

Formulation of the Torsion Problem in Stresses and Solution by Boundary Element Method

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Abstract

The paper presents a boundary element method approach to solving the torsion problem for rods based on a stress formulation. This approach has a key advantage over classical torsion formulations, as it enables direct evaluation of stresses in the cross-section without numerical differentiation of auxiliary functions. A second-kind boundary integral equation is derived for the stress-based formulation. The boundary element solution procedure is described, and the resulting system of governing equations is constructed. The applicability of the proposed formulation for stress evaluation in multiply connected cross-sections is demonstrated using test problems. Comparison with available analytical solutions confirms the accuracy and reliability of the proposed approach, showing that the results meet engineering requirements.

Keywords

torsion problem, stress formulations, boundary element method, theory of elasticity, integral equations

1 Introduction

1.1 Historical overview

The problem of torsion of elastic prismatic rods is among the classical problems of elasticity theory. Its mathematical formulation was proposed by Saint-Venant in the middle of the 19th century. Before the advent of powerful computers, researchers were able to obtain analytical solutions for many cases – mainly for rods with relatively simple cross-sectional shapes. The results of these studies are summarized in a monograph [1].

The advent of computers opened up possibilities for the numerical solution of the torsion problem for rods with cross-sections of arbitrary shape. This provided a strong stimulus for the intensive development of numerical methods applicable not only to torsion problems, but also to analogous problems in potential theory with similar mathematical formulations.

One of the widely used numerical methods for solving torsion problems is the boundary element method (BEM) [2–4]. Using classical formulations and the corresponding boundary

integral equations, numerical results have been obtained for bars with various cross-sectional shapes, including open ones [5–8], as well as for composite bars consisting of several materials [9–10]. The boundary element method has also been employed to analyze nonuniform torsion problems [11–12], to solve dynamics problems [13–14], and for elastic-plastic torsion [15].

In all of the above references, the numerical solution of the torsion problem is based on determining the stress or displacement functions [1], with the stresses themselves subsequently obtained by additional differentiation of the sought functions.

1.2 Classic formulations

Consider the torsion problem of a prismatic bar of arbitrary cross-section subjected to two external moments applied in the planes of its end cross-sections. The body forces are assumed to be zero, and the lateral surface of the bar is free of external loads.

The coordinate system is chosen as follows: the z-axis coincides with the torsion axis, i.e., the axis that remains stationary when the bar is twisted; the x and y axes are mutually orthogonal and arbitrarily oriented in the plane of the cross-section. The cross-section of the bar is a multiply connected domain, meaning that the bar has longitudinal cylindrical cavities (Fig. 1).

The torsion problem for a bar with cross-section S , outer contour Γ_0 and inner contours $\Gamma_1, \Gamma_2, \dots, \Gamma_p$ is formulated in terms of the Prandtl stress function Φ as follows [16]:

$$S : \Delta \Phi = -2, \quad (1)$$

$$\Gamma_0 : \Phi = 0, \quad (2)$$

$$\Gamma_k : \Phi = \Phi_k = \text{const} (k=1, 2, \dots, p), \quad (3)$$

and an additional condition on the boundary Γ_k

$$\Gamma_k : \int_{\Gamma_k} \frac{\partial \Phi}{\partial n} d\Gamma = -2S_k (k=1, 2, \dots, p), \quad (4)$$

where $\Delta = \partial_x^2 + \partial_y^2$ – the Laplace operator in the plane, $\partial_x \equiv \partial/\partial x$, $\partial_y \equiv \partial/\partial y$; S_k – the area enclosed by the contour Γ_k .

Equation (4) represent an analytical expression of Bredt's theorem on the circulation of shear stress in torsion of a bar.

The unknown shear stresses τ_{zx} and τ_{zy} are expressed through the stress function Φ as follows:

$$\tau_{zx} = \mu \theta \partial_y \Phi, \quad \tau_{zy} = \mu \theta \partial_x \Phi, \quad (5)$$

where μ – shear modulus; θ – the angle of twist per unit length of the bar.

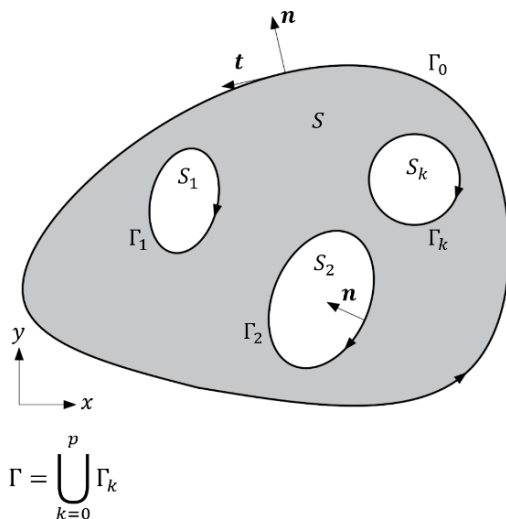


Fig. 1 Multiply connected cross-section of the bar

At the same time, the solution to the analogous torsion problem can be obtained in another form, as was done by Saint-Venant [1]:

$$S : \Delta \varphi = 0, \quad (6)$$

$$\Gamma_{k,k \geq 0} : \frac{\partial \varphi}{\partial n} = y n_x - x n_y. \quad (7)$$

The shear stresses τ_{zx}, τ_{zy} in this case are expressed as follows

$$\tau_{zx} = \mu \theta (\partial_x \varphi - y), \quad \tau_{zy} = \mu \theta (\partial_y \varphi + x), \quad (8)$$

and they are independent of the location of the origin of coordinates, since the expressions in parentheses are invariant with respect to the translation of the coordinate axes [16].

From the perspective of the numerical solution of the torsion problem using the boundary element method and the determination of stresses in the cross-section, both described formulations have a number of significant drawbacks.

1. Solving the problem in Eqs. (1)–(4) is only possible when the values of the function Φ on the inner contours $\Gamma_{k,k>0}$ are known. Consequently, before solving the system of Eqs. (1)–(3), it is necessary to solve p additional boundary value problems to determine Φ on the inner contours $\Gamma_{k,k>0}$.
2. Since the numerical solution of the problem in Eqs. (6)–(7) yields the values of the function φ at the boundary element nodes, the determination of shear stresses on the boundary reduces to their numerical differentiation. This introduces an additional source of computational error.
3. The boundary integral equation for the stress function Φ in Eqs. (1)–(4) is a numerically unstable equation of the first kind.
4. The function φ in Eqs. (6)–(7) is a solution of the Neumann problem. As is well known [17], a solution of the Neumann problem exists only if the following condition is satisfied

$$\int_{\Gamma} \frac{\partial \varphi}{\partial n} d\Gamma = 0, \quad (9)$$

and it is not unique, but determined only up to an arbitrary additive constant. In numerical implementation, this leads to a singular influence matrix and requires the introduction of an additional condition, for example, fixing the value of the function at a single point [18].

This paper presents a formulation of the torsion problem in terms of stresses that avoids the above-mentioned drawbacks. The corresponding boundary integral equations in

stresses are obtained, and a solution procedure based on the boundary element method is described. Unlike classical formulations, the proposed approach yields integral equations of the second kind, whose numerical solution is stable. The proposed formulation is verified by comparing the numerical solution with the known analytical solution for two example cross-sections.

2 Methods

2.1 Formulation of the torsion problem regarding stress

To formulate the problem in terms of stresses, we utilize Eqs. (6)–(7), as it does not require solving auxiliary boundary value problems. Since shear stresses depend on the gradient of the function φ , we formulate a problem where the unknowns are the first derivatives of the solution to Eqs. (6)–(7). Let us introduce the vector $\mathbf{q} = \nabla\varphi$ with the following components:

$$q_x = \partial_x \varphi, \quad q_y = \partial_y \varphi. \quad (10)$$

Then, Eq. (6) can be written as follows:

$$\partial_x q_x + \partial_y q_y = 0. \quad (11)$$

The functions satisfy the equation:

$$\partial_x q_y - \partial_y q_x = 0, \quad (12)$$

which can be naturally called the compatibility equation.

Next, we formulate the boundary conditions in terms of the unknown functions q_x and q_y . Since, by the definition of the normal derivative:

$$\frac{\partial \varphi}{\partial n} = n_x \partial_x \varphi + n_y \partial_y \varphi = n_x q_x + n_y q_y, \quad (13)$$

where n_x, n_y are the projections of the unit normal vector onto the coordinate axes (Fig. 1), then boundary condition Eq. (7) takes the following form:

$$n_x q_x + n_y q_y = y n_x - x n_y. \quad (14)$$

To provide the necessary number of boundary conditions, we require that Eq. (12) be satisfied at the points of the boundary Γ .

Consequently, the vector formulation of the problem takes the following form:

$$S: \begin{cases} \nabla \cdot \mathbf{q} = 0 \\ \nabla \cdot \tilde{\mathbf{q}} = 0 \end{cases}, \quad (15)$$

$$\Gamma: \mathbf{n} \cdot \mathbf{q} \equiv q_n = \mathbf{n} \cdot \mathbf{C} \cdot \mathbf{r}, \quad (16)$$

and the shear stresses at an arbitrary point $m \in S \cup \Gamma$, according to Eq. (8), take the form:

$$\boldsymbol{\tau}(\mathbf{r}) = \mu \theta (\mathbf{q} - \mathbf{C} \cdot \mathbf{r}), \quad (17)$$

where ∇ is the gradient operator on the plane; the vector $\tilde{\mathbf{q}}$ is the vector $\mathbf{q}$ rotated by $\pi/2$ clockwise when viewed from the z -axis; $\mathbf{n}$ is the unit normal vector to the boundary Γ ; $\boldsymbol{\tau}$ is the vector of tangential stresses; $\mathbf{r}$ is the radius vector from the origin; $\mathbf{C}$ is the fundamental skew-symmetric tensor in the plane.

In the case of a multiply connected body, Eq. (12) is a necessary but not sufficient compatibility condition. To formulate an additional condition, we utilize the analytical expression of Bredt's theorem Eq. (4), expressing it in terms of the vector $\mathbf{q}$. By taking the dot product of Eq. (17) and the vector $\mathbf{t}$ (the unit tangent vector to the boundary Γ), we obtain:

$$\tau_t = \boldsymbol{\tau} \cdot \mathbf{t} = \mu \theta (\mathbf{q}_t + \mathbf{n} \cdot \mathbf{r}), \quad (18)$$

where τ_t is the tangential stress at a point on the cross-section boundary.

We represent the stresses τ_t as [16]:

$$\tau_t = -\mu \theta \frac{\partial \Phi}{\partial n}, \quad (19)$$

Then Eq. (4) takes the form:

$$\begin{aligned} \Gamma_k: \int_{\Gamma_k} q_t d\Gamma + \int_{\Gamma_k} \mathbf{n} \cdot \mathbf{r} d\Gamma &= \\ &= 2S_k \quad (k = 1, 2, \dots, p). \end{aligned} \quad (20)$$

The second integral on the left-hand side Eq. (20) equals twice the area S_k enclosed by the boundary Γ_k . Thus, it follows from Eq. (20) that:

$$\Gamma_k: \int_{\Gamma_k} q_t d\Gamma = 0 \quad (k = 1, 2, \dots, p). \quad (21)$$

Equation (21) form p additional conditions that ensure the uniqueness of the solution in terms of derivatives.

Equations (15)–(16) supplemented by conditions Eq. (21) will be referred to as the stress-based formulation (or problem). By solving problem Eqs. (15)–(16), one can find the derivatives of the function φ , without determining the function φ , itself, and then determine the shear stress vector $\boldsymbol{\tau}$, using Eq. (17) through purely algebraic operations.

2.2 Boundary integral equations for a vector on a plane

To derive the boundary integral equations with respect to the vector $\mathbf{q}$ a corresponding integral representation is required.

Let $\mathbf{A}$ be the unit tensor on the plane, and $\mathbf{C}$ be the previously introduced fundamental skew-symmetric tensor on the plane. If $\mathbf{e}_x, \mathbf{e}_y$ are the unit vectors of the Cartesian coordinate system, then these tensors have the following

dyadic representation:

$$A = e_x e_x + e_y e_y, \quad C = e_x e_y - e_y e_x. \quad (22)$$

The fundamental property of the introduced tensors is:

$$A \cdot b = b, \quad C \cdot b = \tilde{b}, \quad (23)$$

where b is an arbitrary vector, $b = b_x e_x + b_y e_y$,

$\tilde{b} = b_y e_x - b_x e_y$ is the vector b rotated clockwise by $\pi/2$.

We write the following representation of the Laplacian for an arbitrary vector ω :

$$\Delta \omega = \nabla \cdot (A \nabla \cdot \omega + C \nabla \cdot \tilde{\omega}). \quad (24)$$

Taking the dot product of Eq. (24) and an arbitrary vector q , transforming the right-hand side using the equality [16]:

$$q \cdot \nabla \cdot Q = \nabla \cdot (Q \cdot q) - Q^T \cdot \nabla q, \quad (25)$$

where Q is an arbitrary second-order tensor; the symbol "..." denotes the double dot product of two tensors [16].

The obtained equation has the form:

$$q \cdot \nabla \omega = \nabla \cdot (q \nabla \cdot \omega + \tilde{q} \nabla \cdot \tilde{\omega}) - \nabla \cdot \omega \nabla \cdot q - \nabla \cdot \tilde{\omega} \nabla \cdot \tilde{q} \quad (26)$$

Integrating Eq. (26) over the domain S and utilizing the Gauss-Ostrogradsky formula [17]:

$$\int_S \nabla \cdot b dS = \int_\Gamma n \cdot b d\Gamma, \quad (27)$$

which is valid for any vector b , we obtain:

$$\begin{aligned} \int_S q \cdot \Delta \omega dS &= \int_\Gamma (q_n \nabla \cdot \omega + q_t \nabla \cdot \tilde{\omega}) d\Gamma - \\ &- \int_S (\nabla \cdot q \nabla \cdot \omega + \nabla \cdot \tilde{q} \nabla \cdot \tilde{\omega}) dS, \end{aligned} \quad (28)$$

where $q_n \equiv n \cdot q$, $q_t \equiv t \cdot q$ are the projections of the vector q onto the normal and the tangent to the boundary Γ , respectively.

Let m, z be two arbitrary points on the plane, and let r_m, r_z be their radius vectors (Fig. 2). Let us introduce the following notation:

$$r = r_z - r_m, \quad r = |r| = \sqrt{r \cdot r}. \quad (29)$$

We also introduce the function $v(x, y)$, which is the fundamental solution to the Laplace equation in the plane [17]:

$$v = \ln r. \quad (30)$$

By the definition of the fundamental solution:

$$\Delta_z v = 2\pi \delta(m, z), \quad (31)$$

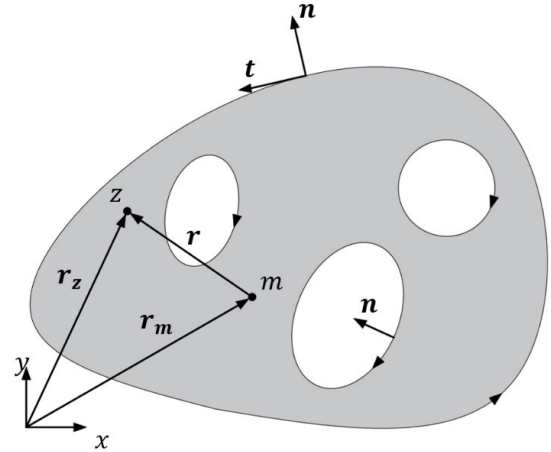


Fig. 2 Definition of the radius vectors

where Δ_z is the Laplace operator with differentiation with respect to the coordinates of point z ; $\delta(m, z)$ is the Dirac delta function with the following fundamental property [18]:

$$\int_S g(z) \delta(m, z) dS(z) = g(m) c(m), \quad (32)$$

where $g(z)$ is an arbitrary continuous function; the notation $dS(z)$ underlines that the integration is performed with respect to the coordinates of point z ;

$$c(m) = \begin{cases} 1, & m \in S, \\ 0.5, & m \in \Gamma, \\ 0, & m \notin S + \Gamma. \end{cases} \quad (33)$$

Using Eqs. (29)–(31), let us set $\omega = b \ln r$, in Eq. (28), where b is an arbitrary constant vector. For the vector ω we have:

$$\Delta_z \omega = b 2\pi \delta(m, z), \quad (34)$$

$$\nabla_z \cdot \omega = b \cdot \frac{r}{r^2}, \quad (35)$$

$$\nabla_z \cdot \tilde{\omega} = -b \cdot \frac{\tilde{r}}{r^2}. \quad (36)$$

Substituting these into Eq. (28) and using property Eq. (32), we obtain Eq. (37) (after omitting the arbitrary vector b):

$$\begin{aligned} 2\pi c(m) q(m) &= \int_\Gamma \left(q_n \frac{r}{r^2} - q_t \frac{\tilde{r}}{r^2} \right) d\Gamma(z) + \\ &+ \int_S \left(\frac{\tilde{r}}{r^2} \nabla \cdot \tilde{q} - \frac{r}{r^2} \nabla \cdot q \right) dS(z). \end{aligned} \quad (37)$$

Equation (37) is the desired integral representation of the vector q in the plane.

If by the vector q we mean the vector of derivatives of the function $\varphi(q = \nabla \varphi)$, then, according to Eq. (15), Eq. (37) takes the following form:

$$2\pi c(m) q(m) = \int_\Gamma \left(q_n \frac{r}{r^2} - q_t \frac{\tilde{r}}{r^2} \right) d\Gamma(z). \quad (38)$$

Substituting the integral representation Eq. (38) into Eq. (17), we obtain:

$$\tau(m) = \mu\theta \left[\frac{1}{2\pi c(m)} \int_{\Gamma} \left(q_n \frac{\tilde{\mathbf{r}}}{r^2} - q_t \frac{\tilde{\mathbf{r}}}{r^2} \right) d\Gamma(z) - \mathbf{C} \cdot \mathbf{r}_m \right]. \quad (39)$$

Using Eq. (39), the shear stress vector τ can be determined at any point of the cross-section S , if the tangential projection q_t is known on the boundary Γ . To find the function q_t we take the dot product of Eq. (38) and the vector $\mathbf{t}_m$ (the unit tangent vector to the boundary Γ at point m) and impose the condition $m \in \Gamma$. Taking into account the boundary condition Eq. (16), we obtain the following boundary integral equation with respect to the function q_t ($\mathbf{t}_m \cdot \tilde{\mathbf{r}} = -\mathbf{n}_m \cdot \mathbf{r}$, $\mathbf{t} = \mathbf{n} \cdot \mathbf{C}$):

$$\begin{aligned} \pi q_t(m) - \int_{\Gamma} q_t \frac{\mathbf{n}_m \cdot \mathbf{r}}{r^2} d\Gamma(z) = \\ = \int_{\Gamma} \mathbf{t}_z \cdot \mathbf{r}_z \frac{\mathbf{t}_m \cdot \mathbf{r}}{r^2} d\Gamma(z). \end{aligned} \quad (40)$$

Equation (40) is an integral equation of the second kind. The solution to Eq. (40) is the tangential component q_t of the vector $\mathbf{q}$.

Using Eq. (18), the tangential stresses τ_t acting on the boundary Γ can also be expressed in terms of the function q_t , among which is the maximum shear stress in the cross-section [16].

The boundary integral Eq. (40), together with the additional conditions Eq. (21), forms a system of equations with respect to the function q_t for numerical solution by the boundary element method.

2.3 Boundary element method

Let us write the discrete form of the integral Eq. (40) and the additional conditions Eq. (21) using so-called 'constant' boundary elements. The values of the function q_t are assumed to be constant over each element and equal to their values at the element's midpoint. By partitioning the boundary Γ into N elements, we transform Eq. (21) and Eq. (40) into the following form:

$$\begin{cases} \pi q_{ii} - \sum_{j=1}^N \int_{\Gamma_j} q_{ij} \frac{\mathbf{n}_i \cdot \mathbf{r}_{ij}}{r_{ij}^2} d\Gamma(j) = \\ = \sum_{j=1}^N \int_{\Gamma_j} q_{nj} \frac{\mathbf{t}_i \cdot \mathbf{r}_{ij}}{r_{ij}^2} d\Gamma(j), \end{cases} \quad (41)$$

$$\Gamma_k : \sum_{j, \Gamma_j \in \Gamma_k} \int_{\Gamma_j} q_{ij} d\Gamma(j) = 0 \quad (k=1, 2, \dots, p), \quad (42)$$

where i, j are the boundary element indices.

The functions q_{ij} , q_{nj} , appearing under the integral sign in Eq. (41) and (42) have constant values over each element

and can therefore be taken outside the integral. If points i and j are located on the same element, then the integral

$$\int_{\Gamma_j} \frac{\mathbf{n}_i \cdot \mathbf{r}_{ij}}{r_{ij}^2} d\Gamma(j) \quad (43)$$

is identically zero due to the orthogonality of the vectors $\mathbf{n}_i$, $\mathbf{r}_{ij}$. In turn, the integral

$$\int_{\Gamma_j} \frac{\mathbf{t}_i \cdot \mathbf{r}_{ij}}{r_{ij}^2} d\Gamma(j) \quad (44)$$

is singular and can be evaluated analytically. In a local coordinate system passing through the center of the boundary element (Fig. 3)

$$\mathbf{r}(s) = \mathbf{r}_i + s\mathbf{t}_i, \quad s \in \left[-\frac{L}{2}, \frac{L}{2} \right], \quad (45)$$

where L is the boundary element size.

The vector $\mathbf{r}_{ij}$, its projection, and its magnitude in local coordinates take the following form:

$$\mathbf{r}_{ij} = s\mathbf{t}_i, \quad \mathbf{t}_i \cdot \mathbf{r}_{ij} = s, \quad r_{ij}^2 = s^2, \quad (46)$$

then

$$\frac{\mathbf{t}_i \cdot \mathbf{r}_{ij}}{r_{ij}^2} = \frac{1}{s}. \quad (47)$$

Thus, the integral Eq. (44) understood in the sense of the Cauchy principal value is equal to:

$$v.p. \int_{-L/2}^{L/2} \frac{ds}{s} = \ln \frac{L}{L} = 0. \quad (48)$$

Let us introduce the following notation:

$$H_{ij} = \pi \delta_{ij} - \int_{\Gamma_j} \frac{\mathbf{n}_i \cdot \mathbf{r}_{ij}}{r_{ij}^2} d\Gamma(j), \quad (49)$$

$$G_{ij} = \int_{\Gamma_j} \frac{\mathbf{t}_i \cdot \mathbf{r}_{ij}}{r_{ij}^2} d\Gamma(j), \quad (50)$$

where δ_{ij} is the Kronecker delta.

The system of Eq. (41) with additional conditions Eq. (42) is solved using the method of Lagrange multipliers. In matrix form, the system takes the following form:

$$\begin{bmatrix} H & C \\ C^T & 0 \end{bmatrix} \begin{pmatrix} \mathbf{Q} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} GF \\ 0 \end{pmatrix}, \quad (51)$$



Fig. 3 Evaluation of the singular integral

where Q is the column vector of unknown nodal values of the function q , λ are the Lagrange multiplier column,

$$F = \begin{pmatrix} q_{n1} \\ q_{n2} \\ \vdots \\ q_{nN} \end{pmatrix} = \begin{pmatrix} t_1 \cdot r_1 \\ t_2 \cdot r_2 \\ \vdots \\ t_N \cdot r_N \end{pmatrix}, \quad (52)$$

$$C = \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1p} \\ C_{21} & C_{22} & \dots & C_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ C_{N1} & C_{N2} & \dots & C_{Np} \end{pmatrix}, \quad (53)$$

where

$$C_{jk} = \begin{cases} L, & \Gamma_j \in \Gamma_k \\ 0, & \Gamma_j \notin \Gamma_k \end{cases}.$$

The number of Lagrange multipliers in system Eq. (51) is equal to the number of holes in the analyzed cross-section; therefore, each hole adds one additional unknown.

3 Results and discussion

Based on the BEM algorithm in stresses presented in the previous section, we will perform numerical calculations of the shear stresses that arise during the torsion of bars with multiply connected cross-sections. In both examples, we discretize the boundary using N equal constant boundary elements of length L . To simplify the interpretation of the results, we take the values of the cross-section twist angle θ and the shear modulus μ as equal to one, since their values do not affect the stress distribution in the cross-section. Therefore, we will conduct further assessment of the stress state using dimensionless shear stresses $\tau/\mu\theta$.

We will perform the calculations with a successive reduction of the boundary element size. Due to the symmetrical distribution of stresses across the section, we will consider only part of the boundaries.

3.1 Torsion of a rectangular cross-section rod with an asymmetrical rectangular hole

In the first example, we calculate the tangential stresses in a rectangular cross-section rod with an asymmetrical rectangular hole. The dimensions of the section are shown in Fig. 4.

Fig. 5 and Fig. 6 show the graphs of the distribution of shear stresses along the internal and external contours of the rod and their comparison with the analytical solution [19].

Table 1 shows the numerical values of stresses at certain points of the section. The numerical values in Table 1 correspond to the stress diagrams shown in Fig. 5 and 6.

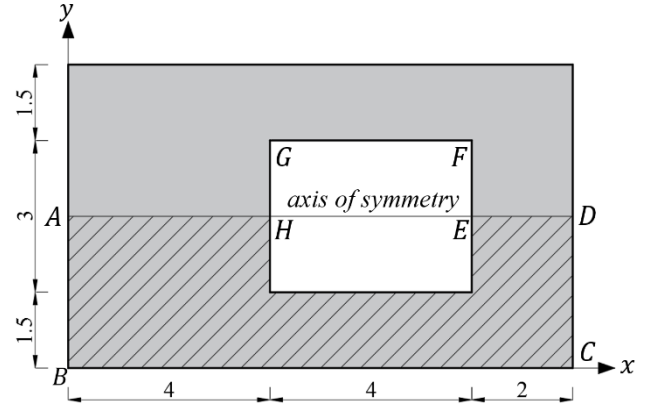


Fig. 4 Cross-section of a rod with a hole

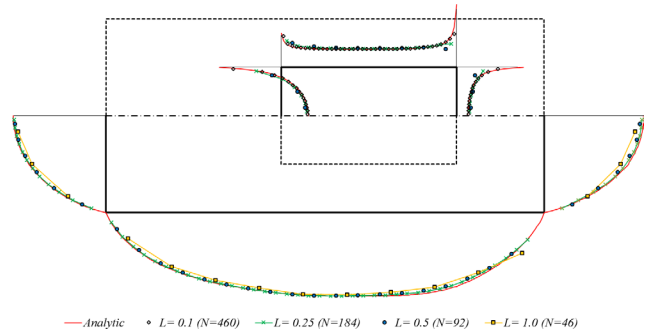


Fig. 5 Distribution of dimensionless tangential stresses along the boundaries of the calculated section. Stress diagrams are not drawn to scale

For example, the stress diagram for the inner contour with 46 boundary elements is not shown because it degenerates into a straight line. To preserve the readability of the graphs, the stress diagram for the outer contour, corresponding to 460 boundary elements, is not shown. Its difference from the presented results is negligible. Based on this table, conclusions can be drawn about the accuracy of the boundary element method in the formulation with respect to stresses.

3.2 Torsion of a rectangular rod with two symmetrical rectangular holes

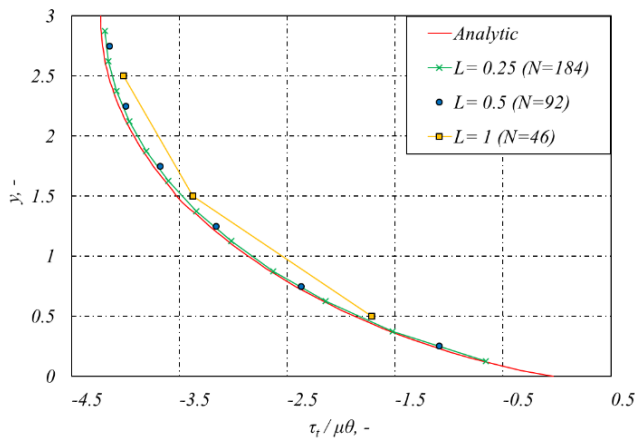
In the second example, a solution to the problem of the torsion of a rod with two symmetrical holes is presented. The cross-sectional dimensions are given in Fig. 7.

The calculation results and their comparison with the analytical solution [20] are shown in Fig. 8 and Fig. 9.

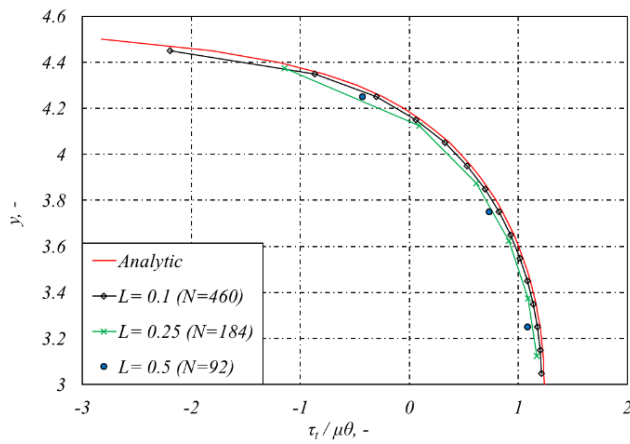
Table 2 shows the numerical values of the stresses at the characteristic points of the section.

4 Conclusions

The paper presents a stress-based formulation of the torsion problem for an arbitrary multiply connected cross-section. An integral representation and the corresponding boundary integral equations are formulated



(a)



(b)

Fig. 6 Diagram of dimensionless tangential stresses τ_t for different sizes (L) and numbers (N) of boundary elements: (a) on contour AB; (b) on contour HG

Table 1 Numerical values of the tangential stresses τ_t in the rod for various values N

Point x, y	Number of boundary elements N				Analytical solution
	46	92	184	460	
0, 1 (external contour)	-2.5487	-2.7654	-2.8351	—	-2.8638
4, 4 (internal contour)	—	0.1503	0.3506	0.4577	0.473

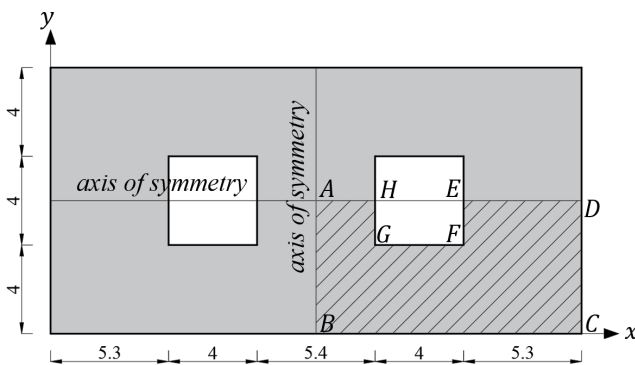
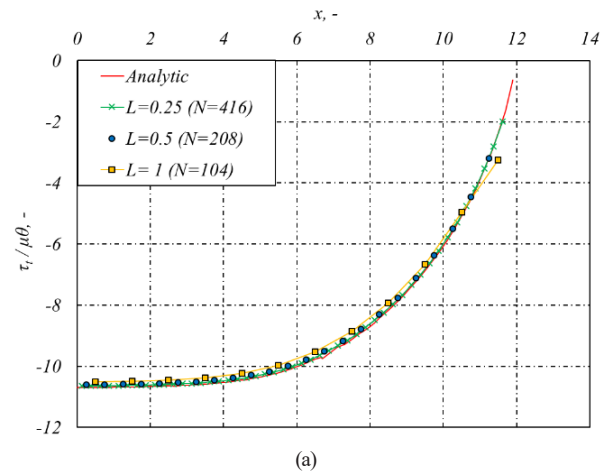
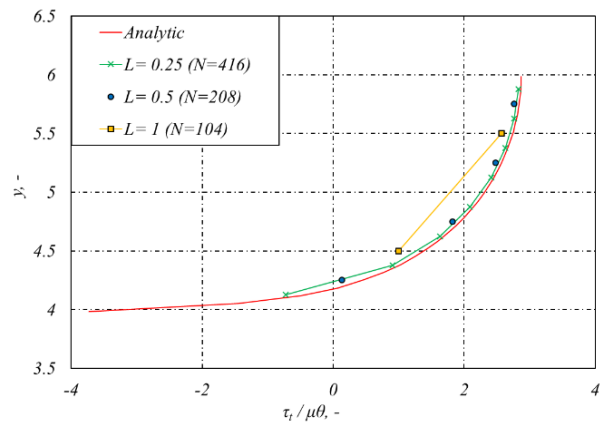


Fig. 7 Cross section of a rod with two holes



(a)



(b)

Fig. 8 Diagram of dimensionless tangential stresses τ_t for different sizes (L) and numbers (N) of boundary elements: (a) on contour BC; (b) on contour HG

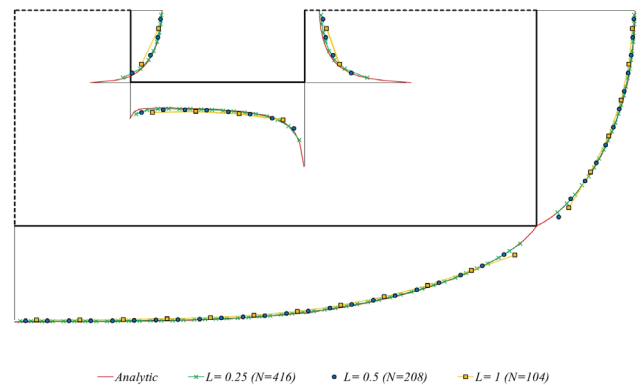


Fig. 9 Distribution of dimensionless shear stresses along the boundaries of the analyzed cross-section. Stress diagrams are not drawn to scale

Table 2 Numerical values of the tangential stresses τ_t in the rod for various values N

Point x, y	Number of boundary elements N			Analytical solution
	104	208	416	
14.5, 0 (external contour)	-10.4452	-10.545	-10.6007	-10.647
14.7, 6 (internal contour)	1.7869	2.1526	2.2603	2.2929

for the stress-based approach. Two boundary value problems for multiply connected sections are solved using the boundary element method. The results demonstrate acceptable engineering accuracy of the stress-based boundary element method solution.

The main conclusions drawn from this study are as follows:

1. Unlike the formulation based on Prandtl's stress functions, the proposed approach does not require solving auxiliary boundary value problems when analyzing multiply connected cross-sections.
2. The stress-based solution provides a more accurate determination of the tangential shear stresses τ_t at the cross-section boundaries. Unlike the classical

formulation, where the determination of τ_t relies on additional numerical differentiation of the nodal values of function φ , the stress-based formulation allows τ_t to be determined directly from the system of boundary element equations.

3. The boundary integral equation for the stress-based formulation is a numerically stable integral equation of the second kind.
4. The presence of multiple connectivity in the cross-section of a twisted bar leads to a negligible increase in the number of unknowns during numerical solution. Each degree of connectivity adds only one additional unknown.

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