{\mathrm{d}y}\,[\bar{c}(u_c' v_c')^*] - g\varrho_c i\bar{c} + R_x = 0 \tag{56}$$

$$\frac{\varkappa u_{\max}^2}{h}\, \varrho_c \frac{\mathrm{d}}{\mathrm{d}y}\,[\bar{c}(1 - k_2 \bar{c})] + g(\varrho_c - \varrho)\,\bar{c} + R_y = 0\,. \tag{57}$$

$$-g\varrho_i \frac{\mathrm{d}}{\mathrm{d}y}\,[(1 - \bar{c})(1 - k_3 \bar{c})(1 - y)] - g(1 - \bar{c})\,\varrho i - R_x = 0 \tag{58}$$

$$\frac{\varkappa u_{\max}^2}{h}\, \varrho \frac{\mathrm{d}}{\mathrm{d}y}\,[(1 - \bar{c})(1 - k_1 \bar{c})] - R_y = 0\,. \tag{59}$$

The system of equations (56) through (59) is thus completely closed, since the number of unknowns (R_x, R_y, $(u_c' v_c')$, $\bar{c}$) equals that of the equations.

For determining the value of $\bar{c}$ combine the terms of Eqs. (57) and (59):

$$\frac{\mathrm{d}\bar{c}}{\mathrm{d}y}\left\{\bar{c}\left(\frac{2\,\varkappa u_{\max}^2}{gh}\;\frac{\varrho_c k_2 - \varrho k_1}{\varrho_c - \varrho}\right) - \frac{\varkappa u_{\max}^2}{gh}\left(\frac{\varrho_c - (1 + k_1)\,\varrho}{\varrho_c - \varrho}\right)\right\} - \bar{c} = 0 \tag{60}$$

which becomes, after the introduction of simplifying notations:

$$\frac{\mathrm{d}\bar{c}}{\mathrm{d}y}\,(B\bar{c} - A) - \bar{c} = 0\,. \tag{60'}$$

The integration of Eq. (60′) yields

$$B(\bar{c} - \bar{c}_0) - A\, ln \frac{c}{c_0} = y - y_0\,, \tag{61}$$

where $\bar{c}_0$ is the concentration at depth y_0, the so-called bottom concentration.

Accordingly, the resistance coefficients may be expressed from Eqs. (58) and (59), as

$$R_x = -\varrho g i \bar{c}\left[(1-\bar{c})\,k_3 + \frac{(1-y)(1+k_3-2\,\bar{c}\,k_3)}{A-B\bar{c}}\right] \tag{62}$$

$$R_y = \varrho g \bar{c}\,\frac{(1+k_1)(\varrho_c-\varrho)}{\varrho_c \cdot (1+k_1)\,\varrho} = -\frac{\varkappa \varrho u_{\max}^2}{h}\,\frac{(1+k_1-2\,k_1\bar{c})\,\bar{c}}{A-b\bar{c}}\,, \tag{63}$$

or, the resultant resistance due to the presence of the solid particles is

$$R = \sqrt{R_x^2 + R_y^2}\,. \tag{64}$$

Considering the above solution it may be concluded that realistic results are to be expected primarily in the range of high concentrations, i.e., in the case of slurry flow. In fact, the coefficients k_1, k_2 and k_3 depend in this case on the average particle diameter alone, although obviously, these will vary with concentration and grain-size distribution and probably with the intensity of turbulence, etc. Furthermore, it is hardly to be expected that the relationships expressed by Eqs. (53) through (55) could be represented in a more generalized case as linear functions of the said coefficients. This situation is unimaginable, unless in the case of high slurry concentrations the distribution along the vertical is nearly uniform, the intensity of turbulence is practically zeroed, and the behaviour of the so-called gravity medium is governed fundamentally by the weight (volume, diameter) of the entrained material.

5. Solution of the system of equations for quasi-steady flow and random concentration

Owing to the shortcomings of the presented partial solutions of the generalized fundamental equation, it was decided to seek a substantially different approach. The extent to which the variation of pulsational velocity components — controlling substantially the entire process of suspension — can be described in terms of concentration, is obviously critical for the success of the approach.

In turbulent flow the mixing processes are highly involved and transform the mechanical energy into other (mainly thermal) forms of energy. In steady flow of clear water, mechanical losses of energy are induced by (external) friction on the boundaries, as well as by internal friction.

In cases where solid particles are also contained in the flowing medium, its component parts (second phase) participate themselves in the process of turbulent mixing. As a result of friction of sediment particles on the boundary surfaces, friction (and possibly impact) of sediment particles among themselves and on the fluid particles *additional* losses of mechanical energy occur. Furthermore, because of the difference in specific weight (non-uniform distribution) and the difference between the velocities of solid and fluid particles, minute *turbulent wakes* develop behind the solids. The energy of these wakes is converted directly into thermal energy, presenting special cases of *energy dissipation*.

At the same time it is evident that a uniform turbulent field of motion occurs in turbulent, sediment-laden flow, in which the mean velocities of the solid and fluid media may be equal at particular points. As indicated by experimental evidence available so far, the condition

$$\overline{v'_x c'} > 0 ; \quad \overline{v'_x v'_y} < 0$$

is in general satisfied, the physical interpretation is that fluid particles moving at a lower velocity are more frequently encountered by the solid particles than such moving at a higher velocity. This is the reason for the *statistical* (and not primarily hydrodynamical) lag of solids, which applies to *groups of particles* (and not primarily to individual particles).

One of the possible solutions resulting from the above considerations will be outlined subsequently.

5.1 Choice of the fundamental equations

The solution will be based on the mathematically exact fundamental equations (19) and (20) without neglections. The influence enhanced by the presence of the solid phase will be expressed in a manner similar to the conditions in Eqs. (53) through (55), with the difference, however, of applying concentration influence functions, given, for the time being, in a general form $\Phi(\bar{c})$. Thus

$$(u_2'^2)^* = \alpha u_{\max}^2 \Phi_1(\bar{c}) \tag{65}$$

$$(u_{c2}'^2)^* = \alpha u_{\max}^2 \Phi_2(\bar{c}) \tag{66}$$

$$(u_1' u_2')^* = - gi(h - x_2)\, \Phi_3(\bar{c}) \tag{67}$$

$$(u_{c1}' u_{c2}')^* = - gi(h - x_2)\, \Phi_4(\bar{c}) \tag{68}$$

It is assumed further that

$$x_1 = xh ; \quad x_2 = yh \tag{69}$$

and

$$\frac{dp}{dx_2} = - g\varrho . \tag{70}$$

The fundamental equations (19) and (20) assume thus, with allowance for Eqs. (65) through (70) the following form

$$g\varrho_c\, i \frac{\mathrm{d}}{\mathrm{d}y}\left\{c(1-y)\,\Phi_4(\bar{c})\right\} - g\varrho_c\, i\bar{c} + R_1 = 0 \tag{71}$$

$$\frac{\varkappa u_{\mathrm{max}}^2}{h}\,\varrho_c \frac{\mathrm{d}}{\mathrm{d}y}\left[c\Phi_3(\bar{c})\right] + g(\varrho_c - \varrho)\,\bar{c} + R_2 = 0 \tag{72}$$

$$g\varrho i \frac{\mathrm{d}}{\mathrm{d}y}\left[(1-\bar{c})(1-y)\,\Phi_3(\bar{c})\right] - g(1-\bar{c})\,\varrho i - R_1 = 0 \tag{73}$$

$$\frac{\varkappa u_{\mathrm{max}}^2}{h}\,\varrho \frac{\mathrm{d}}{\mathrm{d}y}\left[(1-\bar{c})\,\Phi_1(\bar{c})\right] - R_2 = 0\,. \tag{74}$$

Adding Eqs. (72) and (74), as well as Eqs. (71) and (73) yields

$$\frac{\varkappa u_{\mathrm{max}}^2}{h}\left\{\varrho_c \frac{\mathrm{d}}{\mathrm{d}y}\left[\bar{c}\,\Phi_2(\bar{c})\right]\varrho + \frac{\mathrm{d}}{\mathrm{d}y}\left[(1-\bar{c})\,\Phi_1(\bar{c})\right]\right\} - g(\varrho_c - \varrho)\,\bar{c} = 0 \tag{75}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\left\{(1-y)\left[\varrho_c\bar{c}\,\Phi_4(\bar{c}) + \varrho\,(1-\bar{c})\,\Phi_3(\bar{c})\right]\right\} - \left[\bar{c}\,\varrho_c + (1-\bar{c})\right] = 0. \tag{76}$$

Let

$$\mu = \frac{\varrho_c}{\varrho}$$

from Eq. (75) it is:

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[\varrho_c\bar{c}\,\Phi_2(\bar{c}) + \varrho(1-\bar{c})\,\Phi_1(\bar{c})\right] = \frac{g(\varrho_c - \varrho)\,h}{\varkappa u_{\mathrm{max}}^2}\,. \tag{75'}$$

With the notations

$$U(\bar{c}) = \frac{\mu\bar{c}\,\Phi_2(\bar{c}) + (1-\bar{c})\,\Phi_1(\bar{c})}{\bar{c}} \tag{77}$$

$$V(\bar{c}) = \mu\bar{c}\left[\Phi_4(\bar{c}) + 1\right] + (1-\bar{c})\left[\Phi_3(\bar{c}) + 1\right]. \tag{78}$$

Eqs. (75′) and (76) assume the form

$$\frac{\mathrm{d}y}{\mathrm{d}c} = \frac{\varkappa u_{\mathrm{max}}^2}{(\mu - 1)\,gh}\,\frac{1}{\bar{c}}\,\frac{\mathrm{d}}{\mathrm{d}c}\left[\bar{c}U(\bar{c})\right] \tag{79}$$

$$\frac{\mathrm{d}y}{\mathrm{d}c} = (1-y)\,\frac{V'(\bar{c}) - (\mu - 1)}{V(\bar{c})}\,. \tag{80}$$

Thus, in the foregoing equation three functions, namely

$$y(\bar{c}), \quad U(\bar{c}) \quad \text{and} \quad V(\bar{c})$$

are involved, so that two equations are insufficient for their determination. The approach adopted will consist of determining one of them from measurement data. The function to be determined by measurement be the function $U(\bar{c})$.

In this case the function on the right-hand side of Eq. (79) may be regarded as known and denoted by $Y(\bar{c})$. In this manner $y(\bar{c})$ and $V(\bar{c})$ can be determined from Eq. (79) and Eq. (80), respectively.

Consequently, by introducing on the basis of Eq. (79)

$$\frac{\mathrm{d}y}{\mathrm{d}c} = Y(\bar{c})$$

into Eq. (80), this becomes

$$Y(\bar{c}) = (1-y)\frac{V'(\bar{c}) - (\mu - 1)}{V(c)} \,. \tag{81}$$

The function y is then obtained as the solution of Eq. (79):

$$\frac{\mathrm{d}y}{\mathrm{d}c} = \frac{\alpha u_{\max}^2}{(\mu-1)\,gh}\,\frac{1}{\bar{c}}\,\frac{\mathrm{d}}{\mathrm{d}c}\left[\bar{c}U(\bar{c})\right],$$

whence upon integration

$$y - y_0 = \frac{\alpha u_{\max}^2}{(\mu-1)\,gh}\left[U(\bar{c}) - U(\bar{c}_0) + \int_{c_0}^{c}\frac{U(\bar{c})}{\bar{c}}\,\mathrm{d}c\right]. \tag{82}$$

Consider now Eq. (81). Upon rearrangement of terms

$$V'(\bar{c}) = \frac{Y(\bar{c})}{1-y}\,V(\bar{c}) = \mu - 1. \tag{81'}$$

Introducing the notation

$$W(\bar{c}) = -\frac{Y(\bar{c})}{1-y} = \frac{-\dfrac{\mathrm{d}y}{\mathrm{d}c}}{1-y} \tag{83}$$

the solution of Eq. (81′) is obtained as

$$V(\bar{c}) = e^{-\int_{c_0}^{c} W(\bar{c})\mathrm{d}c} \left[V(\bar{c}_0) + (\mu - 1) \int_{c_0}^{c} e^{\int_{c_0}^{c} W(\bar{c})\mathrm{d}c} \mathrm{d}c \right]. \tag{84}$$

5.2 *Expanding the system of equations*

On the basis of experimental evidence it has been concluded that concentration had almost the same effect on the variation of the coefficients either $(u_2'^2)^*$ or $(u_{c2}'^2)^*$. Identical trends were observed also in the variation of the terms $(u_1', u_2')^*$ and $(u_{c1}', u_{c2}')^*$ so that the simplifying assumptions

$$\Phi_1(\bar{c}) \cong \Phi_2(\bar{c}) = \Phi(\bar{c})$$

and

$$\Phi_3(\bar{c}) \cong \Phi_4(\bar{c}) = \Psi(\bar{c})$$

appeared permissible. At the same time it was found from measurement data that

$$\Phi(\bar{c}) = e^{-kc}. \tag{85}$$

Consequently

$$U(\bar{c}) = \left(\mu - 1 + \frac{1}{\bar{c}}\right) e^{-k\bar{c}} \tag{86}$$

$$V(\bar{c}) = \left(\mu - 1 + \frac{1}{\bar{c}}\right) \bar{c}\,[\Psi(\bar{c}) + 1]. \tag{87}$$

Thus from Eq. (82)

$$y = y_0 + \frac{\varkappa u_{\max}^2}{gh} \left[e^{-k\bar{c}} - e^{-k\bar{c}_0} + \left(1 - \frac{k}{\mu - 1}\right) \int_{C}^{c} \frac{e^{-k\bar{c}}}{\bar{c}} \mathrm{d}c \right].$$

Introducing the exponential integral function

$$\int_{u}^{\infty} \frac{e^{-u}}{u} \mathrm{d}u = - E_i(-u)$$

the expression for the function $y(\bar{c})$ becomes

$$y = y_0 + \frac{\varkappa u_{\max}^2}{gh} \left\{ e^{-k\bar{c}} - e^{-k\bar{c}_0} + \left(\frac{k}{\mu - 1} - 1\right) [-E_i(-k\bar{c}) + E_i(-k\bar{c}_0)] \right\}. \tag{88}$$

The function $V(\bar{c})$ is obtained from Eq. (84) by introducing Eq. (83). Since

$$\int_{c_0}^{c} W(\bar{c})\,\mathrm{d}c = \int_{c_0}^{c} \frac{-\dfrac{\mathrm{d}y}{\mathrm{d}c}}{1-y}\,\mathrm{d}c = \Big[ln(1-y)\Big]_{c_0}^{c} = ln\,\frac{1-y(\bar{c})}{1-y_0},$$

it follows that

$$V(\bar{c}) = \frac{1}{1-y(\bar{c})}\left\{(1-y_0)\,V(\bar{c}_0) + (\mu-1)\int_{c_0}^{c}[(-y(\bar{c}))]\,\mathrm{d}c\right\},$$

where the expression for $y(\bar{c})$ should be introduced from Eq. (82). Under the condition in Eq. (85) and with regard to Eq. (88) the following expression is obtained;

$$V(\bar{c}) = \frac{1}{1-y(\bar{c})}\left\{(1-y_0)\,V(\bar{c}_0) + \left[(1-y_0) + \frac{\alpha u_{\max}^2}{gh}\,e^{-k\bar{c}_0}\right](\mu-1)(\bar{c}-\bar{c}_0)\right\} +$$
$$+ \frac{\alpha u_{\max}^2}{gh}\left\{e^{-k\bar{c}} - e^{-k\bar{c}_0} - (\mu-1)\left(\frac{k}{\mu-1}-1\right)\bar{c}\,[-E_i(-k\bar{c}) + E_i(-kc_0)]\right\}. \tag{84''}$$

5.3 *Determination of the relationships*

Let us return now to the expression of the function $y(\bar{c})$. Eq. (88) was obtained by assuming that $\Phi_1(\bar{c}) = \Phi_2(\bar{c})$. Furthermore, in Eqs. (65) and (66) it has been accepted that $u_{\max}^2$ (the square of the maximum velocity in the direction of travel observed at a particular point) is not directly related to concentration. In lack of more accurate measurements assume temporarily that

$$(u_2'^2) = \alpha u(\bar{c})\,\Phi(\bar{c}),$$

or

$$u(\bar{c}) = u_0^2 + \beta\bar{c}$$

and

$$q = \frac{\beta}{u_0^2}; \qquad \frac{n^2(c)}{gh} = Fr.$$

With these in mind the following result is obtained:

$$\frac{y-y_0}{\alpha Fr} = \left[1 + q\frac{1}{\mu-1} - \frac{1}{\mu} + q\bar{c}\right]e^{-k\bar{c}} +$$
$$+ \left(\frac{k-q}{\mu-1}-1\right)[-E_i(-k\bar{c})] - \left\{\left[1 + q\left(\frac{1}{\mu-1}-\frac{1}{k}\right) + \right.\right. \tag{89}$$
$$\left.\left. + q\bar{c}_0\right]e^{-k\bar{c}_0} + \left(\frac{k-q}{\mu-1}-1\right)[-E_i(-k\bar{c}_0)]\right\},$$

whence, with the substitution $q = 0$, obviously Eq. (88) is obtained.

Determine now the mean concentration along a vertical.

In a general form this becomes

$$c_{fk} = \frac{1}{1 - y_0} \int_{y_0}^{1} c \, dy$$

since from Eqs. (75′) and (79)

$$\bar{c} \, dy = \frac{\varkappa u_{\max}^2}{(\mu - 1) gh} \, d[\bar{c} \, U(\bar{c})] \tag{90}$$

thus

$$c_{fk} = \frac{1}{1 - y_0} \, \frac{\varkappa u_{\max}^2}{(\mu - 1) gh} \int_{c}^{c_m} d[\bar{c} U(\bar{c})]$$

and

$$c_{fk} = Fr \frac{\varkappa}{1 - y_0} \left\{ c_m e^{-kc_m} - \bar{c}_0 e^{-kc_0} + \frac{1}{\mu - 1} (e^{-k\bar{c}_m} - e^{-k\bar{c}_0}) \right\}. \tag{91}$$

Concentration at the surface ($\bar{c}_m$) is obtained at $y = 1$ from Eq. (88), i.e.

$$y_0 - 1 + \varkappa Fr \left\{ e^{-kc_m} - e^{-kc_0} + \left(\frac{k}{\mu - 1} - 1 \right) [-E_i(-k\, c_m) + E_i(-c_0)] \right\} = 0. \tag{92}$$

For facility of computation the following notations will be introduced:

$$z = k\bar{c}\,; \; z_0 = k\bar{c}_0\,; \; z_m = k\bar{c}_m\,; \; z_k = kc_{fk}$$

$$F(z) = e^{-z} + \left(\frac{k}{\mu - 1} - 1 \right) [-E_i(-z)] \tag{93}$$

$$G(z) = \left(\frac{k}{\mu - 1} + z \right) e^{-z}.$$

Eqs. (88), (91) and (92) can then be rewritten into the following form:

— for the concentration at any point along a vertical:

$$y - y_0 = \varkappa Fr \, [F(z) - F(z_0)] \,, \tag{88′}$$

— for the concentration at the surface:

$$1 - y_0 = \varkappa Fr \, [F(z_m) - F(z_0)] \,, \tag{92′}$$

— for the mean concentration in the vertical:

$$1 - y_0 = \frac{\alpha Fr}{z_k} [G(z_m) - G(z_0)] . \qquad (91')$$

In view of the fact that the above equations are transcendental ones, double point-row diagrams have been prepared for ease of computation. Computation is impossible unless the concentration at the bottom, the distribution of velocities and the water depth are known.

In the solution presented above the main inconsistencies of previous theories could thus be resolved. In fact, by introducing and determining the concentration influence functions, we succeeded in describing the influence exerted by the presence of the solid phase on the pulsation velocity characteristics, decisive for the suspension process. The resulting computation formulae are at the same time easy to handle and yield practical relationships.

Nevertheless, for a number of additional practical problems (e.g. settling tanks, slow and high-rate filters, colmatation, etc.) it appeared advisable to continue the study of the resisting forces R_i (which were replaced in the foregoing study, i.e., under the definitely turbulent conditions prevailing in open watercourses, by the concentration influence functions), since the problems mentioned above involve mostly flow conditions in the laminar or transition ranges.

6. Determination of the resisting forces

The core of the problem is essentially that the vector

$$R_i = \sum_{k=1}^{3} \overline{c' \frac{\partial p'_{ik}}{\partial x_k}} \qquad (94)$$

(the force resisting the movement of the fluid) involved in the exact dynamical equation (19), represents, together with the generalized Archimedian force

$$A_i = \bar{c} \sum_{k=1}^{3} \frac{\partial \bar{p}_{ik}}{\partial \bar{x}_k} \qquad (95)$$

(resulting from microscopic stresses $\bar{p}_{ik}$) the interaction of the two phases.

Since the effect of micro-stresses due to friction is practically insignificant indeed, it is sufficient to consider the term R_i alone.

For the sake of completeness it should be remembered that the above expression results from the assumption that the stress tensors p_{ik}, p_{in}, as well as their derivatives with respect to the co-ordinates are continuous functions of the coordinates within and along the boundaries of the areas $F(\bar{x}_1, \ldots) < 0$ and $\Phi(\bar{x}_1, \ldots) < 0$. In fact, applying them to the resulting force the *Gauss-Ostrogradinsky* theorem yields

$$\iint\limits_{\Phi(\bar{x}_1,\ldots)=0} p_i \, d\Phi = \iiint\limits_{\Phi(x_1,\ldots)<0} \sum_{k=1}^{3} \frac{\partial p_{ik}}{\partial x_k} dx_1 \, dx_2 \, dx_3 = \iiint\limits_{F(x_1,\ldots)<0} \sum_{k=1}^{3} \frac{\partial p_{ik}}{\partial x_k} dx_1 \, dx_2 \, dx_3. \qquad (96)$$

In agreement with Eq. (16), after the averaging operations according to Eqs. (2) and (3) it follows for the solid phase

$$\iiint\limits_{F(x_1,\ldots)<0} \sum_{k=1}^{3} \frac{\partial p_{ik}}{\partial x_k} dx_1 \, dx_2 \, dx_3 = \iiint\limits_{F(x_1,\ldots)<0} \bar{c} \sum_{k=1}^{3} \frac{\partial \bar{p}_{ik}}{\partial x_k} dx_1 \, dx_2 \, dx_3 + \iiint\limits_{F(x_1,\ldots)<0} \sum_{k=1}^{3} \overline{c' \frac{\partial p'_{ik}}{\partial x_k}} dx_1 \, dx_2 \, dx_3 \qquad (97)$$

and for the fluid phase

$$\iiint\limits_{F(x_1,\ldots)<0} (1-\bar{c}) \sum_{k=1}^{3} \frac{\partial p_{ik}}{\partial x_k} dx_1 \, dx_2 \, dx_3 = \iiint\limits_{F(x_1,\ldots)<0} (1-\bar{c}) \sum_{k=1}^{3} \frac{\partial \bar{p}_{ik}}{\partial x_k} dx_1 \, dx_2 \, dx_3 - \iiint\limits_{F(x_1,\ldots)<0} \sum_{k=1}^{3} \overline{c' \frac{\partial p'_{1k}}{\partial x_k}} dx_1 \, dx_2 \, dx_3 \qquad (98)$$

It was seen in the foregoing that, for approaching the second terms sought for on the right-hand sides of the above equations FRANKL applied Eq. (42), SANOYAN and ANANYAN obtained Eqs. (62) and (63), resp., whereas, according to the solution suggested by the author in 1962, in conformity with Eqs. (73), 74) and (85), it is:

$$R_1 = \frac{d}{dy} [(1-y) \Psi(\bar{c})] - 1 \qquad (99)$$

$$R_2 = - \frac{d}{dy} [(1-\bar{c}) e^{-k\bar{c}}] \alpha \gamma Fr. \qquad (100)$$

It should be noted that several investigations have been performed in recent years to determine the numerical value of coefficient k. E.g. the coefficient k was found by DSHRBASYAN [5] to depend primarily on particle size (settling velocity).

On the basis of the foregoing relationships the magnitude of the resistance factors can be computed and consequently the effect of the solid phase on the pulsational velocity components expressed.

Summary

The development history of suspended sediment transportation is reviewed. Purely empirical theories and semi-empirical approaches have been neglected, since the deterministic model on which their development relied upon, was necessarily poorly founded. The present study relied on the theoretical basic equations of classical mechanics and hydrodynamics, as well as on the so-called microscopic systems of equations developed in theoretical physics, to derive the theory of turbulent sediment movement. Based on research results of the author, and on those published in the literature, the resulting equations are solved by different approaches.

It is concluded that the microscopic system of equations suggested by *Frankl* may be regarded at present as the most suitable foundation for the development of the theory of turbulent sediment movement. In the development of the theory presented in Chapter 5 and in the determination of the resisting forces shown in Chapter 6, empirical relationships were necessarily introduced in order to make the results suitable for practical applications. It follows, that the theory of turbulent sediment movement would be difficult to develop further using the classical approach followed so far. The uncertainties involved in the determination of parameters call for a radical revision of earlier methods of measurement and for the statistical analysis of data series intended for use.

The development of the classical deterministic approach seems to have attained its practical limits and further efforts should instead be directed towards the development of stochastic-mathematical models, using recent results of probability theory and mathematical statistics. The next step should be to develop suitable computer programs.

Nevertheless, the study presented above is still useful for solving problems practically not fully understood such as diffuse or turbulent movement of substances with specific weights other than that of water. Such problems are encountered in different systems of settling tank, various flow-through basins used in water and sewage treatment, etc. Knowledge of achievements in the theory of turbulent sediment movement may furthermore be useful in the study of hydraulic conditions, oxidation basins, various shafts, etc. with predominantly turbulent flow.

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