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RESEARCH ARTICLE

Stochastic compliance constrained topology optimization based on optimality critera method

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Abstract

The aim of this research is to introduce a new type of stochastic optimal topology design method with iterative solution technique. The paper presents stochastic topology design procedure and compares the achieved results with optimal obtained topologies on deterministic way. The standard mathematical programming problem is based on a minimum volume design procedure subjected to a bounded compliance constraint given in stochastic form. In the numerical method an optimality criteria procedure is used.

Keywords

Topology optimization · stochastic programming · optimality criteria · compliance · optimal design · minimum volume design.

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1 Introduction

Recently the topology optimization is very popular topics in the expanding field of optimal design but the majority of the papers deal with deterministic problems or reliability analysis. The reason for introducing stochastic programming theory, more generally, probabilistic notation is to attempt to consider in a more rational way the fact that the precise strength of a structure is not known, among the constraints there are probabilistic inequalities and perhaps even more importantly, that the loadings applied to the structure are not known with any degree of precision. There is an extensive and expanding literature in this area. Marti made significant achievements in this expanding field [1]-[4]. His Phd student, Stöckl gives a very wide study on stochastic optimization by mathematical programming [5]. Melchers used significant simplification in his optimality criteria based reliability design [6]. Recently Kharmanda et al. [7] have integrated the reliability analysis into a deterministic topology optimization problem by the introduction of the reliability constraint into the standard SIMP procedure, but this one is fundamentally different from the presented method here.

The aim of this research is to introduce a new type of probability based topology design iterative procedure and to compare the obtained results with optimal topologies calculated on deterministic way. This paper is a revised and extended version of reference [12]. The paper is divided into three parts: the first part deals with the deterministic topology optimization briefly, the second part presents the probabilistic based design and the third part compares the topologies obtained by the use of stochastic and deterministic approach.

Introducing the deterministic problem an iterative technique (SIMP) and the connected numerical examples will be discussed briefly. The object of the design (ground structure) is a rectangular disk with given loading and support conditions. The material is linearly elastic. The design variables are the thicknesses of the finite elements. To obtain the correct optimal topology some filtering method (the ground elements are subdivided into further elements) has to be applied to avoid the so-called “checkerboard pattern” [10]. The optimization problem is to minimize the penalized weight of the structure subjected to a given compliance and side constraints.

In the proposed probabilistic topology optimization method: the minimum penalized weight design of the structure is subjected to compliance constraint which has uncertainties and side constraints. The compliance design is very often applied in the field of topology optimization due to its simple formulations but there are a significant number of researchers who state that the method is not acceptable in practice due to the uncertainties of the compliance. This study makes an attempt to get closer to the reality in case of compliance design. If the compliance value is given by the distribution function, the mean value and variance, then the deterministic topology problem can be modified and the compliance constraint is substituted by a probability constraint. This probabilistic expression can be used as a constraint in the original problem. By the use of the first order optimality criteria a redesign formula of the stochastically constrained topology optimization problem can be derived.

The new classes of optimal topologies with their analytical and numerical confirmation are presented. The standard FEM computer program with quadrilateral membrane elements is applied in the numerical calculation. Through the numerical examples the paper compares the deterministic optimal topologies and optimal topologies obtained in case of uncertain situations.

2 Optimization problem

2.1 Deterministic problem determination

The deterministic compliance design procedure is known from literature (e.g. Lógó [8]). For illustration purposes, by the use of the FEM, let us consider the following simple case:

- the linearly elastic, 2D structure (disk) is subdivided into ($g = 1, \ldots, G$) ground elements with constant thicknesses (t_g) which are either $t_g = t_{\min} = 0$ or $t_g = t_{\max} = 1$, such that each ground element (g) contains several sub-elements ($e=1,\ldots,E_s$), whose stiffness coefficients are linear homogeneous functions of the ground element thickness t_g. Practically it means that the meshing consists of two parts, a primary and a secondary one.
- single static loading,
- given boundary conditions and
- compliance constraints.

The above-normalized formulation is equivalent with the problems with a different prescribed maximum thickness $t_g = t_{\max}$. Due to the linear relations this is done by multiplying all loads by $t_{\max}$, whilst stresses, strains and displacements do not alter their values.

The weight (W) of the structure is given by

$$W = \sum_{g=1}^{G} \gamma_g A_g t_g. \tag{1}$$

Where γ_g is the specific weight and A_g the area of the ground element g.

The compliance constraint can be expressed as

$$\mathbf{u}^{\mathrm{T}}\mathbf{K}\mathbf{u} - C \leq 0; \tag{2}$$

where $\mathbf{K}$ is the system stiffness matrix, $\mathbf{u}$ is the nodal displacement vector associated with the load $\mathbf{F}$. In Eq. (2) the nodal displacement vector $\mathbf{u}$ is calculated from the $\mathbf{Ku=F}$ linear system.

The side constraints can be stated as

$$\begin{aligned} -t_g + t_{\min} \leq 0; \quad (for\ g = 1, ..., G)\ ; \\ t_g - t_{\max} \leq 0; \quad (for\ g = 1, ..., G)\ . \end{aligned} \tag{3}$$

In order to suppress the intermediate thicknesses, the weight calculation formulation is replaced by $\tilde{W} = \sum_{g=1}^{G} \gamma_g A_g t_g^{\frac{1}{p}}$, where p is the penalty parameter and $p \geq 1$. This gives the exact weight value for t_g =0 and t_g =1 in case any p value. The use of the penalty parameter has similar effect in the later formulations as its role was in the classical optimality criteria method.

The deterministic optimization problem is to minimize the penalized weight of the structure which is subjected to a given compliance and side constraints.

$$\tilde{W} = \sum_{g=1}^{G} \gamma_g A_g t_g^{\frac{1}{p}} = \min!$$

$$\text{subject to} \begin{cases} \mathbf{u}^{\mathrm{T}}\mathbf{K}\mathbf{u} - C \leq 0; \\ -t_g + t_{\min} \leq 0; (for\ g = 1, ..., G)\ , \\ t_g - t_{\max} \leq 0; (for\ g = 1, ..., G)\ . \end{cases} \tag{4}$$

2.2 Stochastic problem determination

Let us suppose that in case of probabilistic design the lower and upper compliance bounds are random variables and they follow normal distribution. They are given by their distribution functions $\Phi\,(C_1)$, $\Phi\,(C_2)$, mean values and variances ($C_{\min}$, $\sigma_{\min}$, $C_{\max}$, $\sigma_{\max}$), respectively. The compliance constraint has the modified form

$$\mathrm{P}(C_1 \leq \mathbf{u}^{\mathrm{T}}\mathbf{K}\mathbf{u} \leq C_2) \geq q; \tag{5}$$

where $q \geq 0$ is the given probability value. Substituting Eq. (5) in Eq. (4) one can obtain the following optimal design formulation:

$$\tilde{W} = \sum_{g=1}^{G} \gamma_g A_g t_g^{\frac{1}{p}} = \min! \tag{6.a}$$

$$\text{subjectto} \begin{cases} q - \mathrm{P}(C_1 \leq \mathbf{u}^{\mathrm{T}}\mathbf{K}\mathbf{u} \leq C_2) \leq 0; \\ -t_g + t_{\min} \leq 0; (\text{for } g = 1, ..., G)\ , \\ t_g - t_{\max} \leq 0; (\text{for } g = 1, ..., G)\ . \end{cases} \tag{6.b-d}$$

Let us suppose that C_1 and C_2 are independent random variables, so Eq. (6.b) can be written as

$$\mathrm{P}\left(C_1 \leq \mathbf{u}^{\mathrm{T}}\mathrm{K}\mathbf{u}\right) \mathrm{P}\left(\mathbf{u}^{\mathrm{T}}\mathbf{K}\mathbf{u} \leq C_2\right) \geq q\ . \tag{7}$$

Then the minimum penalized weight problem subjected to probabilistic constraint is defined as follows:

$$\tilde{W}=\sum_{g=1}^{G}\gamma_g A_g t_g^{\frac{1}{p}}=\min! \tag{8.a}$$

$$\text{subject to}\begin{cases} q-P(C_1\le \mathbf{u}^{\mathrm{T}}\mathbf{Ku})P(\mathbf{u}^{\mathrm{T}}\mathbf{Ku}\le C_2)\le 0;\\ -t_g+t_{\min}\le 0;\ \ (\text{for } g=1,\dots,G)\,;\\ t_g-t_{\max}\le 0;\ \ (\text{for } g=1,\dots,G)\,. \end{cases} \tag{8.b-d}$$

To simplify the probabilistic constraint (8.b) the following standardized forms [9] can be introduced for the random variables:

$$q-\mathrm{P}\left(\frac{C_1-C_{\min}}{\sigma_{\min}}\le x\right)P\left(y\le\frac{C_2-C_{\max}}{\sigma_{\max}}\right)\le 0; \tag{9}$$

where

$$x=\frac{\mathbf{u}^{\mathrm{T}}\mathbf{Ku}-C_{\min}}{\sigma_{\min}}\ \text{ and }\ y=\frac{\mathbf{u}^{\mathrm{T}}\mathbf{Ku}-C_{\max}}{\sigma_{\max}}.$$

The density functions and the distribution functions are the following:

$$dsx=\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}},\ dsy=\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}},\ drx=\int_{-\infty}^{x}\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}z,\ dry=\int_{y}^{\infty}\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}z.$$

By the use of the standardized forms of the random variables constraint (8.b) can be written as follows:

$$q-\int_{-\infty}^{x}\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}z\int_{y}^{\infty}\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}z\ \le 0; \tag{10}$$

The final form of minimum penalized weight problem subjected to probabilistic constraint is defined as follows:

$$\tilde{W}=\sum_{g=1}^{G}\gamma_g A_g t_g^{\frac{1}{p}}=\min! \tag{11.a}$$

$$\text{subject to}\begin{cases} q-\int_{-\infty}^{x}\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}z\int_{y}^{\infty}\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}z\ \le 0;\\ -t_g+t_{\min}\le 0;\ (\text{for } g=1,\dots,G);\\ t_g-t_{\max}\le 0;\ (\text{for } g=1,\dots,G). \end{cases} \tag{11.b-d}$$

2.2.1 Lagrange function

Using the Lagrange multipliers υ, α_g, β_g and slack variables h_1, h_{2g}, h_{3g} for the constraints in problem (11), the following Lagrange function can be written:

$$\begin{aligned}&\mathcal{L}\left(t_g,\upsilon,\alpha_g,\beta_g,h_1,h_{2g},h_{3g}\right)=\\ &\sum_{g=1}^{G}\gamma_g A_g t_g^{\frac{1}{p}}+\upsilon\left(q-\int_{-\infty}^{x}\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}z\int_{y}^{\infty}\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}z\ +h_1^2\right)+\\ &\sum_{g=1}^{G}\alpha_g\left(-t_g+t_{\min}+h_{2g}^2\right)+\sum_{g=1}^{G}\beta_g\left(t_g-t_{\max}+h_{3g}^2\right)\end{aligned} \tag{12}$$

2.2.2 Kuhn-Tucker conditions

Neglecting the details, one can obtain

$$\begin{aligned}&\frac{\partial\mathcal{L}}{\partial t_g}=\frac{1}{p}\gamma_g A_g t_g^{\frac{1-p}{p}}+\nu\left(\frac{dsx\cdot dry}{\sigma_{\min}^2}+\frac{dsy\cdot drx}{\sigma_{\max}^2}\right)\times\\ &\left(\frac{\partial\mathbf{u}^T}{\partial t_g}\mathrm{K}\mathbf{u}+\mathbf{u}^T\frac{\partial\mathbf{K}}{\partial t_g}\mathbf{u}+\mathbf{u}^T\mathbf{K}\frac{\partial\mathbf{u}}{\partial t_g}\right)-\alpha_g+\beta_g=0;\\ &(g=1,\dots,G).\end{aligned} \tag{13.a}$$

Due to the symmetry of the stiffness matrix $\mathbf{K}$ and other simplification Eq. (13.a) can be replaced by the following relation

$$\begin{aligned}&\frac{\partial\mathcal{L}}{\partial t_g}=\frac{1}{p}\gamma_g A_g t_g^{\frac{1-p}{p}}-\nu\left(\frac{dsx\cdot dry}{\sigma_{\min}^2}+\frac{dsy\cdot drx}{\sigma_{\max}^2}\right)\times\\ &\sum_{e=1}^{E_s}u_{ge}^{T}\frac{\partial\mathbf{K}_{ge}}{\partial t_g}u_{ge}-\alpha_g+\beta_g=0;\\ &(g=1,\dots,G)\,,\end{aligned} \tag{13.b}$$

where the subscript ge refers to the e-th finite element of the g-th ground element.

If the "normalized" element stiffness matrix is $\tilde{\mathbf{K}}_{ge}$ (e.g. calculated for a unit thickness ($t_g = 1$)), then due to the linear relation the element stiffness matrix $\mathbf{K}_{ge}$ for the actual thickness t_g is expressed as $\mathbf{K}_{ge}=t_g\tilde{\mathbf{K}}_{ge}$ and $\frac{\partial\mathbf{K}_{ge}}{\partial t_g}=\tilde{\mathbf{K}}_{ge}$. Introducing the following notation $R_g=t_g^2\sum_{e=1}^{E_g}\mathbf{u}_{ge}^T\tilde{\mathbf{K}}_{ge}u_{ge}$ the Eq. (13.b) becomes very simple

$$\begin{aligned}&\frac{1}{p}\gamma_g A_g t_g^{\frac{1-p}{p}}-\upsilon\frac{R_g\left(\frac{dsx\cdot dry}{\sigma_{\min}^2}+\frac{dsy\cdot drx}{\sigma_{\max}^2}\right)}{t_g^2}-\alpha_g+\beta_g=0;\\ &(g=1,\dots,G).\end{aligned} \tag{13.c}$$

Continuing the further derivations:

$$\frac{\partial\mathcal{L}}{\partial\nu}=q-\int_{-\infty}^{x}\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}z\cdot\int_{y}^{\infty}\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}z\ +h_1^2=0; \tag{14.a}$$

$$\frac{\partial\mathcal{L}}{\partial h_1}=2\nu h_1=0; \tag{14.b}$$

$$\frac{\partial\mathcal{L}}{\partial\alpha_g}=-t_g+t_{\min}+h_{2g}^2=0; \tag{15.a}$$

$$\frac{\partial\mathcal{L}}{\partial h_{2g}}=2\alpha_g h_{2g}=0; \tag{15.b}$$

$$\frac{\partial\mathcal{L}}{\partial\beta_g}=t_g-t_{\max}+h_{3g}^2=0; \tag{16.a}$$

$$\frac{\partial\mathcal{L}}{\partial h_{3g}}=2\beta_g h_{3g}=0; \tag{16.b}$$

Omitting the details from Eqs. (13.c), (14.a-b), (15.a-b) and (16.a-b) the values of the Lagrange multipliers, slack variables and the thickness values t_g can be calculated iteratively.

As it is in COC type methods, before the calculation of the Lagrange multiplier ν, one needs to define two sets of the thicknesses: a set of active and a set of passive thicknesses.

There exist three possibilities:

If $t_{\min} < t_g < t_{\max}$ (or by other words, the ground element is "active", $g \in A$) then $\alpha_g = \beta_g = 0$ and by (13.c) the following formula can be obtained

$$t_g = \left(\frac{\nu p R_g \left(\frac{dsx \cdot dry}{\sigma_{\min}^2} + \frac{dsy \cdot drx}{\sigma_{\max}^2} \right)}{A_g \gamma_g} \right)^{\frac{p}{p+1}} \tag{17}$$

In case of $t_g = t_{\min}$, $\alpha_g \geq 0$, $h_{2g} = 0$ and (13.c) becomes

$$t_g \geq \left(\frac{\nu p R_g \left(\frac{dsx \cdot dry}{\sigma_{\min}^2} + \frac{dsy \cdot drx}{\sigma_{\max}^2} \right)}{A_g \gamma_g} \right)^{\frac{p}{p+1}} \tag{18}$$

This means that if (17) gives a t_g-value which is smaller than $t_{\min}$ then (13.c) is satisfied by $t_g = t_{\min}$. Similarly, in case of $t_g = t_{\max}$, $\beta_g \geq 0$, $h_{3g} = 0$ and then (13.c) implies

$$t_g \leq \left(\frac{\nu p R_g \left(\frac{dsx \cdot dry}{\sigma_{\min}^2} + \frac{dsy \cdot drx}{\sigma_{\max}^2} \right)}{A_g \gamma_g} \right)^{\frac{p}{p+1}} \tag{19}$$

This allows $t_g = t_{\max}$ when (17) gives a t_g-value which is greater than $t_{\max}$. If $t_g = t_{\min}$ or $t_g = t_{\max}$ we call the ground element "passive" ($g \in P$).

2.2.3 Optimality criteria and the final iterative formulas

If a ground element is "active",($g \in A$) then Eq. (13.c) can be written as follows

$$1 - \nu \frac{\left(\frac{dsx \cdot dry}{\sigma_{\min}^2} + \frac{dsy \cdot drx}{\sigma_{\max}^2} \right) \sum_{e=1}^{E_s} u_{ge}^{T} \frac{\partial \mathbf{K}_{ge}}{\partial t_g} u_{ge}}{\frac{1}{p} \gamma_g A_g t_g^{\frac{1-p}{p}}} = 0; \quad (g = 1, \ldots, G). \tag{20.a}$$

By words it means that in case of optimal solution the stochastically modified average strain energy variation of all active elements are same and constant. This expression eq. (20.a) can be called as optimality criteria of the stochastic compliance design.

According to the simplification in Eq. (13.c) the Eq. (20.a) is equivalent with the following expression

$$1 - \nu \frac{R_g \left(\frac{dsx \cdot dry}{\sigma_{\min}^2} + \frac{dsy \cdot drx}{\sigma_{\max}^2} \right)}{\frac{1}{p} \gamma_g A_g t_g^{\frac{1+p}{p}}} = 0; \quad (g = 1, \ldots, G). \tag{20.b}$$

The value of the Lagrange multiplier ν during the iteration process may be found from Eq. (20.b) by minimizing the sum of the squares of the residuals at iteration n:

$$\mathrm{Res}_n = \sum_{g=1}^{G} \left(1 - \nu \frac{R_g \left(\frac{dsx \cdot dry}{\sigma_{\min}^2} + \frac{dsy \cdot drx}{\sigma_{\max}^2} \right)}{\frac{1}{p} \gamma_g A_g t_g^{\frac{1+p}{p}}} \right)^2 \tag{21}$$

Since the thickness value for passive elements ($g \in P$) is given, the "effect" of the zero thickness elements can be handled in Eq. (21) and for active elements ($g \in A$), it can be calculated by the use of (21), then at iterate (n) the Lagrange multiplier ν can be formed as follows:

$$\nu_n = \frac{\sum_{g=1}^{G} \left(\frac{R_{g,n-1} \left(\frac{dsx \cdot dry}{\sigma_{\min}^2} + \frac{dsy \cdot drx}{\sigma_{\max}^2} \right)}{\frac{1}{p} \gamma_g A_g t_{g,n-1}^{\frac{1+p}{p}}} \right)}{\sum_{g=1}^{G} \left(\frac{R_{g,n-1} \left(\frac{dsx \cdot dry}{\sigma_{\min}^2} + \frac{dsy \cdot drx}{\sigma_{\max}^2} \right)}{\frac{1}{p} \gamma_g A_g t_{g,n-1}^{\frac{1+p}{p}}} \right)^2}, \tag{22}$$

where $R_{g,n-1} = t_{g,n-1}^2 \sum_{e=1}^{E_g} \mathbf{u}_{ge}^T \tilde{\mathbf{K}}_{ge} \mathbf{u}_{ge}$ was calculated by the results of the $(n-1)$-th iterate.

If the probabilistic compliance constraint is active in problem (8.a-d) (e.g. satisfies the equality sign) the following form holds

$$\int_{-\infty}^{x} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \mathrm{d}z \int_{-\infty}^{y} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \mathrm{d}z - q = 0 \tag{23}$$

where x and y are given by (9). This equation can be used as a termination condition.

The optimal solution can be obtained by evaluating iteratively the thickness values t_g and the Lagrange multiplier ν from (17) and (22).

2.2.4 Stochastic SIMP algorithm

The *Applied stochastic SIMP Algorithm* can be defined as follows:

1 Specify the Max and Min value of t_g, ($t_{g\,\max} = 1$, $t_{g\,\min} = 10^{-6}$).

2 Specify the $C_{\min}$, $\sigma_{\min}$, $C_{\max}$, $\sigma_{\max}$ values.

3 Specify a maximum of C (compliance), of say $C = C_{\max} + 4 \cdot \sigma_{\max}$.

4 Specify design domain, including supports and loading.

5 Set the penalty value, $p = 1$, later this value will be incremented to $p = 1.25$, 2, etc.

6 Carry out FEM.

7 Extract displacement field for entire structure $\mathbf{u}^T$.

8 Calculate elemental compliance $\bar{C}_e$ and R_g with displacement vector based on current element solution set t_g, but using the stiffness matrix for the elements as if it had t_g=1.

$$\bar{C}_e = \{\mathbf{u}_e\}^T \left[\tilde{\mathbf{K}}_e\right] \{\mathbf{u}_e\}.$$

9 Calculate the densities (*dsx, dsy*), and probability values (*drx, dry*).

10 Calculate step length multiplier ν_{new}:

$$\nu_{\text{new}} = \frac{\sum_{g=1}^{G} \left(\frac{R_{g,\text{old}} \left(\frac{dsx \cdot dry}{\sigma_{\min}^2} + \frac{dsy \cdot drx}{\sigma_{\max}^2} \right)}{\frac{1}{p} \gamma_g A_g t_{g,\text{old}}^{\frac{1+p}{p}}} \right)}{\sum_{g=1}^{G} \left(\frac{R_{g,\text{old}} \left(\frac{dsx \cdot dry}{\sigma_{\min}^2} + \frac{dsy \cdot drx}{\sigma_{\max}^2} \right)}{\frac{1}{p} \gamma_g A_g t_{g,\text{old}}^{\frac{1+p}{p}}} \right)^2}.$$

11 Calculate new element solution set:

$$t_{g,\text{new}} = \left(\frac{\upsilon_{\text{new}} p \left(R_{g,\text{old}} \left(\frac{dsx \cdot dry}{\sigma_{\min}^2} + \frac{dsy \cdot drx}{\sigma_{\max}^2} \right) \right)}{A_g \gamma_g} \right)^{\frac{p}{p+1}},$$

where υ is the step length multiplier calculated in step 11.

12 Determine the set of active and passive elements by the following element limit set:

$$\begin{aligned} t_{g,\text{new}} &= t_{\min} \quad \text{if} \quad t_{g,\text{new}} \le t_{\min} = 10^{-6}; \quad e \in P, \\ t_{g,\text{new}} &= t_{\max} \quad \text{if} \quad t_{g,\text{new}} \ge t_{\max} = 1; \quad e \in P, \\ t_{g,\text{new}} &= t_{g,\text{new}} \quad \text{if} \quad t_{\min} \le t_g \le t_{\max}; \quad e \in A. \end{aligned}$$

13 Calculate the probability value $P_{\text{new}} = \int_{-\infty}^{x} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \mathrm{d}z \int_{y}^{\infty} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \mathrm{d}z$.

14 If the probability value is $P_{\text{new}} < q$ and the active set has changed in the previous iteration, go to step 8, else if active set has not changed and the probability value is still $P_{\text{new}} < q$ from pervious iteration, increase the penalty parameter $p = p + \textit{increment}$ (step size is controlled), and go to step 6.

15 If the probability value is $P_{\text{new}} < q$ and all the elements are passive increase the penalty limit and go to step 6 with $p = p + \textit{increment}$, else if the probability value is $P_{\text{new}} = q$ and active set has not changed then stops.

Then we get the optimal solution of problem (11).

In topology optimization the checker board pattern frequently happens. To avoid this as an optimal solution a simple procedure was used which was suggested by Gáspár, Lógó, Rozvany [10]. The key point is that all the ground elements (a primary meshing provides the so-called ground elements) should sub-divide into further finite elements (secondary elements). For the subdivision it is enough to use 2 by 2 elements. Further number of sub-elements cannot improve significantly the final result.

To speeding the iterative process it is common to use bi-level algorithm [12] which means it is advised to use the deterministic SIMP algorithm[8] until the calculated probability value reaches a certain value (say 50% of q).

3 Numerical examples

A very often investigated test structure is a rectangular domain (Fig. 1) with two simple supports and a point load. The height/length ratio is 0.5 and the supports are located at the bottom of the left and right edges, respectively. The load (F_1) is constant (100 units) and located at middle of the bottom line. The rectangular ground structure is dimensionless (40x20 units). 80x40 ground elements with 2x2 sub-elements are used. (Total number of finite elements is 12800.) The Poisson's ratio is 0.

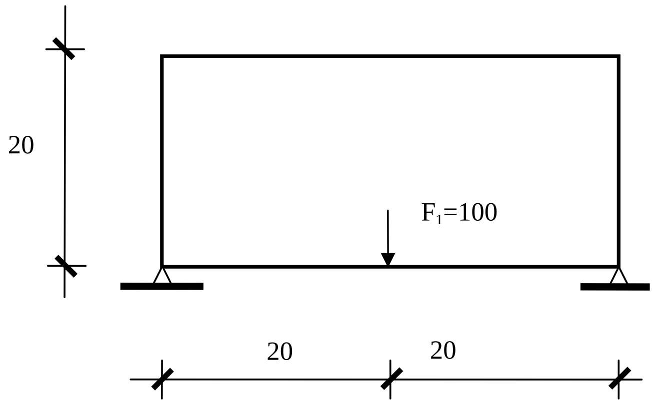

Fig. 1. Rectangular design domain.

The exact analytical solution can be seen on Fig. 2. which was proved by Rozvany [11] and the original topology comes from Hemp's work [13].

In the following, two cases are investigated, where at first the deterministic problem is presented while secondly the stochastic optimal topology is calculated.

3.1 Deterministic topology optimization

The penalty parameter p was run from $p = 1$ to $p = 1.5$ with smooth increasing (*increment is 0.1*) and later to $p = 2.5$ with *increment*=0.25. The compliance limit is 180000. The numerically obtained optimal topologies can be seen on Fig. 3 what is in a good agreement with the analytical solution.

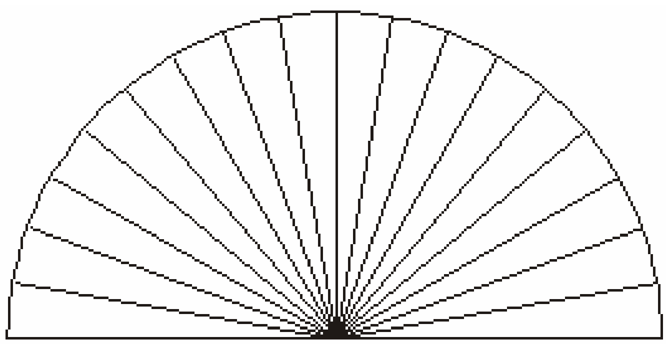

Fig. 2. Analytical solution

3.2 Stochastic topology optimization

In case of stochastic topology optimization there are several data to describe the evolution of the compliance limits ($C_1,\ C_2$) what are random variables. The mean values and the variances are $C_{\min} = 166000$, $\sigma_1 = 3000$, $C_{\max} = 176000$, $\sigma_2 = 2000$. The evolution of the normalized density functions can be seen on Fig. 4. The required probability value is q =0.954 in Eq. (11.a).

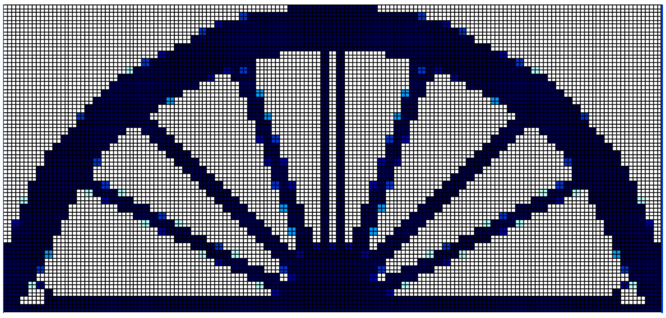

Fig. 3. Deterministic optimal topology

Applying the iterative procedure presented in Chapter 2.2.4 the optimal topology can be computed. It was 897 major iteration steps to obtain the solution shown on Fig. 5. One can see that the character of figure of the stochastic optimal topology is in good agreement with the deterministic optimal topology. The difference comes that the black and white solution (1-0) of the deterministic topology becomes grey as it usually happen in case of stochastic optimization. To stabilize the introduced iterative algorithm the first part of the calculation is based on the deterministic algorithm until the calculated probability is different from zero (to start with a feasible solution).

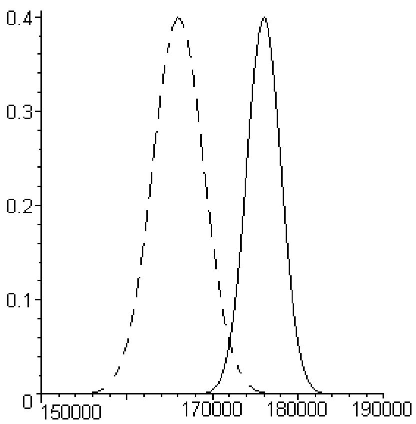

Fig. 4. Distribution functions

4 Conclusions

The stochastic optimization problem was solved. The introduced algorithm provides an iterative tool which allows to use thousands of design variables what is impossible by the use of a conventional optimization program. By the use of secondary meshing of ground elements the amount of the checker-board pattern was neglected on acceptable level.

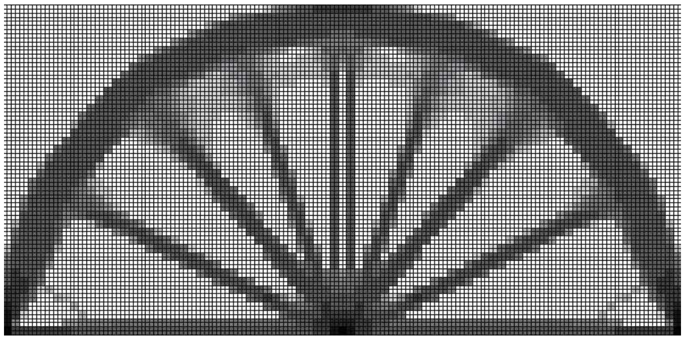

Fig. 5. Stochastic optimal topology

The present stage of the research shows that due to iterative formulation of thicknesses of the ground elements obtained in the stochastic problem it is advised to start the computation with the deterministic SIMP algorithm and when the calculated probability of the solution is not zero, is needed to turn to the stochastic algorithm. The data of the random variables in the problem can create such cases where the problem is non-convex. The "so-called grey" solution means that the probabilistic optimal topology is naturally lighter than the corresponding deterministic optimal topology, but the optimal shape of the structure is practically the same. Some other advantage is that the designer can take into consideration some initial design uncertainties with the probabilistically given compliance limit.

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