DETERMINATION OF THE HORIZONTAL NATURAL FREQUENCY OF MULTISTOREY PANEL BUILDINGS

by

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1. Introduction

Floor loads of panel buildings are supported on vertical reinforced concrete slabs rather than on columns or traditional brickwalls. Multistorey buildings constructed in this system have become very popular recently, because they are economical both structurally and technologically. Several problems, however, of the structural analysis of panel structures are still uncleared, and it can be said without exaggeration that theoretical research work lags behind practical demands. This is even more true for the always more frequent dynamic problems, part of which either are not at all solved yet or the available solution is unsatisfactory in many respects, such as that of the determination of the horizontal natural frequency of multistorey panel buildings. The structure is designed in most cases with non-symmetrical structural walls, and from the point of view of the vibration theory this means that the building cannot perform pure flexural or pure torsional vibrations.

In a simple girder of non-symmetrical section, where the centre of gravity does not coincide with the shear centre, the free vibration is composed of simultaneous flexural and torsional vibrations. Vibrations of this character are termed „coupled" vibrations. The „coupled" vibration of the girder can be described by an equation system of three simultaneous partial differential equations of the fourth order ([1, 4, 11]) the solution of which is very difficult even in case of simple type girders.

As to the horizontal free vibrations of panel buildings, they are evidently „coupled" vibrations, similar to those of the girder. Nevertheless because of the different character of the structures they cannot be approached in the same way. The most decisive structural difference between the two is that while every point of a girder section – conform to the theory based on the *Bernoulli-Navier* hypothesis – can only perform movements suiting the given geometrical conditions and not independently of the adjacent points. the load-bearing walls forming the slab structure – consisting of elements connected in most cases at skew angles – can perform some movements only in interaction, while others independently of each other. It is clear therefrom that this vibration system is a very complex one, and the existing methods are unappropriate to solve similar problemes.

In the following, a computation method is presented to determine the horizontal natural frequency of multistorey panel buildings lending itself for general cases of the mentioned type of buildings, provided certain simplifying conditions are met.

2. Computation principles and the assumed model

The considered building consists of horizontal floors and of perpendicular structural walls between. The cross-sectional elevation of such a building is shown in Fig.1. These walls at skew angles and structurally properly connected, form a single wall unit, in lack of such a connection, however, each wall has to be considered a separate element.

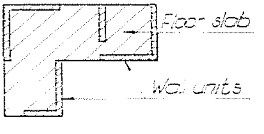

Fig.1.

In computation the following assumptions are made:

a) The load-bearing walls behave elastically in vibration.

b) Each floor forms in his own plane a plate to be considered infinitely rigid, but normally to it, they are perfectly flexible. This means that in the plane of the floor each wall element must displace and rotate by the same amount but normally to it they may deform independently of each other.

This condition is closely satisfied in reality and it is also in accordance with the actual design practice. In fact, floors develop an essentially lower resistance to moments causing bending normally to their plane than wall elements in their own plane, because in precast floor units usually no connections to bear bending moments are provided above the supports. Monolithic floor structures, however, to be built in the usual structural panel system, consist of moderately thick reinforced concrete slabs with negligible transverse rigidity.

c) Floor plan layout and thickness of the wall elements are quite arbitrary, through identical throughout the building. Interaction of superposed wall elements between storeys with themselves and the foundation is assured by means of rigid connections at the joints. Thus, the superposed wall elements constitute vertical cantilevers.

d) The torsional rigidity of the individual wall elements is very low as compared to the flexural rigidity, and can therefore be neglected in deformation analyses.

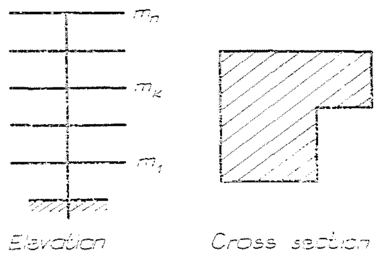

Fig.2.

Wall elements with open cross-section have in fact very low torsional rigidities compared to the bending rigidity. hence this assumption is quite justified. In case of closed (box-type) cross-sections the torsional rigidity of the wall element might be important, therefore neglecting it bears on the results; however this kind of wall element occurs but seldom in practice. It should be noted that taking into account the torsional rigidity of a closed cross-section does not involve changes in the solution principle; it is only tedious to compute each factor intervening in the equations.

e) The mass of the structure for each storey, including the mass of the floor of the walls and of other components, is assumed at the floor level but there is no restriction for the mass distribution itself.

f) The effect of damping modifying the natural frequency has been neglected.

The model used for computation purposes is shown in Figure 2. The cross-section – realized by cutting the building by a horizontal plane immediately above any floor – contains only the floor, considered as a disc, rigid in its plane, without the supporting wall elements. Such a section will be called in the following the building cross-section. As mentioned in the introduction. the cross-section of the building differs essentially from the cross-section of the ordinary girders in bending and therefore the usual cross-section characteristics have to be interpreted. In the following the *centre of gravity* (S) *of the building cross-*

section means the common centre of gravity of all sections of the wall elements in that cross-section. This generally does not coincide with the *mass centre* (S_M) of the cross-section, which represents the centre of gravity of the mass concentrated on floor level and continuously distributed in the horizontal plane.

Similarly as the section of the girder, the building cross-section has also a point, which – when acted upon by a horizontal external force acting in the plane of the floor, the cross-section (floor) is only displaced but does not rotate, whereas a torque acting on the floor would produce rotation around this point. This point is named in the following the *rotation centre of the building cross-section* (O).

A horizontal force passing through the rotation centre displaces the floor only in its plane, in general, however, the direction of the displacement does not agree with the direction of the force. As seen below, among the infinity of directions there are two orthogonal directions so that the force acting in these directions incites only displacement in direction of the force, and these will be termed *principal directions.*

Notations:

F	cross-sectional area of the wall element;
J_1, J_2	principal inertia moments of the wall element cross-section;
E	modulus of elasticity;
s	weakening coefficient of the wall element;
S	centre of gravity of cross-section;
S_M	mass centre;
O	centre of shear or torsion;
p_{xx}, p_{yy}, p_{xy}	$= p_{yx}$ rigidity coefficients of the wall element;
$A =$	$\sum_{i=1}^{m} p_{ixx}$; $\quad B = \sum_{i=1}^{m} p_{iyy}$; $\quad C = \sum_{i=1}^{m} p_{ixy} = \sum_{i=1}^{m} p_{iyx}$;
K	$= AB - C^2$;
x, y	co-ordinate system in the floor plane;
z	co-ordinate axis normal to the plane of the floors;
u, v	co-ordinate system in the principal directions of the building cross-section and displacement along the axes;
u_M, v_M	co-ordinates of the mass centre in the u, v system;
Φ	angle of rotation around the centre of rotation of the building cross-section;
E_h	potential energy;
J_o	inertia moment of the building cross-section mass acting in the rotation centre;

E_m	kinetic energy;
M_i	numerical value of torque inducing unit rotation of building cross-section;
m_i	mass of floor concentrated at storey level;
p_u, p_v	numerical value of force acting in principal direction and inducing unit displacement.

3. Determination of cross-section characteristics

Centres of gravity and of mass of the building cross-section, as defined above, are determined by the well-known computation method for centroids of planes and masses, respectively. New notions are principal directions and rotation centre, functions of the cross-sectional rigidity and related with the rigidity of the individual wall elements, considered as rigidly restrained cantilevers. Their determination will be discussed next.

3.1 Determination of rigidity factors of the wall elements

As known from the theory of strength, there is always a characteristic point in integral beam cross-sections, which if passed by a force parallel to the plane of the cross-section, this cross-section is only displaced, and if acted upon by a torque, the cross-section rotates around this point. The point itself is termed the *centre of shear or torsion*. This characteristic point is always on the symmetry axis of the cross-section, therefore in cases of bisymmetry it coincides with the centre of gravity, though it may also be determined for general cases [9].

The feature common in both the principal directions of inertia of the beam cross-section and of the building cross-section is that an external force parallel to any of them will displace the cross-section parallelly to the direction of the force. As the principal directions of inertia for the cross-section (1,2) and its principal inertia moments (J_1, J_2) can be easily determined by means of relationships known from the strength theory [7], they will be assumed to be known in the following.

Computation of rigidity coefficients of each wall element is also known from the literature [7,9] and will be but shortly treated.

Assume a prismatic bar rigidly clamped at one end, made of homogeneous material obeying Hooke's law. To displace cross-section i at a distance z from the clamping, by magnitudes Δ_1 and Δ_2, parallelly to principal directions 1 and 2, resp., forces P_1 and P_2, parallel to the respective displacements, have to be applied in the shear centre of the cross-section k at a distance c ($c \geqq z$) from the clamping point. Force values can be determined from a relationship known from the theory of strength, as:

$$P_1 = \frac{J_2}{H} \Delta_{1ik} \tag{1}$$

and

$$P_2 = \frac{J_1}{H} \Delta_{2ik} \tag{2}$$

respectively.

In the formula

$$H = \frac{s}{E}\left(c\frac{z^2}{2} - \frac{z^3}{6}\right)$$

where E is the modulus of elasticity of the bar material and s a factor expressing the effect of wall openings (doors, windows), the determination of which will be discussed in chapter 7.

Assume now a co-ordinate system xy at a distance z from the clamping and determine the external force to be applied in cross-section k to obtain unit displacement of the cross-section shear centre in direction of axis x. If the x axis is not a principal direction, the displacing force is not parallel with the x axis, but can be described by components p_{xx} and p_{xy} in directions x and y, respectively, which can be computed as follows. The components in the principal directions 1 and 2, respectively, of the unit displacement in direction of the x axis, at an angle α to the principal direction 1 – as seen in Figure 3 – are 1. cos α and -1. sin α, respectively. These can be produced by forces

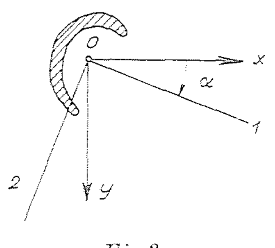

Fig.3.

$$Q_1 = \frac{J_2}{H} \cos \alpha \tag{3}$$

and

$$Q_2 = -\frac{J_1}{H} \sin \alpha \tag{4}$$

acting in direction of axes 1 and 2, respectively, according to formulae (1) and (2). The required force components p_{xx} and p_{xy} are given by the sum of projections in directions x and y, respectively, hence:

$$p_{xx} = Q_1 \cos \alpha - Q_2 \sin \alpha = \frac{1}{H}\left(J_1 \sin^2 \alpha + J_2 \cos^2 \alpha\right) \tag{5}$$

$$p_{xy} = Q_1 \sin \alpha + Q_2 \cos \alpha = -\cos \alpha \sin \alpha \frac{J_1 - J_2}{H}. \tag{6}$$

For a unit displacement in direction y of the considered cross-section, forces p_{yy} and p_{yx} in direction of axes y and x, resp., should be applied. They can be determined as above, namely:

$$p_{yy} = \frac{1}{H}\left(J_1 \cos^2 \alpha + J_2 \sin^2 \alpha\right) \tag{7}$$

$$p_{yx} = -\frac{1}{H} \cos \alpha \sin \alpha \,(J_1 - J_2) = p_{xy}. \tag{8}$$

The quantities p_{xx}, p_{yy} and $p_{xy}=p_{yx}$ will be named in the following the *rigidity coefficients of the wall element.*

3.2 Determination of the principal directions, the centre of rotation and the dynams inducing unit displacement of the building cross-section.

The principal directions of the building cross-section can be determined by supposing that the displacement due to a force parallel to the principal direction and passing through the rotation centre will also be parallel to the direction of the force. The reverse is also true, namely that the so-called restoring force produced by the displacement in the principal direction has its influence line also in the principal direction. In the so far unknown centre of rotation of the building cross-section scheme in Fig.4 a co-ordinate system of arbitrary xy

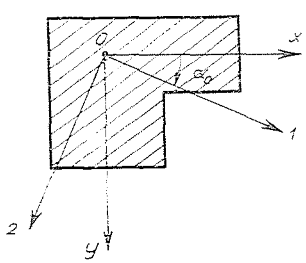

Fig.4.

axes has been assumed. Axis x includes with the principal direction an angle α_o. Let us displace the cross-section supported by m wall elements along the principal direction by Δ. The influence line of the force R necessary for the displacement is also in the principal direction. The components $\Delta \cos \alpha_o$ and $\Delta \sin \alpha_o$ of displacement Δ in direction of the x and y axes, respectively, and forces necessary to produce these displacements are equal to the components of R in direction of the corresponding axis. Between the forces necessary to induce the displacement components and the corresponding components of R, the following relationships can be written by means of the rigidity coefficient of each wall element:

$$\Delta \cos \alpha_o \sum_{i=1}^{m} p_{ixx} + \Delta \sin \alpha_o \sum_{i=1}^{m} p_{ixy} = R \cos \alpha_o$$

$$\Delta \sin \alpha_o \sum p_{iyy} + \Delta \cos \alpha_o \sum p_{ixy} = R \sin \alpha_o . \tag{9}$$

In the equations p_{ixx}, p_{iyy} and $p_{ixy} = p_{iyx}$ stand for the rigidity coefficients of the i-th wall element and the summation comprises all m wall elements.

Introducing notations

$$A = \sum_{i=1}^{m} p_{ixx}; \qquad B = \sum_{i=1}^{m} p_{iyy}; \qquad C = \sum_{i=1}^{m} p_{ixy}$$

and multiplying the first equation by $\sin \alpha_o$ and the second one by $-\cos \alpha_o$ we obtain:

$$\Delta \sin \alpha_o \cos \alpha_o A + \Delta \sin^2 \alpha_o C = R \cos \alpha_o \sin \alpha_o$$

$$-\Delta \sin \alpha_o \cos \alpha_o B - \Delta \cos^2 \alpha_o C = -R \cos \alpha_o \sin \alpha_o . \tag{10}$$

Reducing the two equations and simplifying by Δ:

$$(A - B) \sin \alpha_o \cos \alpha_o - C(\cos^2 \alpha_o - \sin^2 \alpha_o) = 0 .$$

The obtained relationship can be written in a more expedient form by introducing functions of double angles. As it is known:

$$\sin 2\alpha_o = 2 \sin \alpha_o \cos \alpha_o$$

$$\cos 2\alpha_o = \cos^2 \alpha_o - \sin^2 \alpha_o$$

$$\cos^2 \alpha_0 = \frac{1 + \cos 2\alpha_0}{2} \quad \text{and} \quad \sin^2 \alpha_0 = \frac{1 - \cos 2\alpha_0}{2} .$$

After substituting and arranging:

$$\frac{A - B}{2} \sin 2\alpha_0 - C \cos 2\alpha_0 = 0.$$

Finally, dividing the equation by $\cos \alpha_o$, it can be written:

$$\boxed{\operatorname{tg} 2\alpha_0 = \frac{2C}{A-B}} \tag{11}$$

From this relationship it appears that for α_o two solutions exist. Namely, if an α_0' satisfies the equation, also the angle $\alpha_0'' = \alpha_0' + \frac{\pi}{2}$ will satisfy it since $\operatorname{tg} 2\alpha_0'' = \operatorname{tg}(2\alpha_0' + \pi) = \operatorname{tg} 2\alpha_0'$. This means that *two directions normal to each other are found for which the determination of the principal direction is valid, and thereby the existence of the principal directions is proved.*

As seen, in the formula for α_o the distance between the clamping and the considered cross-section does not intervene, therefore the relationship (11) can be used equally for the building cross-section considered at any storey.

Later on, also the force inducing a unit displacement in the principal direction will be needed, therefore determination of this force will be considered next. The force causing a displacement $\Delta = 1$ in the principal direction including with the x axis an angle α_o is denoted by p. Accordingly, equations (9) can be written in the following form:

$$\begin{aligned} A \cos \alpha_o + C \sin \alpha_o &= p \cos \alpha_o \\ B \sin \alpha_o + C \cos \alpha_o &= p \sin \alpha_o . \end{aligned} \tag{12}$$

Multiplying the first equation by $\cos \alpha_o$ and the second one by $\sin \alpha_o$, subtracting the second from the first, the following relationship is obtained:

$$A \cos^2 \alpha_o - B \sin^2 \alpha_o = p(\cos^2 \alpha_o - \sin^2 \alpha_o).$$

Hence:

$$p = \frac{A \cos^2 \alpha_0 - B \sin^2 \alpha_0}{\cos^2 \alpha_0 - \sin^2 \alpha_0} .$$

Introducing again the relationships for the double angles as described above, the obtained equation can be written as:

$$p = \frac{1}{2 \cos 2\alpha_0} \left[A(1 + \cos 2\alpha_0) - B(1 - \cos 2\alpha_0) \right] . \tag{13}$$

For the force p causing unit displacement in the principal direction also a direct relationship might be deduced, by substituting the value for $2\alpha_o$ according to (11) into formula (13). This latter, however, includes the trigonometric function $\cos 2\alpha_o$, therefore the relationship between $\cos 2\alpha_o$ and $\operatorname{tg} 2\alpha_o$ may be applied. According to formulae known from trigonometry, $\cos 2\alpha_0$ can be expressed by $\text{tg}\, 2\alpha_0$ as follows:

$$\cos 2\alpha_0 = \frac{1}{\sqrt{1+\text{tg}^2\, 2\alpha_0}}.$$

Replacing $\text{tg}^2\, 2\alpha_0$ by its value obtained from (11) it can be written:

$$\cos 2\alpha_0 = \frac{1}{\sqrt{1+\frac{4C^2}{(A-B)^2}}} = \frac{A-B}{2}\frac{1}{\sqrt{\left(\frac{A-B}{2}\right)^2+C^2}} = \frac{A-B}{2}\frac{1}{D} \qquad (14)$$

where $D = \sqrt{\left(\frac{A-B}{2}\right)^2+C^2}$.

Now the relationship (13) can be simplified by replacing $\cos 2\alpha_0$ in (13) by its form in (14):

$$p = \frac{D}{A-B}\left[A\left(1+\frac{A-B}{2D}\right)-B\left(1-\frac{A-B}{2D}\right)\right].$$

After reduction we obtain

$$p = \frac{A-B}{2} \pm D = \frac{A+B}{2} \pm \sqrt{\left(\frac{A-B}{2}\right)^2+C^2}\ . \qquad (15)$$

The double sign of the square root results in two values, corresponding to the two principal directions. The higher and lower values (p_u and p_v in the following) are obtained by taking into account the square root with its positive and negative value, respectively. Thus, using the symbols in Fig.4, the forces inducing unit displacement along the principal directions can be computed as follows:

$$\boxed{\begin{aligned} p_u &= \frac{A+B}{2}+\left[\left(\frac{A-B}{2}\right)^2+C^2\right]^{1/2} \\ p_v &= \frac{A+B}{2}-\left[\left(\frac{A-B}{2}\right)^2+C^2\right]^{1/2} \end{aligned}} \qquad (16)$$

Now the determination of the rotation centre of the building cross-section will be considered according to [7] and [9]. Let us apply to the floor, at a distance z from the clamping, a horizontal force $R_x=1$ Mp, parallel to the x axis in Fig.4, acting in a still unknown rotation centre. By definition it does not rotate the floor, only displace it by Δ_{xy} and Δ_{xy} in direction of axes x and y, respectively. Similarly as for the computation of the principal directions, the following projection equations can be written for directions x and y:

$$\left.\begin{aligned} \Delta_{xx}A + \Delta_{xy}C &= 1 \\ \Delta_{xx}C + \Delta_{xy}B &= 0 \end{aligned}\right\}. \tag{17}$$

Introducing the simplification $K = AB - C^2$, from the obtained equation system the floor displacements can be expressed as:

$$\Delta_{xx} = \frac{B}{K}; \qquad \Delta_{xy} = -\frac{C}{K}.$$

Knowing the displacements, forces in directions x and y acting on each wall element are easily computed from external force $R_x = 1$ Mp. Forces acting on the i-th wall element in directions x and y are, respectively:

$$r_{ixx} = \Delta_{xx} p_{ixx} + \Delta_{xy} p_{ixy} = \frac{1}{K}\left(B\, p_{ixx} - C\, p_{ixy}\right) \tag{18}$$

$$r_{ixy} = \Delta_{xx} p_{ixy} + \Delta_{xy} p_{iyy} = \frac{1}{K}\left(B\, p_{ixy} - C\, p_{iyy}\right). \tag{19}$$

The external force being their resultant, it can be written that its moment, in a point (in our case in the origin O' of co-ordinate system $x'y'$ in Fig.5) equals the sum of moments of the forces applied at the same point for each wall element.
Therefore:

$$\sum_{i=1}^{m} (x'_i r_{ixy} - y'_i r_{ixx}) = y'_i.$$

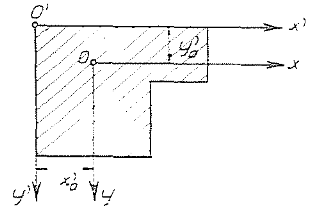

Fig.5.

Here x'_i and y'_i are distances of the shear centre of the i-th wall element from axes y' and x', respectively; and y' is the distance of the rotation centre from the x' axis. This latter distance is, using (18) and (19):

$$y'_0 = \frac{B}{K}\left[\sum_{i=1}^{m} y'_i\, p_{ixx} - \sum_{i=1}^{m} x'_i\, p_{ixy}\right] + \frac{C}{K}\left[\sum_{i=1}^{m} x'_i\, p_{iyy} - \sum_{i=1}^{m} y'_i\, p_{ixy}\right]. \tag{20}$$

Similarly, examining the effect of force $R_y = 1$ Mp acting in direction of the y axis, the distance of the rotation centre from the y' axis can be deduced as:

$$x_0' = \frac{A}{K}\left[\sum_{i=1}^{m} x_i' p_{iyy} - \sum_{i=1}^{m} y_i' p_{ixy}\right] + \frac{C}{K}\left[\sum_{i=1}^{m} y_i' p_{ixx} - \sum_{i=1}^{m} x_i' p_{ixy}\right].$$

Finally it will be determined what moment is needed in the rotation centre of the building cross-section to produce unit rotation of the cross-section. To this purpose it is supposed that the floor pertaining to the considered cross-section undergoes unit rotation. In this case the shear centre of the i-th wall element is displaced in directions x and y by x_i and y_i, resp. and to induce the displacement, in the shear centre forces

$$r_{i\varphi x} = (x_i p_{ixy} - y_i p_{ixx}) \tag{22}$$

and

$$r_{i\varphi y} = (x_i p_{iyy} - y_i p_{ixy}) \tag{23}$$

have to act in directions x and y, respectively. In these formulae $x_i = x_i' - x_0'$ and $y_i = y_i' - y_0'$ stand for the ordinatae of the shear centre of the i-th wall element in the co-ordinate system at the rotation centre. The sum of the moments pertaining to the rotation centre and due to forces developed by the rotation must be equal to the torque of the couple producing the rotation. Thus:

$$M = \sum_{i=1}^{m} (x_i r_{i\varphi x} - y_i r_{i\varphi y}) = \sum_{i=1}^{m} y_i^2 p_{ixx} - 2\sum_{i=1}^{m} x_i y_i p_{ixy} + \sum_{i=1}^{m} x_i^2 p_{iyy}. \tag{24}$$

4. Differential equation system of the vibration and its solution

The differential equation of a complex vibrating system may often be directly expressed by the *Lagrange equation*, using the relationships including kinetic and potential energy of the structure. For the determination of the natural frequency of multistorey panel buildings the same method is chosen, starting from the displacements of the floor i of the building. As mentioned above, the vibrational motion of the floor can be described by its rotation about the rotation centre and a simultaneous displacement. Instead of the displacement of the rotation centre of the floor, further on its components u and v in the principal directions will be considered and the rotation denoted by Φ as shown in Fig.6. As seen, the displacement causes the rotation centre of the floor to move into O' and its mass centre into S_M'. At a given instant the kinetic energy can be obtained as the momentum of a rigid body performing rotating and advancing motion.

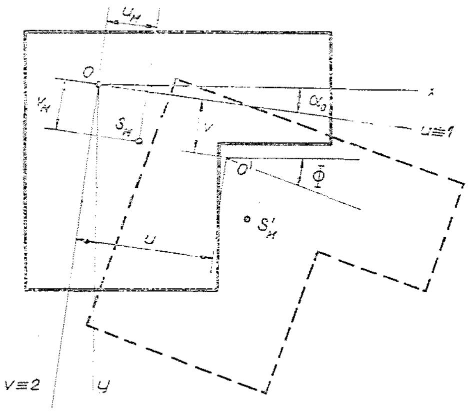

Fig.6.

The kinetic energy of the mass of the i-th storey, concentrated in the plane of the floor, is:

$$E_{mi} = \frac{m_i \dot{u}_i^2}{2} + \frac{m_i \dot{v}_i^2}{2} + \frac{J_{0i}\dot{\Phi}_i^2}{2} + m_i \dot{\Phi}_i (u_M \dot{v}_i - v_M \dot{u}_i). \tag{25}$$

(The superscript point denotes the time-dependent derivative of the displacement.)

The kinetic energy of the whole building, expressed in matrix form, is:

$$E_m = \frac{1}{2}\,\dot{\mathbf{u}}^* \mathbf{M} \dot{\mathbf{u}} + \frac{1}{2}\,\dot{\mathbf{v}}^* \mathbf{M} \dot{\mathbf{v}} + \frac{1}{2}\,\dot{\mathbf{\Phi}}^* \mathbf{J}_0 \dot{\mathbf{\Phi}} + \mathbf{M}\dot{\mathbf{\Phi}}(u_M \dot{v} - v_M \dot{u}) \tag{26}$$

with vectors:

$$\dot{\mathbf{u}} = \begin{bmatrix} \dot{u}^1 \\ \cdot \\ \cdot \\ \cdot \\ u_i \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{bmatrix}; \qquad \dot{\mathbf{v}} = \begin{bmatrix} \dot{v}^1 \\ \cdot \\ \cdot \\ \cdot \\ v_i \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{bmatrix}; \qquad \dot{\mathbf{\Phi}} = \begin{bmatrix} \dot{\Phi}^1 \\ \cdot \\ \cdot \\ \cdot \\ \Phi_i \\ \cdot \\ \cdot \\ \cdot \\ \Phi_n \end{bmatrix}$$

and diagonal matrices:

$$\mathbf{M} = < m_1, m_2 \ldots m_n >; \qquad \mathbf{J}_o = < J_{o1}, J_{oi}, \ldots J_n > .$$

The potential energy of the floor is in the same instant:

$$E_{hi} = \frac{1}{2}\left[k_{ui}\, u_i^2 + k_{vi} v_i^2 + k_{\varphi i} \Phi_i^2\right]. \tag{27}$$

Here k_{ui} and k_{vi} denote the spring constants of the floor displacement in directions u and v, respectively, whereas $k_{\varphi i}$ denotes its torsional spring constant.

The potential energy of the whole building, similarly expressed as for the kinetic energy:

$$E_h = \frac{1}{2}\left[\mathbf{u}^* \mathbf{K}_u \mathbf{u} + \mathbf{v}^* \mathbf{K}_v \mathbf{v} + \mathbf{\Phi}^* \mathbf{K}_\varphi \mathbf{\Phi}\right] \tag{28}$$

with vectors:

$$\dot{\mathbf{u}} = \begin{bmatrix} u_1 \\ \cdot \\ \cdot \\ u_i \\ \cdot \\ \cdot \\ u_n \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ \cdot \\ \cdot \\ v_i \\ \cdot \\ \cdot \\ v_n \end{bmatrix} \qquad \mathbf{\Phi} = \begin{bmatrix} \Phi_1 \\ \cdot \\ \cdot \\ \Phi_i \\ \cdot \\ \cdot \\ \Phi_n \end{bmatrix}$$

and diagonal matrix:

$$\mathbf{K} = \langle k_1,\ k_2,\ \ldots . k_n \rangle .$$

The Lagrange differential equation of motion is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E_m}{\partial \mathbf{q}} - \frac{\partial E_m}{\partial \mathbf{q}} - \frac{\partial E_h}{\partial \mathbf{q}} = \mathbf{f}. \tag{29}$$

In this equation $\mathbf{q}(\mathbf{u}, \mathbf{v}, \mathbf{\Phi})$ are vectors of the so-called generalized co-ordinates characterizing motion of the floor mass centre, and $\mathbf{f}$ the active dynam vectors, $\mathbf{f}$ being zero in the considered case of free vibration.

The derivatives of the Lagrange equation are:

$$\frac{\partial E_m}{\partial \dot{\mathbf{u}}} = \mathbf{M}\dot{\mathbf{u}} - v_M \mathbf{M}\dot{\mathbf{\Phi}}$$

$$\frac{\partial E_m}{\partial \dot{\mathbf{v}}} = \mathbf{M}\dot{\mathbf{v}} + u_M \mathbf{M}\dot{\mathbf{\Phi}}$$

$$\frac{\partial E_m}{\partial \dot{\mathbf{\Phi}}} = \mathbf{J}_o \dot{\mathbf{\Phi}} + \mathbf{M}(u_M \dot{\mathbf{v}} - v_M \dot{\mathbf{u}})$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E_m}{\partial \dot{\mathbf{u}}} = \mathbf{M}\ddot{\mathbf{u}} - v_M \mathbf{M}\ddot{\mathbf{\Phi}}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E_m}{\partial \dot{\mathbf{v}}} = \mathbf{M}\ddot{\mathbf{v}} + u_M \mathbf{M}\ddot{\mathbf{\Phi}}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E_m}{\partial \dot{\mathbf{\Phi}}} = -v_M\mathbf{M}\ddot{\mathbf{u}} + u_M\mathbf{M}\ddot{\mathbf{v}} + \mathbf{J}_0\ddot{\mathbf{\Phi}}$$

$$\frac{\partial E_m}{\partial \mathbf{u}} = \frac{\partial E_m}{\partial \mathbf{v}} = \frac{\partial E_m}{\partial \mathbf{\Phi}} = 0,$$

thus

$$\frac{\partial E_h}{\partial \mathbf{u}} = \mathbf{K}_u \mathbf{u}$$

$$\frac{\partial E_h}{\partial \mathbf{v}} = \mathbf{K}_v \mathbf{v}$$

$$\frac{\partial E_h}{\partial \mathbf{\Phi}} = \mathbf{K}_\varphi \mathbf{\Phi}.$$

The three derivatives, based on $\frac{\partial E_h}{\partial \mathbf{q}_0}$ are seen to be actually the magnitudes of the restoring force or moment in vibration, due to the elastic support of the floor. In single mass systems, with one degree of freedom, they are easy to determine; in the present case, however, as there are several masses, the magnitude of the restoring dynams acting on a certain mass is influenced by the displacement of the other masses as well, namely the displacement of one mass entrains the other mass and *vice versa*. A method for computing the restoring dynams will be considered in the following for such cases.

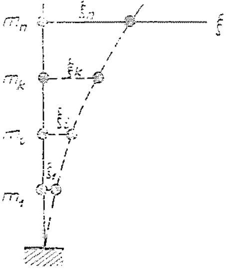

Fig. 7.

The cantilever shown in Fig. 7 carries concentrated masses. Denote displacements of direction ξ at the i-th and k-th mass produced by unit horizontal force acting at i, by a_{ii} and a_{ik}, respectively. Similarly, the unit horizontal force acting at mass k produces displacements a_{kk} and a_{ki} at k and i, respectively. As known from the interchangeability theorem of *Maxwell*, $a_{ki} = a_{ik}$. In the knowledge of the preceding, the forces R inciting jointly a displacement ξ_i at mass i can be computed. Yet these forces yield the elastic restoring force acting on the mass m_i during the simultaneous displacement ξ_i.

The elastic restoring forces acting on each mass will be determined by the following system of equations, based on the principle of superposition:

$$\begin{aligned} a_{11}R_1+a_{12}R_2+\ldots+a_{1n}R_n &= \xi_1 \\ a_{21}R_1+a_{22}R_2+\ldots+a_{2n}R_n &= \xi_2 \\ \cdot \quad \cdot \quad \cdots \quad \cdot & \\ \cdot \quad \cdot \quad \cdots \quad \cdot & \\ a_{n1}R_1+a_{n2}R_2+\ldots \quad a_{nn}R_n &= \xi_n. \end{aligned}$$

The equation system written in matrix form:

$$\mathbf{N}\,\mathbf{r} = \boldsymbol{\xi} \qquad (30)$$

with vectors:

$$\mathbf{r} = \begin{bmatrix} R_1 \\ R_2 \\ \cdot \\ \cdot \\ \cdot \\ R_n \end{bmatrix}; \qquad \boldsymbol{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \cdot \\ \cdot \\ \cdot \\ \xi_n \end{bmatrix}$$

and matrix:

$$\mathbf{N} = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots\ldots\ldots & a_{1n} \\ a_{21} & a_{22} \ldots\ldots\ldots\ldots & a_{2n} \\ \cdot & \cdot \ \ldots\ldots\ldots\ldots & \cdot \\ \cdot & \cdot \ \ldots\ldots\ldots\ldots & \cdot \\ \cdot & \cdot \ \ldots\ldots\ldots\ldots & \cdot \\ a_{n1} & a_{n2} \ldots\ldots\ldots\ldots & a_{nn} \end{bmatrix}$$

The solution of the matrix equation:

$$\mathbf{r} = \mathbf{N}^{-1}\boldsymbol{\xi} \qquad (31)$$

where $\mathbf{N}^{-1}$ is the so-called inverse matrix of $\mathbf{N}$. Matrix $\mathbf{N}$ being symmetrical, its inverse is also symmetrical, hence $a'_{ij} = a'_{ji}$.

On the basis of the foregoing, the derivatives with respect to the generalized co-ordinates of the potential energy will be:

$$\frac{\partial E_h}{\partial \mathbf{u}} = \mathbf{r}_u = \mathbf{N}_u^{-1}\,\mathbf{u}$$

$$\frac{\partial E_h}{\partial \mathbf{v}} = \mathbf{r}_v = \mathbf{N}_v^{-1}\mathbf{v}$$

$$\frac{\partial E_h}{\partial \mathbf{\Phi}} = \mathbf{r}_\varphi = \mathbf{N}_\varphi^{-1}\mathbf{\Phi}$$

which give at the same time the restoring dynams acting on the floors. In the above

$$\mathbf{r}_u = \mathbf{N}_u^{-1} u = \begin{bmatrix} a'_{u,11}u_1 + a'_{u,12}u_2 + \ldots + a'_{u,1n}u_n \\ a'_{u,21}u_1 + a'_{u,22}u_2 + \ldots + a'_{u,2n}u_n \\ \vdots \\ a'_{u,n1}u_1 + a'_{u,n2}u_2 + \ldots + a'_{u,nn}u_n \end{bmatrix}$$

$$\mathbf{r}_v = \mathbf{N}_v^{-1}\mathbf{v} = \begin{bmatrix} a'_{v,11}v_1 + a'_{v,12}v_2 + \ldots + a'_{v,1n}v_n \\ a'_{v,21}v_1 + a'_{v,22}v_2 + \ldots + a'_{v,2n}v_n \\ \vdots \\ a'_{v,n1}v_1 + a'_{v,n2}v_2 + \ldots + a'_{v,nn}v_n \end{bmatrix}$$

$$\mathbf{r} = \mathbf{N}^{-1}\mathbf{\Phi} = \begin{bmatrix} a'_{\Phi,11}\Phi_1 + a'_{\Phi,12}\Phi_2 + \ldots + a'_{\Phi,1n}\Phi_n \\ a'_{\Phi,21}\Phi_1 + a'_{\Phi,22}\Phi_2 + \ldots + a'_{\Phi,2n}\Phi_n \\ \vdots \\ a'_{\Phi,n1}\Phi_1 + a'_{\Phi,n2}\Phi_2 + \ldots + a'_{\Phi,nn}\Phi_n \end{bmatrix}$$

(Terms with comma subscript represent the corresponding terms of the inverse $\mathbf{N}^{-1}$ of the matrix $\mathbf{N}$ formed with the load factors.)

Thereby the differential equation system describing the motion will be as follows:

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{u}} - v_M\mathbf{M}\ddot{\mathbf{\Phi}} + \mathbf{N}_u^{-1}\mathbf{u} &= 0 \\ \mathbf{M}\ddot{\mathbf{v}} + u_M\mathbf{M}\ddot{\mathbf{\Phi}} + \mathbf{N}_v^{-1}\mathbf{v} &= 0 \qquad (32) \\ -v_M\mathbf{M}\ddot{\mathbf{u}} + u_M\mathbf{M}\ddot{\mathbf{v}} + \mathbf{J}_0\ddot{\mathbf{\Phi}} + \mathbf{N}_\varphi^{-1}\mathbf{\Phi} &= 0. \end{aligned}$$

Before solving the differential equation system, the coefficients (load factors) of the equation systems used for the determination of the restoring dynams have to be computed. They are easily obtained on hand of chapter 2.

Namely, in the relationship (16) forces p_1 and p_2 occur, necessary for unit displacement in directions u and v, respectively. In the equation systems for $\mathbf{r}_u$ and $\mathbf{r}_v$, coefficients expressing the displacements due to unit forces are simply their reciprocals:

$$a_{u,ik} = \frac{1}{p_{u,ik}} \qquad \text{and} \qquad a_{v,ik} = \frac{1}{p_{v,ik}} \tag{33}$$

where p_{ik} denotes the force acting at k and inducing unit displacement at the i-th point. According to sense, relationship (16) is valid for the determination of any p_{ik}, only the respective distances for points i and k have to be substituted in terms of H in equations (5), (6), (7), (8).

The rotation of the building cross-section due to unit moment is equal to the reciprocal numerical value of the moment acting in the floor plane and inducing unit rotation.

Thus:

$$a_{\Phi ik} = \frac{1}{M} \tag{34}$$

where the value of M can be computed from the relationship (24).

The system being rigidly fixed at one end and unsupported at the other, hence, if the i-th building cross-section is rotated by an angle Φ in its plane, then all cross-sections k between it and the free end will rotate by the same angle, so that it can be written:

$$a_{\Phi ii} = a_{\Phi ik}. \tag{35}$$

The differential equation system will be solved by the usual method for multiple-mass vibration systems. The free vibration of the system is supposed to be a harmonic vibration and can be described by the functions

$$\mathbf{u} = \mathbf{u}_0 \sin \omega t$$

$$\mathbf{v} = \mathbf{v}_0 \sin \omega t$$

$$\mathbf{\Phi} = \mathbf{\Phi}_0 \sin \omega t$$

hence

$$\ddot{\mathbf{u}} = -\omega^2 \mathbf{u}_0 \sin \omega t$$

$$\ddot{\mathbf{v}} = -\omega^2 \mathbf{v}_0 \sin \omega t$$

$$\ddot{\mathbf{\Phi}} = -\omega^2 \mathbf{\Phi}_0 \sin \omega t\,.$$

Substituting the above into differential equation system (32) and dividing throughout by $\sin \omega t$, we obtain:

$$\begin{array}{|llll|}\hline (\mathbf{N}_u^{-1}-\omega^2\mathbf{M})\mathbf{u}_0+\mathbf{0}\,\mathbf{v}_0 & & +\omega^2 v_M\mathbf{M}\boldsymbol{\Phi}_0 = 0 & \\ \mathbf{0}\quad \mathbf{u}_0+(\mathbf{N}_v^{-1}-\omega^2\mathbf{M})\mathbf{v}_0-\omega^2 u_M\mathbf{M}\boldsymbol{\Phi}_0 = 0 & & & \\ \omega^2 v_M\mathbf{M}\quad \mathbf{u}_0-\omega_0^2 u_M\mathbf{M}\;\mathbf{v}_0+(\mathbf{N}_\Phi^{-1}-\mathbf{J}_0\omega^2)\boldsymbol{\Phi}_0 = 0 & & & \\ \hline\end{array} \tag{36}$$

Here **0** denotes the zero matrix of n-th order. Introducing furthermore notations

$$(\mathbf{N}_u^{-1}-\omega^2\mathbf{M}) = \mathbf{P}$$

$$\omega^2 v_M\mathbf{M} = \mathbf{Q} \qquad \text{(diagonal matrix)}$$

$$(\mathbf{N}_v^{-1}-\omega^2\mathbf{M}) = \mathbf{R}$$

$$-\omega^2 v_M\mathbf{M} = \mathbf{S} \qquad \text{(diagonal matrix)}$$

$$(\mathbf{N}_\varphi^{-1}-\omega^2\mathbf{J}_0) = \mathbf{T}$$

the equation system takes the following from:

$$\begin{aligned} \mathbf{P}\mathbf{u}_0+\mathbf{0}\mathbf{v}_0+\mathbf{Q}\boldsymbol{\Phi}_0 &= 0 \\ \mathbf{0}\mathbf{u}_0+\mathbf{R}\mathbf{v}_0+\mathbf{S}\boldsymbol{\Phi}_0 &= 0 \\ \mathbf{Q}\mathbf{u}_0+\mathbf{S}\mathbf{v}_0+\mathbf{T}\boldsymbol{\Phi}_0 &= 0. \end{aligned} \tag{37}$$

The obtained homogeneous equation system has a solution other than zero, if the determinant formed of the coefficients is zero. In the considered case the coefficient matrix is given by the hypermatrix:

$$\mathbf{W} = \begin{bmatrix} \mathbf{P} & \mathbf{0} & \mathbf{Q} \\ \mathbf{0} & \mathbf{R} & \mathbf{S} \\ \mathbf{Q} & \mathbf{S} & \mathbf{T} \end{bmatrix} \tag{38}$$

Finally the equation to determine the natural circular frequency of the vibration is as follows:

$$\boxed{\det\mathbf{W} = (\det\mathbf{P})\,(\det\mathbf{R}\det\mathbf{T}-\det\mathbf{S}^2)\det\mathbf{Q}^2\det\mathbf{R} = 0} \tag{39}$$

The obtained equation is, in terms of ω^2, of $3n$-th order, with $3n$ roots. As the matrices in the characteristic equation are symmetrical and hypermatrix **W** itself is symmetrical, the equation has real roots and so the results for ω are either real or pure imaginary. It follows from the physical conditions of the motion that pure imaginary roots and negative real roots are impossible, therefore in fact, ω may have n values. The lowest one is the natural circular frequency of the fundamental vibration and the others the circular frequencies for more complex vibration forms. In the considered case only the fundamental vibration is of interest, as in rather squat buildings more complex forms of vibration do not occur. Thus the natural circular frequency of the building $\omega = \omega_{\min}$, and the natural frequency is:

$$N = \frac{\omega_{\min}}{2\pi}. \tag{40}$$

5. Approximation of the natural frequency

According to the method discussed above, the natural frequency of panel buildings can be determined without difficulty, however for multi-storey buildings the problem can only be solved – economically – by using a digital computer, because of the great number of equations and unknowns. To eliminate this disadvantage an approximation method will be presented yielding a fair approximation even for an arbitrary number of storeys, involving no special computation problem. The computation is based on the approximation method of *Dunkerley*, reducing the problem to determine natural frequencies of n one-mass systems rather than to determine the natural frequency of a vibrating system consisting of n masses:

$$\frac{1}{\omega^2} = \sum_{i=1}^{n} \frac{1}{\omega_i^2}. \tag{41}$$

Here ω_i is the fictitious natural circular frequency of a girder of negligible mass, acted upon by the i-th mass only. The obtained value is 5 to 15 per cent lower than the exact result.

The procedure to adopt is therefore to compute the natural frequency of a single-storey building and to vary the position of this storey according to the considered storey of the building. Namely, there is always a single-mass system, performing coupled vibrations, for which the equation system (36) might be written as well, however in an essentially simpler form, as for a single-mass system the restoring dynam is the product of the spring constant and the displacement. The spring constant is equal to the numerical value of the dynam inducing unit displacement, defined already in chapter 3; for p_u and p_v and M in cases of displacement and of rotation, see formulae (16) and (24), respectively.

Accordingly, the equation system expressing the free vibration of the i-th single-mass system is:

$$\left.\begin{array}{ll} (p_{ui} - m_i\omega^2)u_i + \omega^2 v_M m_i \Phi_i & = 0 \\ (p_{vi} - m_i\omega^2)v_i - \omega^2 u_M m_i \Phi_i & = 0 \\ \omega^2 v_M m_i u_i - \omega^2 u_M m_i v_i + (M_i - J_{oi}\omega^2)\Phi_1 & = 0 \end{array}\right\}. \tag{42}$$

The homogeneous equation system has a solution other than zero if its determinant formed of the coefficients is zero, i.e.

$$\begin{vmatrix} (p_{ui}-m_i\omega^2) & 0 & \omega^2 v_M m_i \\ 0 & (p_{vi}-m_i\omega^2) & -\omega^2 u_M m_i \\ \omega^2 v_M m_i & -\omega^2 u_M m_i & (M_i - J_{oi}\omega^2) \end{vmatrix} = 0.$$

After expanding the determinant, the following equation of 3-rd order is obtained for ω^2:

$$a\omega^6 + b\omega^4 + c\omega^2 + d = 0 \tag{43}$$

where

$$\begin{aligned} a &= m_i^2(u_M^2 m_i + v_M^2 m_i - J_{oi}) \\ b &= m_i[J_{oi}(p_{ui}+p_{vi}) - m_i(p_{ui}u_M^2 + p_{vi}v_M^2) + m_i M_i] \\ c &= -[m_i M_i(p_{ui}+p_{vi}) + p_{ui}p_{vi}J_o] \\ d &= p_{ui}p_{vi}M_i. \end{aligned}$$

It follows from the symmetry of the determinant that the equation has real roots. In the considered case the circular frequency pertaking of the fundamental frequency is of concern, so that for n' storeys n different $\Delta_{\min}$ values are obtained, of which the natural circular frequency can be determined by means of formula (41).

6. Consideration of the weakening effect of wall openings

When computing rigidity coefficients of the individual wall elements, the factor s intervening in the H value is related to the weakening effect of doors or windows. This factor should be assumed mainly according to results of experiments [7]. Denoting by ε the relation between the width of the wall opening and the width of the wall itself, then the factor s can be determined as:

for $\varepsilon \leqq 0{,}55$,

$$s = 3{,}46\,\varepsilon + 1 \tag{44}$$

for $0{,}55 \leqq \varepsilon \leqq 0{,}7$,

$$s = \frac{1}{\sqrt[3]{(1-\varepsilon)^4}}\,. \tag{45}$$

Above results are valid for a single wall. In practice, wall elements are mostly assembled of several wall units rigidly joined along the edges and usually each part contains different openings. In this case the factor s pertaining to the whole wall element may be the average value of the factors computed for each wall unit as mentioned above.

7. Numeric example

The approximation of the natural frequency – as described above – is presented on a model of a 4-storey panel building as shown in Figure 8. The dimensions are given in the figure. The model is made of a plastic material Columbia C with a dynamic modulus of elasticity $E = 40\ 000$ kp/cm². The adjacent wall

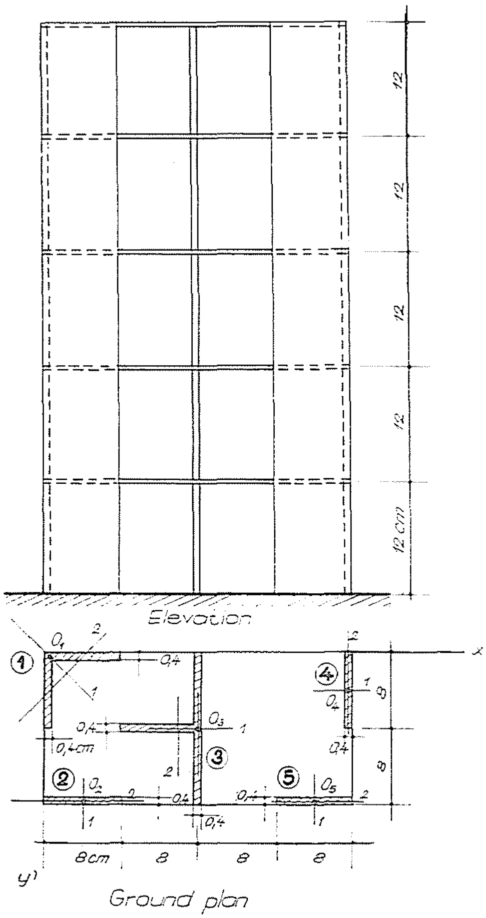

Fig. 8.

units are rigidly connected along the vertical edges, therefore they form wall elements. The model contains 5 such separate wall elements, numbered as seen in the figure, indicating also the principal directions of inertia (1,2) of the cross section, and the shear centre (O) of each wall element.

The principal inertia moments of the wall cross sections are:

Wall element 1:	$J_1 = 57,75$ cm^4	$J_2 = 10,75$ cm^4
Wall element 2:	$J_1 = 17,00$ cm^4	$J_2 = 0$
Wall element 3:	$J_1 = 137,00$ cm^4	$J_2 = 35,9$ cm^4
Wall element 4:	$J_1 = 17,00$ cm^4	$J_2 = 0$
Wall element 5:	$J_1 = 17,00$ cm^4	$J_2 = 0$.

Computation of the rigidity factors according to (5), (6), (7), (8):

Wall element 1

$$p_{1xx} = \frac{1}{H} 34,25$$

$$p_{1xy} = p_{1yx} = \frac{1}{H} 23,5$$

$$p_{1yy} = \frac{1}{H} 34,25$$

Wall element 2

$$p_{2xx} = \frac{1}{H} 17,00$$

$$p_{2yy} = 0$$

$$p_{2xy} = 0$$

Wall element 3

$$p_{3xx} = \frac{1}{H} 137,0$$

$$p_{3yy} = \frac{1}{H} 35,9$$

$$p_{3xy} = 0$$

Wall element 4

$$p_{4xx} = 0$$

$$p_{4yy} = \frac{1}{H} 17,0$$

$$p_{4xy} = 0$$

Wall element 5

$$p_{5xx} = \frac{1}{H} 17,00$$

$$p_{5yy} = 0$$

$$p_{5xy} = 0.$$

Rigidity factor values s related to each storey; one for each wall element.

$$H_1 = \frac{12^3}{3E} = \frac{5 \cdot 75 \cdot 10^2}{E}$$

$$H_{II} = \frac{24^3}{3E} = \frac{4{,}6\cdot 10^3}{E}$$

$$H_{III} = \frac{36^3}{3E} = \frac{1{,}55\cdot 10^4}{E}$$

$$H_{IV} = \frac{48^3}{3E} = \frac{3{,}67\cdot 10^4}{E}$$

$$H_{V} = \frac{60^3}{3E} = \frac{7{,}18\cdot 10^4}{E}$$

Wall element 1;

I[st] storey: $p^{(I)}_{1xx} = 5{,}96\cdot 10^{-2}E$ kp/cm

$p^{(I)}_{1xy} = 4{,}09\cdot 10^{-2}E$ kp/cm

$p^{(I)}_{1yy} = 5{,}96\cdot 10^{-2}E$ kp/cm

II[nd] storey: $p^{(II)}_{1xx} = 7{,}43\cdot 10^{-3}E$ kp/cm

$p^{(II)}_{1xy} = 5{,}12\cdot 10^{-3}E$ kp/cm

$p^{(II)}_{1yy} = 7{,}43\cdot 10^{-3}E$ kp/cm

III[rd] storey: $p^{(III)}_{1xx} = 2{,}19\cdot 10^{-3}E$ kp/cm

$p^{(III)}_{1xy} = 1{,}52\cdot 10^{-3}E$ kp/cm

$p^{(III)}_{1yy} = 2{,}19\cdot 10^{-3}E$ kp/cm

IV[th] storey: $p^{(IV)}_{1xx} = 9{,}4\cdot 10^{-4}E$ kp/cm

$p^{(IV)}_{1xy} = 6{,}3\cdot 10^{-4}E$ kp/cm

$p^{(IV)}_{1yy} = 9{,}4\cdot 10^{-4}E$ kp/cm

V[th] storey: $p^{(V)}_{1xx} = 4{,}8\cdot 10^{-4}E$ kp/cm

$p^{(V)}_{1yy} = 3{,}27\cdot 10^{-4}E$ kp/cm

$p^{(V)}_{1yy} = 4{,}8\cdot 10^{-4}E$ kp/cm

Wall element 2,

I[st] storey: $p^{(I)}_{2xx} = 2{,}96\cdot 10^{-2}E$ kp/cm

II[nd] storey: $p^{(II)}_{2xx} = 3{,}7\ \cdot 10^{-3}E$ kp/cm

III[rd] storey: $p^{(III)}_{2xx}$ $1{,}13\cdot 10^{-3}E$ kp/cm

IV[th] storey: $p^{(IV)}_{2xx} = 4{,}64\cdot 10^{-4}E$ kp/cm

V[th] storey: $p^{(V)}_{2xx} = 2{,}37\cdot 10^{-4}E$ kp/cm

Wall element 3,

I[st] storey: $p^{(I)}_{3xx} = 2{,}39\cdot 10^{-1}E$ kp/cm

$p^{(I)}_{3yy}$ $6{,}24\cdot 10^{-2}E$ kp/cm

II[nd] storey: $p^{(II)}_{3xx} = 2{,}98\cdot 10^{-2}E$ kp/cm

$p^{(II)}_{3yy} = 7{,}8\ \cdot 10^{-3}E$ kp/cm

IIIrd storey:	$p_{3xx}^{(III)} = 8{,}85 \cdot 10^{-3} E$ kp/cm
	$p_{3yy}^{(III)} = 2{,}51 \cdot 10^{-3} E$ kp/cm
IVth storey:	$p_{3xx}^{(IV)} = 3{,}73 \cdot 10^{-3} E$ kp/cm
	$p_{3yy}^{(IV)} = 9{,}75 \cdot 10^{-4} E$ kp/cm
Vth storey:	$p_{3xx}^{(V)} = 1{,}93 \cdot 10^{-3} E$ kp/cm
	$p_{3yy}^{(V)} = 5{,}0 \cdot 10^{-4} E$ kp/cm

Wall element 4,

Ist storey:	$p_{4yy}^{(I)} = 2{,}96 \cdot 10^{-2} E$ kp/cm
IInd storey:	$p_{4yy}^{(II)} = 3{,}7 \cdot 10^{-3} E$ kp/cm
IIIrd storey:	$p_{4yy}^{(III)} = 1{,}13 \cdot 10^{-3} E$ kp/cm
IVth storey:	$p_{4yy}^{(IV)} = 4{,}64 \cdot 10^{-4} E$ kp/cm
Vth storey:	$p_{4yy}^{(V)} = 2{,}37 \cdot 10^{-4} E$ kp/cm

Wall element 5,

Ist storey:	$p_{5xx}^{(I)} = 2{,}96 \cdot 10^{-2} E$ kp/cm
IInd storey:	$p_{5xx}^{(II)} = 3{,}7 \cdot 10^{-3} E$ kp/cm
IIIrd storey:	$p_{5xx}^{(III)} = 1{,}13 \cdot 10^{-3} E$ kp/cm
IVth storey:	$p_{5xx}^{(IV)} = 4{,}64 \cdot 10^{-4} E$ kp/cm
Vth storey:	$p_{5xx}^{(V)} = 2{,}37 \cdot 10^{-4} E$ kp/cm

$A^{I} = 35{,}78 \cdot 10^{-2} E$	$B^{I} = 15{,}9 \cdot 10^{-2} E$	$C^{I} = 4{,}09 \cdot 10^{-2} E$
$A^{II} = 44{,}63 \cdot 10^{-3} E$	$B^{II} = 18{,}93 \cdot 10^{-3} E$	$C^{II} = 5{,}12 \cdot 10^{-3} E$
$A^{III} = 13{,}30 \cdot 10^{-3} E$	$B^{III} = 5{,}63 \cdot 10^{-3} E$	$C^{III} = 1{,}52 \cdot 10^{-3} E$
$A^{IV} = 55{,}98 \cdot 10^{-4} E$	$B^{IV} = 23{,}79 \cdot 10^{-4} E$	$C^{IV} = 6{,}3 \cdot 10^{-4} E$
$A^{V} = 28{,}84 \cdot 10^{-4} E$	$B^{V} = 12{,}17 \cdot 10^{-4} E$	$C^{V} = 3{,}27 \cdot 10^{-4} E.$

Determination of the principal directions according to formula (11):

$$\operatorname{tg} 2\alpha_0 = \frac{2C}{A-B} = \frac{2 \cdot 4{,}09 \cdot 10^{-2}}{19{,}88 \cdot 10^{-2}} = 0{,}412$$

$$2\alpha_0 = 22°26' \qquad \alpha_0 = 11°13'.$$

This angle indicates the direction u of the greater displacing force; the other principal direction is its normal.

Computation of forces inducing unit displacement in the principal directions, according to formula (16):

Ist storey:

$$p_u^{\mathrm{I}} = E\left\{\left(\frac{35,78\cdot 10^{-2}+15,9\cdot 10^{-2}}{2}\right) + \left[\left(\frac{35,78\cdot 10^{-2}-15,9\cdot 10^{2}}{2}\right) + \right.\right.$$

$$\left.\left. + \left(4,09\cdot 10^{-2}\right)^2\right]\right\}^{1/2} = E\,36,59\cdot 10^{-2} \text{ kp/cm}$$

$$p_v^{\mathrm{I}} = E(25,84\cdot 10^{-2}-10,75\cdot 10^{-2}) = E15,09\cdot 10^{-2} \text{ kp/cm.}$$

IInd storey:

$$p_u^{\mathrm{II}} = E\left\{\left(\frac{44,63\cdot 10^{-3}+18,93\cdot 10^{-3}}{2}\right) + \left[\left(\frac{44,63\cdot 10^{-3}-18,93\cdot 10^{-3}}{2}\right)^2 + \left(5,12\cdot 10^{-3}\right)^2\right]^{1/2}\right\} =$$

$$= E\,45,58\cdot 10^{-3} \text{ kp/cm}$$

$$p_v^{\mathrm{II}} = E(31,78\cdot 10^{-3}-13,8\cdot 10^{-3}) = E17,98\cdot 10^{-3} \text{ kp/cm.}$$

IIIrd storey:

$$p_u^{\mathrm{III}} = E\left\{\frac{13,3\cdot 10^{-3}+5,63\cdot 10^{-3}}{2} + \left[\left(\frac{13,3\cdot 10^{-3}-5,63\cdot 10^{-3}}{2}\right)^2 + \left(1,53\cdot 10^{-3}\right)^2\right]^{1/2}\right\} =$$

$$= E\,13,57\cdot 10^{-3} \text{ kp/cm}$$

$$p_v^{\mathrm{III}} = E(9,46\cdot 10^{-3}-4,11\cdot 10^{-3}) = E\,5,35\cdot 10^{-3} \text{ kp/cm.}$$

IVth storey:

$$p_u^{\mathrm{IV}} = E\left\{\frac{23,79\cdot 10^{-4}+55,98\cdot 10^{-4}}{2} + \left[\left(\frac{55,98\cdot 10^{-4}-23,79\cdot 10^{-4}}{2}\right)^2 + \left(6,3\cdot 10^{-4}\right)^2\right]^{1/2}\right\} =$$

$$= E\,56,98\cdot 10^{-4} \text{ kp/cm}$$

$$p_v^{\mathrm{IV}} = E\,(39,88\cdot 10^{-4}-17,1\cdot 10^{-4}) = E\,22,78\cdot 10^{-4} \text{ kp/cm.}$$

Vth storey:

$$p_u^{\mathrm{V}} = E\left\{\frac{28,84\cdot 10^{-4}+12,17\cdot 10^{-4}}{2} + \left[\left(\frac{28,84\cdot 10^{-4}-12,17\cdot 10^{-4}}{2}\right)^2 + \left(3,27\cdot 10^{-4}\right)^2\right]^{1/2}\right\} =$$

$$= E\,29,43\cdot 10^{-4} \text{ kp/cm}$$

$$p_v^{\mathrm{V}} = E\,(20,5\cdot 10^{-4}-8,93\cdot 10^{-4}) = 11,57\cdot 10^{-4} \text{ kp/cm.}$$

Determination of the rotation centre according to formulae (20) and (21):

$A = 205,25$; $B = 87,15$; $C = 23,5$; $K = AB - C^2 = 17\,850 - 550 = 17\,300$.

$$y_0' = \frac{87,15}{17\,300}\,(1699-11,75) + \frac{23,5}{17\,300}\,(1200,5-11,75) = 10,1 \text{ cm}$$

$$x_0' = \frac{205}{17\,300}(1200{,}5-11{,}75)+\frac{23{,}5}{17\,300}(1699-11{,}75) = 14{,}1+2{,}3 = 16{,}4 \text{ cm.}$$

Determination of the torsional moment of the couple inducing unit rotation, according to formula (24):

I[st] storey

i	x_i	y_i	x_i^2	y_i^2	$x_i\ y_i$	$x_i^2\ p_{iyy}/E$	$y_i^2\ p_{ixx}/E$	$x_i y_i p_{ixy}/E$
1	−16,2	−9,9	212,0	98,0	160,5	12,6	5,84	6,55
2	−12,4	5,7	154,0	32,4	−70,7	0	0,96	0
3	− 0,4	−2,1	0,16	4,4	0,8	0	0,15	0
4	15,6	−6,1	244,0	37,2	−95,0	7,2	0	0
5	11,6	5,7	134,0	32,4	62,7	0	0,98	0
Σ						19,8	7,93	6,55

$M^{\mathrm{I}} = E(19{,}8+7{,}93-13{,}1) = E\ 14{,}63$ kpcm

II[nd] storey

i	$x_i^2\ p_{iyy}/E$	$y_i^2\ p_{ixx}/E$	$x_i y_i p_{ixy}$
1	$15{,}7\cdot10^{-1}$	$7{,}3\cdot10^{-1}$	$8{,}53\cdot10^{-1}$
2	0	$1{,}2\cdot10^{-1}$	0
3	0	$1{,}32\cdot10^{-1}$	0
4	$9{,}0\cdot10^{-1}$	0	0
5	0	$1{,}2\cdot10^{-1}$	0
Σ	$24{,}7\cdot10^{-1}$	$11{,}02\cdot10^{-1}$	$8{,}53\cdot10^{-1}$

$M^{\mathrm{II}} = E(2{,}47+1{,}1-1{,}74) = E\ 1{,}87$ kpcm

III[rd] storey

i	$x_i^2\ p_{iyy}/E$	$y_i^2\ p_{ixx}/E$	$x_i y_i p_{ixy}/E$
1	$4{,}65\cdot10^{-1}$	$2{,}13\cdot10^{-1}$	$2{,}44\cdot10^{-1}$
2	0	$0{,}36\cdot10^{-1}$	0
3	0	$0{,}40\cdot10^{-1}$	0
4	$2{,}75\cdot10^{-1}$	0	0
5	0	$0{,}36\cdot10^{-1}$	0
Σ	$7{,}40\cdot10^{-1}$	$3{,}25\cdot10^{-1}$	$2{,}44\cdot10^{-1}$

$M^{\mathrm{III}} = (0{,}74+0{,}325-0{,}488)E = E\ 0{,}577$ kpcm

IV[th] storey

i	$x_i^2\, p_{iyy}/E$	$y_i^2\, p_{ixx}/E$	$x_i y_i p_{ixy}/E$
1	$1{,}99\cdot 10^{-1}$	$0{,}92\cdot 10^{-1}$	$1{,}01\cdot 10^{-1}$
2	0	$0{,}15\cdot 10^{-1}$	0
3	0	$0{,}17\cdot 10^{-1}$	0
4	$1{,}13\cdot 10^{-1}$	0	0
5	0	$0{,}15\cdot 10^{-1}$	0
Σ	$3{,}12\cdot 10^{-1}$	$1{,}39\cdot 10^{-1}$	$1{,}01\cdot 10^{-1}$

$$M^{IV} = (0{,}312+0{,}139-0{,}202)E = 0{,}249\ E \text{ kpcm}$$

V[th] storey

i	$x_i\ p_{iyy}/E$	$y_i\, p_{ixx}/E$	$x_i y_i p_{ixy}/E$
1	$1{,}02\cdot 10^{-1}$	$0{,}47\ \cdot 10^{-1}$	$0{,}524\cdot 10^{-1}$
2	0	$0{,}076\cdot 10^{-1}$	0
3	0	$0{,}085\cdot 10^{-1}$	0
4	$0{,}58\cdot 10^{-1}$	0	0
5	0	$0{,}076\cdot 10^{-1}$	0
	$1{,}60\cdot 10^{-1}$	$0{,}707\cdot 10^{-1}$	$0{,}524\cdot 10^{-1}$

$$M^{V} = (0{,}16+0{,}07-0{,}102) = 0{,}125 \text{ kpcm}$$

Figure 9 shows the calculated principal directions of displacements u and v and the rotation centre 0. The mass of the building is assumed on each floor level. In the case under consideration the uniformly distributed mass intensity is $8\cdot 10^{-3}$ kg/cm^2 for each storey. The centre of mass S_M is in the intersection of the diagonals of the cross-sectional rectangle and in the co-ordinate system xy at the rotation centre its co-ordinates are

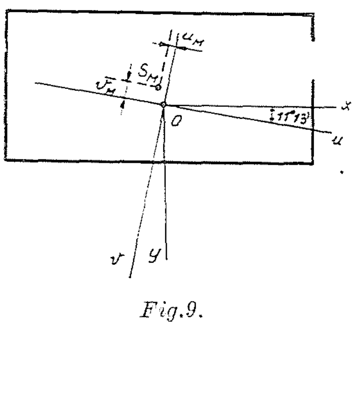

Fig.9.

$x_M = -0,4$ cm and $y_M = -2,1$ cm. The co-ordinates of the mass centre in the co-ordinate system u, v are obtained by co-ordinate transformation:

$$u_M = -0,4 \cos 11°13' - 2,1 \sin 11°13' = -0,4 \cdot 0,980 - 2,1 \cdot 0,194 = -0,8 \text{ cm}$$

$$v_M = 0,4 \sin 11°13' - 2,1 \cos 11°13' = 0,4 \cdot 0,194 -- 2,1 \cdot 0,980 = -1,98 \text{ cm}$$

The mass concentrated on one storey $m_i = 4,0$ kg.

Computation of the inertia moment referred to the rotation centre of the mass concentrated in the cross-section:

Moment of inertia referred to the rotation centre:

$$J_0 = 426 \text{ kg cm}^2$$

After substitution, five different equations of 3-rd order arise for ω^2 according to Eq. (43), yielding five basic natural circular frequencies ω^2:

$$\omega_I^2 = 5,75 \cdot 10^5 \text{ sec}^{-2}$$
$$\omega_{II}^2 = 7,35 \cdot 10^4 \text{ sec}^{-2}$$
$$\omega_{III}^2 = 2,25 \cdot 10^4 \text{ sec}^{-2}$$
$$\omega_{IV}^2 = 8,95 \cdot 10^3 \text{ sec}^{-2}$$
$$\omega_V^2 = 4,45 \cdot 10^3 \text{ sec}^{-2}$$

The natural circular frequency:

$$\frac{1}{\omega^2} = \frac{1}{5,75 \cdot 10^5} + \frac{1}{7,35 \cdot 10^4} + \frac{1}{2,25 \cdot 10^4} + \frac{1}{8,95 \cdot 10^3} + \frac{1}{4,45 \cdot 10^3} = 4,01 \cdot 10^{-4} \text{ sec}^2$$

$$\omega^2 = 2,5 \cdot 10^3 \text{ sec}^{-2}$$

$$\omega = 50 \text{ sec}^{-1}$$

The natural frequency of the considered model:

$$N = \frac{50}{2\pi} = 7,9 \text{ sec}^{-1}$$

8. Model test

A model has been made with dimensions and material as discussed in chapter 7. The design mass for each storey level was simulated by shots uniformly distributed over each floor. The quantity of shots has been established so that the mass of the floor should be 4 kg. The developed model supplied with a vibrometer is shown in Fig.10. Natural frequency was measured by an electronic vibrometer type *SDM3* with pick-ups fixed on each floor level as shown in the figure. To determine the natural frequency, two kinds of vibration inducing effects have been applied. Either the model was pulled horizontally at its top level and then suddenly released or a support rigidly fixed to the model and otherwise immobile was given a small horizontal impact. Both vibration effects had the same result and the following could be concluded from the measurements:

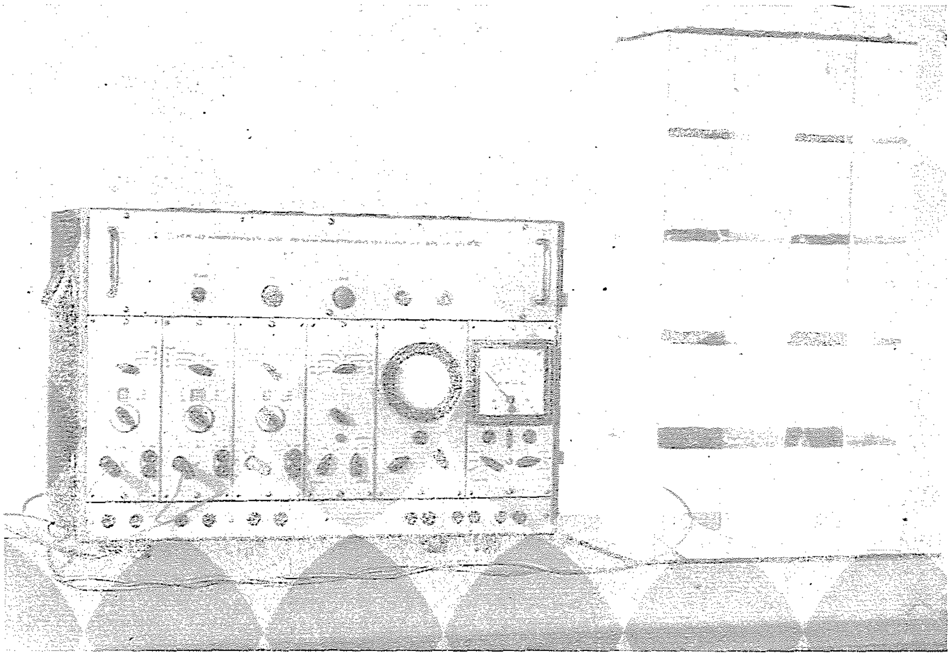

Fig.10.

The model subjected to either effet assumed the vibration form corresponding to the fundamental vibration. No more complex forms of vibration could be observed. The frequency of the simultaneous bending and torsional vibrations was 9 Hz. The vibration pattern recorded on film tape is shown in Fig. 11.

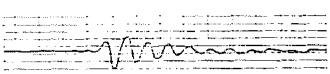

Fig.11.

Summary

Horizontal free vibrations of multistorey panel buildings can be classified as so-called "coupled vibrations", since in general cases the building is unable to perform purely flexural or purely torional vibrations. These vibrations are similar to the corresponding beam vibrations, structurally however the two are quite different. Namely, whereas each point of a beam cross-section can only move in correspondence with given geometrical conditions, and not independently of the adjacent points, structural walls constituting the skeleton can execute some motions in mutual dependence and again some others independently of each other, in accordance with the theory based on the Bernoulli-Navier hypothesis. Determination method for the horizontal natural frequencies of multistorey buildings undergoing such vibrations has been presented.

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