DESIGN OF ELASTIC-PLASTIC FRAMES UNDER PRIMARY BENDING MOMENTS

By

O. Halász

Department of Steel Structures, Budapest Technical University
(Received April 12, 1969)

1. First order approach: Simple plastic design

The analysis of the behaviour of elastic-plastic frames neglecting the change in geometry of the structures while setting up the equations of equilibrium will be referred to as first order approach. The load-deflection diagram of the frame in Fig. 1*a* according to the first-order approach — assuming unit shape factor — is to be seen in Fig. 1, dots and numbers indicating plastic hinges formed at the corresponding cross-sections.

If only failure load P_F is of interest, then the detailed analysis of the structure behaviour can be omitted as the fundamental (static and kinematic) theorems of simple plastic design directly yield the P_F value [1].

The fundamental theorems of simple plastic design can be utilised for two purposes; namely (i) to check the failure load of a given (previously designed) structure or (ii) — if the P_F value is given — to compute the required value of the full-plastic moment M_u of the cross-sections. This latter will be referred to as "direct method of design", illustrated in Fig. 1*c* to *e*. Based on previous consideration an adequate yield mechanism (pattern of plastic hinges) is to be chosen (Fig. 1*e*). Denoting the displacements of the external forces in the yield mechanism by u_i and the hinge rotations by χ_j (Fig. 1*d*), the virtual work equation furnishes:

$$\sum_i \alpha_i P_F u_i = \sum_j |M_u \varkappa_j| . \tag{1}$$

Supposing all the plastic hinges to form under a common value M_u of full-plastic moment, the required M_u value will be:

$$M_u = P_F \frac{\sum_i \alpha_i u_i}{\sum_j |\varkappa_j|} \tag{2}$$

Subsequently — using the equilibrium equations — the entire moment diagram can be determined (Fig. 1*e*) and the structure will be safe if designed so that the bending moments due to the former moment diagram nowhere exceed the full-plastic moment of the corresponding cross-sections.

The basic assumptions in Simple Plastic Design restrict its use to cases [2], [3] where either axial forces or deflections are small (continuous beams, no-sway frames with stocky columns bent in double curvature etc.). In other cases it may give unsafe estimate of the failure load.

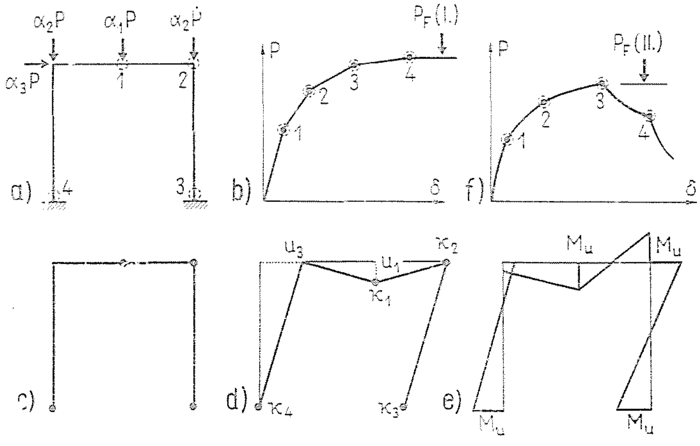

Fig. 1

2. Second order approach

Second order approach can be spoken of where the equilibrium equations are set up taking into account the deformations of the structure. A typical load-deflection diagram according to a second order approach — and supposing again unit shape factor — is illustrated in Fig. 1*f*. It differs basically from that in Fig. 1*b*; (i) branches are curvilinear; (ii) the failure load (peak load) is less than in the simple plastic theory; (iii) failure may occur before the complete yield mechanism has developed and is followed by unstable behaviour. In addition, the location and sequence of the plastic hinges do not necessarily coincide with those in the first-order approach.

Though the elastic-plastic frame analysis based on second order approach is dealt with in the literature [4], [5], [6], its practical application is cumbersome and bound to the use of a computer.

This paper is to offer an approximate solution possible by manual calculation as well.

3. Assumptions

Let a frame — such as that in Fig. 1 — be subject to monotonously increasing loads proportional to a single load factor P. In general, during the loading process the axial forces N_K in the members vary not only by magnitude but by relative proportion to each other as well.

For simplicity's sake we confine us to cases where — up to the failure load — a good approximation can be reached, expressing the axial forces in the form (Fig. 2*a*):

$$N_K = \beta_K P \,. \tag{3}$$

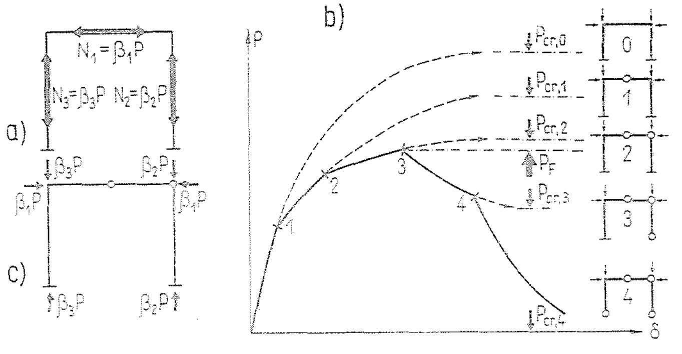

Fig. 2

The constants β_K can be determined from the axial forces delivered either by the first-order elastic analysis or by the simple plastic theory. Further the cross-sections are supposed to have unit shape factor; so the frame may be built up of perfectly elastic members, with plastic hinges at certain points only.

Finally, cases where backward rotation may take place in the plastic hinges are excluded from the analysis. Using these assumptions, the load-deflection curve can be characterised by the diagram in Fig. 2*b*. The subsequent branches of the curve indicate the behaviour of the frame containing an increasing number of plastic hinges. Each branch continued beyond its range of validity (broken lines in Fig. 2*b*) approaches asymptotically a certain value $P_{cr,n}$. These values are referred to in the literature as "deteriorated critical loads" [4], [8]. They represent the load factor causing buckling of an assumed completely elastic frame loaded by axial forces $N_K = \beta_K P$ only (Fig. 2*c*), with real hinges at locations where plastic hinges have developed in the actual frame. As an equivalent to our basic assumption expressed by Eq. (3), the "deteriorated" critical loads are to be computed, assuming that the axial forces remain unchanged during buckling.

4. The special case of the „direct method of design"

According to the Simple Plastic Design, the failure load — as indicated by Eq. (1) — depends upon the value of the full plastic moment only. In a second order approach, however, failure load depends upon two quantities: the full-plastic moment of the cross-sections (the "strength" of the structure) and "flexural rigidity" EJ of the members (E being the Young's modulus and J the moment of inertia of the cross-section). Assuming EJ to increase infinitely, the concept of rigid-plastic material is arrived at, failure will only occur after the formation of a complete yield mechanism (see "mechanism curve" in Fig. 2*a*). With decreasing stiffness the load-deflection curve may reach its peak value after the formation of hinges less than needed for the yield mechanism to develop (lower curves in Fig. 3*a*).

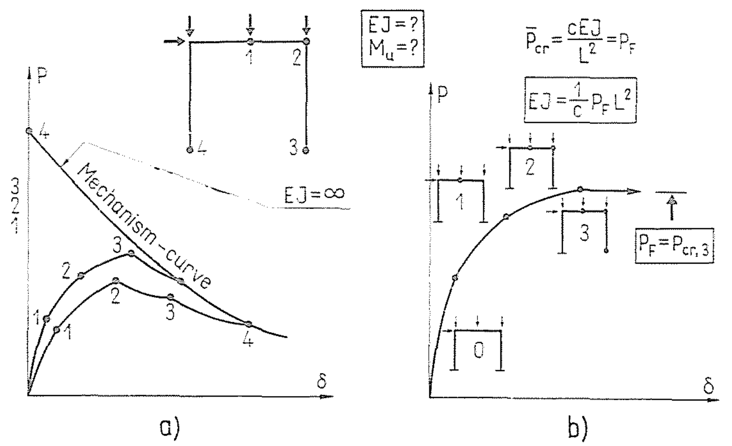

Fig. 3

A special but easy to handle case of the "direct method of design" is to design a structure where the predetermined failure load P_F coincides with one of the "deteriorated critical loads" $P_{cr,n}$. Fig. 3*b* represents a case with $n = 3$, i. e. the structure fails as soon as the third plastic hinge has developed. The "deteriorated" critical load $P_{cr,3}$ (the buckling load of a completely elastic frame with three real hinges and subject to a given set of axial forces) is a function of geometry (L) and rigidity (EJ) data:

$$P_{cr,3} = \frac{cEJ}{L^2} . \tag{4}$$

c being constant. Setting

$$P_F = P_{cr,3}\,5,$$

the required value of the frame rigidity is

$$EJ = \frac{L^2}{c} P_F . \tag{5}$$

The next problem is to calculate the required value of the full-plastic moment M_u. To this aim let us consider two structures (Fig. 4*a*, *b*). The first one is the actual structure just before failure: the load factor is equal to P_F, the third plastic hinge (at cross-section 3) is just about to develop. The displacements and the bending moments at cross-sections $j = 1; 2; 3$ are denoted by u and M_j, respectively.

The second one is the "deteriorated" structure with three real hinges subject to axial forces only, the load factor being $P_{cr} = P_{cr,3}$. This structure

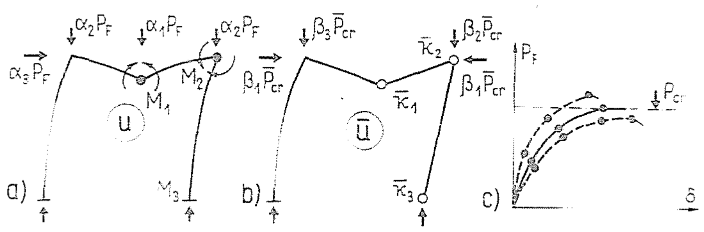

Fig. 4

will buckle under the load $P_{cr,3}$. The displacement and the hinge rotations during buckling are denoted by $\overline{u}$ and $\overline{\chi}_j$, respectively. The axial forces in both structures are defined by Eq. (3), and are supposed to keep unchanged during buckling.

Let us set up two virtual work equations, using two-way combination of loads and displacements of both systems:

$$\sum_i \alpha_i P_F \overline{u}_i - \sum_j M_j \overline{\varkappa}_j + \sum_k \beta_K P_F \int_l u' \overline{u}' \, dx = EJ \int_l u'' \overline{u}'' \, dx$$

$$\sum_k \beta_K P_{cr,3} \int_l \overline{u}' u' \, dx = EJ \int_l \overline{u}'' u'' \, dx. \tag{6}$$

where u' and u'' denote first and second derivatives, respectively. (Displacements u and $\overline{u}$ contain only first order terms, so $u_3 = \overline{u}_3 = 0$.)

After subtraction:

$$\sum_i \alpha_i P_F \overline{u}_i - \sum_j M_j \varkappa_j = \sum_k \beta_K (P_{cr,3} - P_F) \int_l \overline{u}' u' \, dx. \tag{7}$$

Considering Eq. (5)

$$P_F \sum_i \alpha_i \overline{u}_i = \sum_j M_j \overline{\varkappa}_j. \tag{8}$$

Now if failure has to occur at $P_F = P_{cr,3}$ (Fig. 4*c*), bending moments M_j in cross sections $j = 1, 2, 3$ have to equal the full-plastic moment, and thus

$$P_F \sum \alpha_i \overline{u}_i = \sum_j | M_u \overline{\varkappa}_j | . \tag{9}$$

Eq. (9) replaces Eq. (1) of the Simple Plastic Theory and helps to compute the required value of M_u. Supposing all the plastic hinges to form under the same value of M_u, by analogy to Eq. (2):

$$M_u = P_F \frac{\sum_i \alpha_i u_i}{\sum_j |\varkappa_j|} \tag{10}$$

Using Eqs. (5) and (10) the flexural rigidity EJ of the members and the full-plastic moment in the plastic hinge cross-sections can be computed.

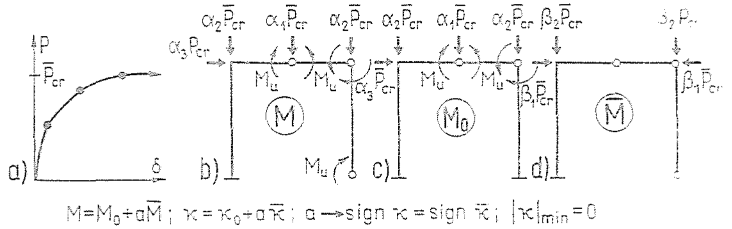

Fig. 5

For designing the rest of the cross-sections the entire bending moment diagram at failure is to be known. This can only be determined indirectly, as the strcuture is in an indifferent state of equilibrium during failure, the failure load P_F being equal to the "deteriorated" critical load (Fig. 5*a*):

$$P_F = \overline{P}_{cr} = P_{cr,3} \tag{11}$$

Without giving detailed prove of the procedure [9], let the bending moment diagram M at failure be composed of two parts:

$$M = M_0 + a\overline{M}. \tag{12}$$

The moment diagram M_0 refers to the structure after removing one plastic hinge chosen arbitrarily (Fig. 5*c*). As thus the frame gets into a state of stable equilibrium and the plastic hinges can be considered as real hinges with external moments equal to the full-plastic moment acting upon them, the diagram M_0 can be determined by a second-order elastic analysis [4]. The second bending moment diagram $\overline{M}$ represents the bending moments arising during buckling of the "deteriorated" structure (Fig. 5*d*) (with three real hinges and subject to axial forces only), which can be determined by known methods of second-order elastic theory [4], at least as far as its shape is concerned. The constant factor a is to be chosen as follows:

Like the bending moments, the displacements — among them plastic hinge rotations χ_j — can be built up of similarly chosen components:

$$\varkappa_j = \varkappa_{0j} + a\bar{\varkappa}_j \tag{13}$$

where χ_j and $\bar{\chi}_j$ are hinge rotations in the two structures defined above. We shall get the actual moments and displacements by selecting the value a so that for all hinge rotations

$$\operatorname{sign} \varkappa_j = \operatorname{sign} \bar{\varkappa}_j ,$$

and

$$|\varkappa_j|_{\min} = 0 . \tag{14}$$

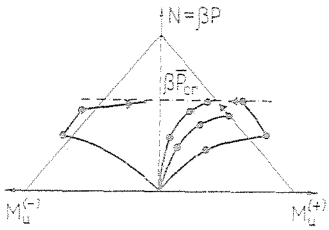

Fig. 6

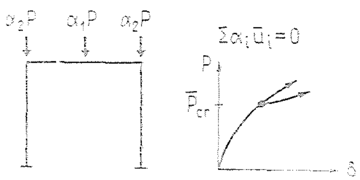

Fig. 7

Put in words, all hinge rotations should be of the same sign as hinge rotations of the deteriorated structure during buckling, but one of them (of course the ast plastic hinge, just forming at failure) should be zero.

Knowing the bending moment diagram M at failure the frame can be designed.

5. Additional remarks

The direct design method suggested above is based on the supposition that plastic hinges do not form but at predetermined cross-sections (indicated in the example by numbers 1, 2 and 3). The design according to the bending moment diagram $M = M_0 + aM$ at failure will make it certain to avoid other plastic hinges to exist at failure load, but further prove is needed, that no other plastic hinges develop in previous stages of the loading process. In conformity with the condition that the cross-sections are of unit shape factor, the interaction curve between axial force and full-plastic moment will be a straight line like that indicated in Fig. 6. It is to prove that bending moments at any value of the load factor P will not exceed the interaction curve. A method for this prove will be given in a subsequent publication.

The suggested direct design method is not applicable if failure occurs after formation of a complete yield mechanism. This case is dealt with in [10] in detail.

Finally, a special case emerges if in Eq. (9)

$$\sum' \alpha_i \bar{u}_i = 0 , \tag{15}$$

as e. g. in case of the frame indicated in Fig. 7, the actual deformations being normal to the buckling deformation. This case may lead to a bifurcation under stable conditions [11].

Summary

The use of the Simple Plastic Design is restricted to cases where change in geometry of the structure has negligible effect, otherwise it may give unsafe estimate of the failure load. Paper offers a direct method of design to be used when the structure fails by instability of the whole structure before a complete yield mechanism has developed. Attention is drawn to a special application of Shanley's phenomenon as well.

References

1. PRAGER, W.: An Introduction to Plasticity. Addison—Wesley Publishing Company, Inc. Reading, Massachusetts, USA.
2. GALAMBOS, T. V.: Structural Members and Frames. Prentice-Hall, Inc. New York, 1968.
3. HALÁSZ, O.: Adoption of the Theory of Plasticity to Steel Structures. Acta Technica Acad. Sci. Hung. T. **59**. 1967.
4. HORNE, M. R.—MERCHANT, W.: The Stability of Frames. Pergamon Press 1965.
5. HORNE, M. R.: The Stability of Elastic-Plastic Structures. Progress in Solid Mechanics, Vol. II. North-Holland Publishing Co. Amsterdam 1961.
6. LIVESLEY, R. K.: Symposium on the Use of Electronic Computers in Structural Engineering, University of Southampton, 1959.
7. WOOD, R. H.: The Stability of Tall Buildings, Proc. I. C. E. **11**, 69. 1957.
8. BOLTON, A.: Ph. D. Thesis, Manchester University 1957.
9. HALÁSZ, O.: Thesis. Manuscript, Budapest, 1969.
10. VOGEL, U.: Die Traglastberechnung stählerner Rahmentragwerke nach der Plastizitätstheorie II. Ordnung. Stahlbau-Verlag Köln, 1965.
11. HALÁSZ, O.: A Generalization of Shanley's Phenomenon. Symposium of IUTAM Hungarian National Group, Miskolc, 1967.

Professor Dr. OTTO HALÁSZ Budapest XI., Műegyetem-rkp. 3. Hungary