LIMIT ANALYSIS OF PRESTRESSED CONCRETE BEAMS*

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1. Introduction

In general, prestressed concrete beams may be designed with reduced cross-sectional dimensions as compared to those of normal reinforced beams. The requirements for prestressed beams are often more rigorous, e.g. the requirement to avoid cracks or excessive crack width. Therefore, in spite of a relatively high rigidity of prestressed structures, it is often necessary to analyse post-cracking behaviour and deformations, while for ordinary reinforced concrete beams generally a limit analysis of load capacity is sufficient. Namely, restrictions on deformation usually require no detailed analysis and uncracked performance is seldom a requirement.

There are several regulations for this analysis. Some of them apply cross-sectional deformation to characterize ultimate moment, while in others the computation method is based on equilibrium conditions for known stress distribution, given for some limit condition.

Several methods have been developed for determining cracking and ultimate moment. There is no general method, however, for the stress-strain condition due to an arbitrary external moment in a cracked prestressed beam. Elastic analysis of a cracked beam has been made by KÁRMÁN [4]. Analysis based on assumptions by DMITRIEV and KALATUROV has been elaborated by ROZENBLUMAS [6].

The methods based on the cross-sectional deformation accept the *Bernoulli—Navier* hypothesis, fundamental also for the ultimate moment computation in the *Hungarian Specification for Highway Bridges* and for different Soviet computation methods, e.g. that by ZHDANOV [9]. DMITRIEV and KALATUROV [2] assume a given stress distribution, while the position of the neutral axis and other parameters are determined from the equilibrium conditions.

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Symbols

Cross-sectional characteristics

b [cm]	width of cross-section
h [cm] φ [ ] α [ ]	see Fig. 1.
$\varkappa$ [ ] $\varkappa_1$ [ ] $\varkappa_2$ [ ] β_1 [ ] β_2 [ ]	see Fig. 6.

Material characteristics

ε_t [ ]	ultimate concrete strain;
ε_c [ ]	ultimate concrete compression;
σ_t [kp/cm^2]	ultimate concrete tensile stress;
σ_c [kp/cm^2]	ultimate concrete compressive stress;
E [kp/cm^2]	initial modulus of elasticity of concrete;
σ_s [kp/cm^2]	theoretical compressive stress at $\varepsilon = -\infty$ for a stress-strain curve described by a second-order fractional function;
ε_M [ ]	strain at maximum tensile stress for a stress-strain diagram described by a second-order fractional function;
B [kp/cm^2]	parameter of the rational function with second degree denominator describing the stress-strain curve;
R [kp/cm^2] S [kp/cm^2]	constants depending on the parameters of the stress-strain curve (see Table 2).

Characteristics of prestressing

ε_p [ ]	strain due to effective prestress;
e [cm]	excentricity of prestressing force with respect to the reference axis;
$\vartheta = \frac{e}{h}$ [ ]	relative distance of tendon centroid from the reference axis.

Moment of external loads

M_e [kp cm]	reference axis bending moment due to external loads in the section considered.

Variables of the deformation diagram

ε [ ]	concrete strain in a fibre;
ε_ϑ [ ]	concrete strain at the steel centroid level;
ε_{st} [ ]	steel strain;
ξ [ ]	relative depth of compressed concrete;
γ [ ]	relative distance between fibre of strain ε_t and the reference axis;
γ_c [ ]	relative distance of a fibre of strain ε_t to the reference axis, if top fibre strain is ε_c;
η [ ]	relative distance of a fibre of strain ε to the reference axis.

Stresses, internal forces

σ [kp/cm^2]	axial normal stress in any concrete fibre;
P [kp]	sum of axial stresses σ;
M [kp cm]	reference axis moment of stresses σ.

2. General stress-strain conditions for a cross-section

General relationships based on the principle of cross-sectional rotations involve the following fundamental assumptions:

1. The *Bernoulli—Navier* hypothesis is valid;
2. Concrete stress and strain are in a given relation;

3. Steel stress and strain are in a given relation;

4. Values of the initial prestress strain and of its reduction due to losses are given;

5. No repeated load acts;

6. Concrete cracks at ultimate tensile strain and fails at ultimate compressive strain;

7. Prestressing steel is bonded;

8. External forces produce no internal axial force.

Let us consider a cross-section of a given geometry (Fig. 1). A given strain diagram of a concrete section defines a unique stress diagram, based on the relationship $\sigma = \sigma(\varepsilon)$.

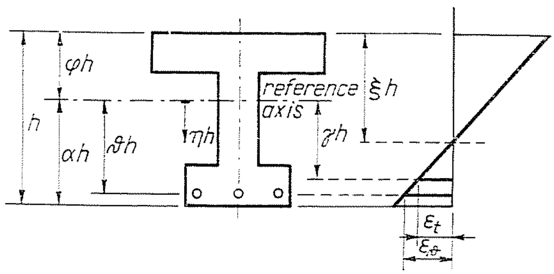

Fig. 1

Force P and its moment M to the reference axis can be written as:

$$\left.\begin{aligned} P &= h \int_{-\varphi}^{\gamma} b(\eta)\, \sigma[\varepsilon(\eta)]\, d\eta\,, \\ M &= h^2 \int_{-\varphi}^{\gamma} b(\eta)\, \eta\, \sigma[\varepsilon(\eta)]\, d\eta. \end{aligned}\right\} \qquad (1)$$

The strain ε at a section level, for ξ and γ given in function of η, is expressed by the relationship:

$$\varepsilon = f(\eta; \xi, \gamma) = \varepsilon_t \left(1 + \frac{\eta - \gamma}{\gamma + \varphi - \xi}\right).$$

Nos train beyond the ultimate compression being possible, it is:

$$\frac{\xi}{\gamma + \varphi - \xi} = \frac{\varepsilon_c}{\varepsilon_t}\,,$$

and of course:

$$\varphi + \gamma > \xi\,,$$

where γ like η is sign-dependent.

Integrals (1) co-ordinate values P, M to the arbitrary values ξ, γ, hence:

$$\left.\begin{aligned} P &= u(\xi, \gamma), \\ M &= v(\xi, \gamma). \end{aligned}\right\} \tag{2}$$

At service loads, the moment of external loads and the initial prestress are given, while ξ and γ are unknown still to be determined, together with the corresponding prestress modified by the deformation.

The values ξ, γ will be determined from the expression for the normal force composed by the simultaneously acting prestress strain and the steel strain increment due to the external load. Its horizontal projection balances the resultant of the axial concrete stresses:

$$P(\varepsilon_p + \varepsilon_\vartheta) = -u(\xi, \gamma). \tag{3}$$

The external load moment is balanced by two moments, the one due to the prestress and the other to the concrete stresses:

$$M_e = M + P \cdot e. \tag{4}$$

Substituting Eqs. (3) and (4) into (2) we obtain:

$$M_e = v(\xi,\gamma) + e \cdot u(\xi,\gamma) \tag{5}$$

and

$$P[\varepsilon_p + f(\vartheta; \xi, \gamma)] = -u(\xi, \gamma). \tag{6}$$

Eq. (5) defines a straight line in the co-ordinate system P, M (Fig. 2), while Eq. (6) a curve in the co-ordinate system γ, ξ, pertaining, of course, to given ε_p and ϑ (Fig. 3). This curve can also be plotted in the co-ordinate system P, M since with Eqs. (2) the ξ and γ values define a unique couple of values P, M (Fig. 4).

The intersection of this curve plotted in the co-ordinate system P, M with the straight line given by Eq. (4) defines the couple of values $P = P_I$; $M = M_I$ belonging to the couple of values $\xi = \xi_I$, $\gamma = \gamma_I$. If we have values ξ_I and γ_I, the stress-strain condition of the cross-section is known. The radius of curvature and the crack width can be determined if the MURASHOV [5] ψ value depending on the material properties and the steel type is known. KÁRMÁN [4] has determined the stresses for values ξ_I, γ_I in the special case of an assumed linear elasticity of concrete and steel.

Eqs (2) can be represented as a family of curves in the co-ordinate system P, M; γ is constant along each curve, and each point of this curve belongs to a certain value of ξ. Once this family of curves is put down, the values ξ_I, γ_I

can be read at the intersection of the straight line (5) by the curve (6). Knowing the curve family, values M_I, P_I and ξ_I, γ_I can be obtained without plotting (5) and (6), by simple *regula falsi*, similarly to the *Mörsch* graphical method.

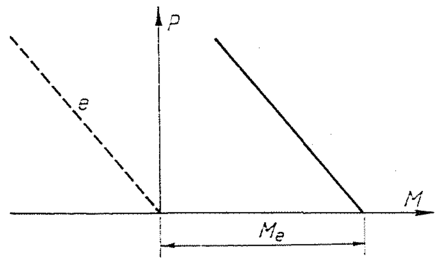

Fig. 2

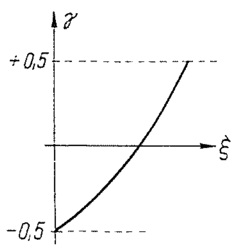

Fig. 3

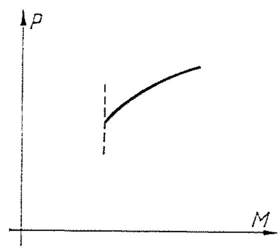

Fig. 4

3. Cracking moment, moment causing given crack width, and ultimate moment

The method presented in Section 2 suits to determine the stress-strain condition for an arbitrary M_e value. The problems of determining the cracking moment, the moment producing a given crack width and the ultimate moment differ essentially from the general case treated in Section 2 in that here the value of γ is known, or given as a function of ξ. Thus, in these special cases the P, M values are defined by one value of ξ, i.e. there is a single curve in the co-ordinate system P, M; and Eq. (6), now with a single unknown, delivers ξ_I. This is why in the considered special cases the moment asked for (either cracking, producing a given crack width, or ultimate moment) can be determined by *regula falsi*. (See [1], [8].)

The determination of the relationship between P and M for a constant γ_c and varying ξ has been treated in detail by SZALAI [7] for the special case of failure, neglecting the concrete tensile stresses and assuming constant compressive stresses. He suggested an application for non-ultimate moment cases, and developed a method for determining the moment maximum and the pertaining force P.

Expressions for general cases can be written according to general formulae, as compiled in Table 1. Relationships determining moments for a given crack width are essentially identical with those for the cracking moment, simply replacing ε_t and the integration limit α by the strain for the corresponding ψ value and the relative spacing for ε_t, respectively.

4. Limit analysis by a single analytic function describing the concrete stress-strain diagram

Eqs (1) are apparently determined by the stress-strain function of the concrete. Various approximations have been applied to describe the test stress-strain diagram of concrete. Mostly, either linear elasticity or rigid-plasticity has been assumed, or a combination of both in the compressed or tensile zone (see e.g. [2]) or possibly another function composed of linear sections.

Even assuming a linear $\sigma = \sigma(\varepsilon)$ function, complex relationships arise for the stress-strain analysis of the cracked section. There are few methods known for the stress-strain condition beyond the cracking moment, assuming a non-linear stress distribution. For instance, ROZENBLUMAS [6] obtained highly complex relationships for linear or constant stress distribution, even applying further approximations. In general cases, the use of computers cannot be avoided. Computer formulae can best be obtained by approaching the concrete stress-strain diagram by a single analytic expression. By this reason it is advisable to eliminate variables from the integration limits arising when functions are composed of different expressions for certain sections. Under these conditions the rational function with second degree denominator seems to be most convenient and most simple to describe the function $\sigma = \sigma(\varepsilon)$ (Fig. 5):

$$\sigma(\varepsilon) = -\sigma_s + \frac{\sigma_s D + B}{(\varepsilon - \varepsilon_M)^2 + D} \tag{7}$$

where

$$B = \sigma_s \varepsilon_M^2 ,$$

$$D = \frac{\sigma_s}{E} 2\,\varepsilon_M - \varepsilon_M^2 ,$$

$$\varepsilon_M = \frac{\varepsilon_t}{2\left(1 - \dfrac{1}{E\varepsilon_t} \dfrac{1}{\dfrac{1}{\sigma_s} + \dfrac{1}{\sigma_t}}\right)} .$$

Table 1

		General case	Special cases: First crack appears $\gamma = \alpha$ *)	Special cases: Failure $\gamma = \gamma_c(\xi) = \left(\frac{\varepsilon_t}{\varepsilon_c} + 1\right)\xi - \varphi$
ε		$\varepsilon = f(\eta\,;\,\gamma, \xi)$	$\varepsilon = f_{cr}(\eta; \xi)$	$\varepsilon = f_f(\eta; \xi)$
		$\varepsilon = \varepsilon_t\left(1 + \frac{\eta - \gamma}{\gamma + \varphi - \xi}\right)$	$\varepsilon = \varepsilon_t\left(1 + \frac{\eta - \alpha}{1 - \xi}\right)$	$\varepsilon = \varepsilon_c\left(\frac{\eta + \varphi}{\xi} - 1\right)$
		$\frac{\xi}{\gamma + \varphi - \xi} \leq \frac{\varepsilon_c}{\varepsilon_t}$	$\xi \leq \frac{\varepsilon_c}{\varepsilon_t + \varepsilon_c}$	
Upper limit of integration		γ	α	$\gamma_c(\xi) = \left(\frac{\varepsilon_t}{\varepsilon_c} + 1\right)\xi - \varphi \approx \xi - \varphi$
Lower limit of integration		$-\varphi$	$-\varphi$	$-\varphi$
(2)	P	$P = u(\xi, \gamma)$	$P = u(\xi, \alpha) = u_{cr}(\xi)$	$P = u[\xi, \gamma_c(\xi)] = u_f(\xi)$
	M	$M = v(\xi, \gamma)$	$M = v(\xi, \alpha) = v_{cr}(\xi)$	$M = v[\xi, \gamma_c(\xi)] = v_f(\xi)$
	ε_ϑ	$\varepsilon_\vartheta = f(\vartheta; \gamma, \xi)$	$\varepsilon_\vartheta = f_{cr}(\vartheta, \xi)$	$\varepsilon_\vartheta = f_f(\vartheta; \xi)$
		$\varepsilon_{st} = \varepsilon_p + \varepsilon_\vartheta$		
(3)		$P(\varepsilon_p + \varepsilon_\vartheta) = -u(\xi, \gamma)$	$P = (\varepsilon_p + \varepsilon_\vartheta) = -u_{cr}(\xi)$	$P(\varepsilon_p + \varepsilon_\vartheta) = -u_f(\xi)$
(4)		$M_e = M + P \cdot e$	$M_{cr} = M + P \cdot e$	$M_f = M + P \cdot e$
(5)		$P\,[\varepsilon_p + f(\vartheta\,;\,\gamma_I, \xi_I)] = -u(\xi_I, \gamma_I)$	$P\,[\varepsilon_p + f_{cr}(\vartheta, \xi_{cr})] = -u_{cr}(\xi_{cr})$	$P[\varepsilon_p + f_f(\vartheta; \xi)] = -u_f(\xi_f)$
(6)		$M_e = v(\xi_I, \gamma_I) + e \cdot v(\xi_I, \gamma_I)$	$M_{cr} = v_{cr}(\xi) + e \cdot u_{cr}(\xi_{cr})$	$M_f = v_f(\xi_f) + e \cdot u_f(\xi_f)$

* Moment for a given crack width is obtained by applying the same relationships to the sense.

The values σ_s, ε_t, E can be assumed so as to obtain realistic values for σ_c, ε_c; σ_t, ε_t; for instance to give a close approximation to the diagrams presented by HALÁSZ [3]. The monotonously increasing diagram at the compression side, in contrast to the empirical decreasing curve, is of no importance, except near the ultimate moment, but also here it is not too significant.

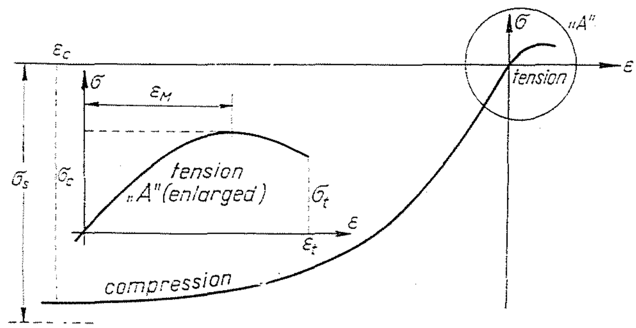

Fig. 5

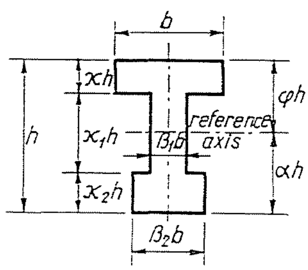

Fig. 6

Eqs (1) corresponding to function $\sigma = \sigma(\varepsilon)$ from Eq. (7) are compiled in Table 2, for the case of Fig. 6.

Correlated values of P and M, given in explicit form, are easy to calculate by means of a computer. P, M curves related to given ε_t values are at our disposal to plot for different cross-section types, and so are curves for a given crack width. Similarly, curves for correlated values of P and M producing ultimate compression in the top fibre can be plotted according to (7).

6. Conclusions

Different stress-strain relationships for prestressed pre-tensioned beams can be given for the general case. Unless the beam is crackfree and of a linearly elastic material, it is rather complicated to determine the stress-strain condition due to a given force, and it is only expedient if the tests are followed by calculations. Cracking moment, moment for a crack width, or ultimate moment are more simple to determine and feasible in engineering practice. It is more advisable therefore to specify in codes of practice that service moment must not exceed the moment for the specified crack width limit, than to give a limit for the crack width for the given (e.g. service) load.

Table 2

$\gamma < \alpha - \varkappa_2$	$P = bh\{\beta_1 G(\gamma) + (1-\beta_1) G(-\varphi+\varkappa) - G(-\varphi)\}$ $M = bh^2\{\beta_1 H(\gamma) + (1-\beta_1) H(-\varphi+\varkappa) - H(-\varphi)\}$
$\gamma > \alpha - \varkappa_2$	$P = bh\{\beta_2 G(\gamma) + (\beta_1-\beta_2) G(\alpha-\varkappa_2) +$ $+ (1-\beta_1) G(a-1+\varkappa) - G(-\varphi)\}$ $M = bh^2\{\beta_2 H(\gamma) + (\beta_1-\beta_2) H(\alpha-\varkappa_2) +$ $+ (1-\beta_1) H(\alpha-1+\varkappa) - H(-\varphi)\}$
	$G(\eta) = -\sigma_s\eta + R(\gamma+\varphi-\xi)\ \mathrm{arctg}\ \dfrac{\varepsilon_t - \varepsilon_M + \varepsilon_t \dfrac{\eta-\gamma}{\gamma+\varphi-\xi}}{\sqrt{D}}$ $H(\eta) = -\dfrac{\sigma_s}{2}\eta^2 + S(\gamma+\varphi-\xi)^2 \left\{ \dfrac{1}{2} \ln\left[\left(\varepsilon_t+\varepsilon_M+\varepsilon_t \dfrac{\eta-\gamma}{\gamma+\varphi-\xi}\right)^2 + D\right] - \right.$ $\left. - \dfrac{\varepsilon_t - \varepsilon_M + \dfrac{\varepsilon_t\gamma}{\gamma+\varphi-\xi}}{\sqrt{D}}\ \mathrm{arctg}\ \dfrac{\varepsilon_t - \varepsilon_M - \varepsilon_t \dfrac{\eta-\gamma}{\gamma+\varphi-\xi}}{\sqrt{D}} \right\}$ $R = \dfrac{\sigma_s D + B}{\varepsilon_t \sqrt{D}}, \quad S = \dfrac{\sigma_s D + B}{\varepsilon_t^2}.$

There exists an analytical function giving a fair approximation to the concrete stress-strain diagram both in compression and in tension. Such a relationship is of great use especially in computer limit analyses of cracking and of given crack width.

Summary

General expressions have been presented for the stress-strain conditions of prestressed concrete beams subject to arbitrary loads, also suitable for determining crack width due to any moment. They can also be applied to determine cracking moment, moment producing a given crack width and ultimate moment as special cases of the general method. By introducing a single analytical function truly fitting the concrete σ—ε diagram both in tension and in compression, it presents a relationship especially useful in digital computation.

References

1. Bölcskei, E.—Tassi, G.: Reinforced Concrete. Prestressed Concrete Structures. Tankönyvkiadó, Budapest, 1962.
2. Дмитриев, С. А. — Калатуров, Б. А.: Расчет предварительно напряженных железобетонных конструкций. Издательство литературы по строительству. Москва, 1965.
3. Halász, I.: Strain and Structural Changes in Loaded Building Materials. Proc. Tech. Univ. Build. Transp. Eng. XIII, No. 3—4. (1967).
4. Kármán, T.: Crack Width in Prestressed Structures. Institute for Building Research Budapest, 1963.
5. Мурашев, В. И.: Трещиноустойчивость, жесткость и прочность железобетона. Машстройиздат, Москва, 1950.
6. Розенблюмас, А. М.: Расчет напряжений и перемещений вызываемых в железобетонных конструкциях эксплуатационными нагрузками. Бетон и железобетон. Москва, 1961. 11.
7. Szalai, K.: Reinforced Concrete Beams Under Bending Moment and Normal Force. Inst. for Post-Grad. Ed. Eng. Budapest, 1967.
8. Tassi, G.: On the Design of Prestressed Post-Tensioned Beams. Inst. Post-Grad. Ed. Eng. Budapest, 1967.
9. Жданов, А. П.: Общий способ расчета по образованию трещин предварительно напряженных элементов с учетом неупругих деформаций в сжатой и растянутой зонах бетона. Бетон и железобетон. Москва, 1966. 12.

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Discussion

P. Lenkei, Institute for Building Research:

This lecture is of high value as it presents general relationships for the limit analysis of prestressed concrete beams by which the behaviour can be followed up to failure during loading.

Digital computers make it possible to apply this method for general cases. I wonder, however, if they do not permit to replace the computational assumptions by more exact ones. Such are for instance the *Bernoulli-Navier* hypothesis (assumption 1) in the lecture, the idealized $\sigma-\varepsilon$ diagram for concrete with no decreasing part of the diagram at the compression side (assumption 2), and the improbability of a repeated load acting on the beam (assumption 5).

Moreover, according to the first half of assumption 6, for the cracking moment the concrete tensile strength is also taken into consideration. It would be more correct to consider the cracking moment for zero crack width as the limit of crack control, since during the lifetime of the beam the cracking moment may be exceeded several times.