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RESEARCH ARTICLE

BEM formulation for plane orthotropic bodies – a modification for exterior regions and its proof

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Abstract

Assuming linear displacements and constant strains and stresses at infinity, our aim is to reformulate the equations of the direct boundary element method for plane problems of elasticity. We shall consider a body made of orthotropic material. The equations we have reformulated make possible to attack plane problems on exterior regions without replacing the region in question by a bounded one.

Keywords

Boundary element method · Somigliana relations · direct method · orthotropic body · plane problems · exterior regions

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1 Introduction

As is well known a large literature studies plane problems for orthotropic bodies, including for instance [9], [8], [7] as well as the books [1, 2] and the references therein. However, as explained by Schiavone [10], the standard formulation for exterior regions has the disadvantage that it is impossible to prescribe a constant stress state at infinity.

The reason is that an assumption about the far field pattern of the displacements is needed in order to establish an appropriate Betti formula and to prove uniqueness and existence for the exterior Dirichlet and Neuman problems. Unfortunately, this assumption excludes those problems from the theory for which the displacements are linear while the strains and stresses are constant at infinity.

To make progress on plane problems with such displacements, we note that if the direct formulation reproduces this displacement field, then the resulting strain and stress conditions must also be constant at infinity. Consequently, plane problems for the exterior regions can be attacked without replacing the region by a bounded one. The work in [11] and [12] presents such direct formulations by assuming constant strains and stresses at infinity for an isotropic body. For exterior regions [11] reformulates the classical approach to plane problems. For the same class of problems but in a dual formulation [12] sets up the equations of the direct method in terms of stress functions of order one.

The present paper is an attempt to clarify how the formulation changes if we apply the ideas presented in paper [11] to orthotropic bodies.

2 Preliminaries

Throughout this paper x_1 and x_2 are rectangular Cartesian coordinates, referred to an origin O. Greek subscripts are assumed to have the range (1,2), summation over repeated subscripts is implied. The doubly connected exterior region under consideration is denoted by A_e and is bounded by the contour $\mathcal{L}_o$. We stipulate that the contour admits a nonsingular parametrization in terms of its arc length s. The outer normal is denoted by n_π. In accordance with the notations introduced, $\delta_{\kappa\lambda}$ is the Kronecker symbol, ∂_α stands for the derivatives taken with respect to x_α and $\epsilon_{3\kappa\lambda}$ is the permutation symbol. Assuming plane problems let u_κ, $e_{\kappa\lambda}$ and $t_{\kappa\lambda}$ be the displacement field and the in plane components of strain and stress, respectively. For orthotropic bodies the material constants are denoted by s_{11}, $s_{12} = s_{21}$, s_{22} and s_{66}.

For homogenous and orthotropic material the plane problem of classical elasticity is governed by the kinematic equations

$$e_{\rho\lambda} = \frac{1}{2}(\partial_\rho u_\lambda + \partial_\lambda u_\rho), \tag{1}$$

Hooke's law

$$\begin{aligned} t_{11} &= c_{11}e_{11} + c_{12}e_{22}\,, \\ t_{22} &= c_{12}e_{11} + c_{22}e_{22}\,, \\ t_{12} &= t_{21} = 2c_{66}e_{12}\,, \end{aligned} \tag{2}$$

where

$$c_{11} = \frac{s_{22}}{d}, \quad c_{12} = c_{21} = -\frac{s_{12}}{d}, \quad c_{22} = \frac{s_{11}}{d}, \qquad c_{66} = \frac{1}{s_{66}}, \quad d = s_{11}s_{22} - s_{12}^2\,; \tag{3}$$

and the equilibrium equations

$$t_{\rho\lambda}\partial_\lambda + b_\rho = 0 \tag{4}$$

which should be complemented with appropriate boundary conditions not detailed here since they play no role in the present investigations. The basic equation for u_λ takes the form

$$\mathcal{D}_{\rho\lambda}u_\lambda + b_\rho = 0\,, \tag{5a}$$

where the differential operator $\mathcal{D}_{\rho\lambda}$ has the form

$$\left[\mathcal{D}_{\rho\lambda}\right] = \begin{bmatrix} c_{11}\partial_1^2 + c_{66}\partial_2^2 & (c_{12}+c_{66})\,\partial_1\partial_2 \\ (c_{21}+c_{66})\,\partial_2\partial_1 & c_{22}\partial_2^2 + c_{66}\partial_1^2 \end{bmatrix}. \tag{5b}$$

Let $Q(\xi_1, \xi_2)$ and $M\,(x_1, x_2)$ be two points in the plane (the source point and the field point). We shall assume temporarily that the point Q is fixed. The distance between Q and M is R, the position vector of M relative to Q is r_κ. The small circle as a subscript (for instance $M_\circ$ or $Q_\circ$) indicates that the corresponding points, i.e., Q or M are taken on the contour.

It is obvious that

$$r_\alpha(M, Q) = x_\alpha(M) - \xi_\alpha(Q) = x_\alpha - \xi_\alpha\,. \tag{6}$$

Let us introduce the following notations

$$\lambda_1 + \lambda_2 = (2s_{12} + s_{66})\,/s_{22}\,, \tag{7}$$

$$\lambda_1\lambda_2 = s_{11}/s_{22}\,, \tag{8}$$

$$A_\alpha = s_{12} - \lambda_\alpha s_{22}\,, \tag{9}$$

$$\rho_\alpha^2 = \lambda_\alpha r_1^2 + r_2^2\,, \tag{10}$$

$$D = \frac{1}{2\pi\,(\lambda_1 - \lambda_2)\,s_{22}}\,. \tag{11}$$

For our later considerations we note that Eqs. (7) and (8) imply

$$\lambda_{1,2} = \frac{2s_{21}+s_{66}}{2s_{22}} \pm \sqrt{(\frac{2s_{21}+s_{66}}{2s_{22}})^2 - \frac{s_{11}}{s_{22}}}\,. \tag{12}$$

The well-known singular fundamental solutions for the basic Eq. (5a) [7, 9] are given by the formulas

$$\begin{aligned} U_{11}(M,Q) &= D\left(\sqrt{\lambda_1}A_2^2\ln\rho_1 - \sqrt{\lambda_2}A_1^2\ln\rho_2\right)\,, \\ U_{12}(M,Q) &= DA_1A_2\arctan\frac{\left(\sqrt{\lambda_1}-\sqrt{\lambda_2}\right)r_1r_2}{\sqrt{\lambda_1}\sqrt{\lambda_2}r_1^2 + r_2^2}\,, \\ U_{21}(M,Q) &= U_{12}(M,Q)\,, \\ U_{22}(M,Q) &= -D\left(\frac{A_1^2\ln\rho_1}{\sqrt{\lambda_1}} - \frac{A_2^2\ln\rho_2}{\sqrt{\lambda_2}}\right) \end{aligned} \tag{13}$$

and

$$\begin{aligned} T_{11}(M,Q) &= D\left[\frac{\sqrt{\lambda_2}A_1}{\rho_2^2} - \frac{\sqrt{\lambda_1}A_2}{\rho_1^2}\right](r_1n_1 + r_2n_2)\,, \\ T_{12}(M,Q) &= D\left\{\left(\frac{\sqrt{\lambda_1}A_1}{\rho_1^2} - \frac{\sqrt{\lambda_2}A_2}{\rho_2^2}\right)r_1n_2 - \left(\frac{1}{\sqrt{\lambda_1}}\frac{A_1}{\rho_1^2} - \frac{1}{\sqrt{\lambda_2}}\frac{A_2}{\rho_2^2}\right)r_2n_1\right\}\,, \\ T_{21}(M,Q) &= D\left\{\left(\frac{\lambda_1\sqrt{\lambda_1}A_2}{\rho_1^2} - \frac{\lambda_2\sqrt{\lambda_2}A_1}{\rho_2^2}\right)r_1n_2 - \left(\frac{\sqrt{\lambda_1}A_2}{\rho_1^2} - \frac{\sqrt{\lambda_2}A_1}{\rho_2^2}\right)r_2n_1\right\}\,, \\ T_{22}(M,Q) &= D\left[\frac{\sqrt{\lambda_1}A_1}{\rho_1^2} - \frac{\sqrt{\lambda_2}A_2}{\rho_2^2}\right](r_1n_1 + r_2n_2)\,, \end{aligned} \tag{14}$$

where

$$u_\lambda(M) = U_{\lambda\kappa}(M,Q)e_\kappa(Q), \text{ and } t_\lambda(M) = T_{\lambda\kappa}(M,Q)e_\kappa(Q)$$

are the displacement vector and stress vector on a line element with a normal $n_\lambda = n_\lambda(M)$ to it at the point M due to the force $e_\kappa = e_\kappa(Q)$ at Q.

3 Basic formulas for exterior regions

Fig. 1 depicts a triple connected region A'_e bounded by the contours $\mathcal{L}_o$, $\mathcal{L}_\varepsilon$ and the circle $\mathcal{L}_R$ with radius ${}_eR$ and center at O. Here $\mathcal{L}_\varepsilon$ is the contour of the neighborhood A_ε of Q with radius R_ε while ${}_eR$ is sufficiently large so that the region bounded by $\mathcal{L}_R$ covers both $\mathcal{L}_0$, and $\mathcal{L}_\varepsilon$. If ${}_eR \to \infty$ and $R_\varepsilon \to 0$ then clearly $A'_e \to A_e$.

Let $u_\kappa(M)$ and $g_\kappa(M)$ be sufficiently smooth – continuously differentiable at least twice – but otherwise arbitrary displacement fields on A_e. The stresses obtained from these displacement fields are denoted by $t_{\lambda\kappa}\left[u_\rho(M)\right]$ and $t_{\lambda\kappa}\left[g_\rho(M)\right]$ respectively.

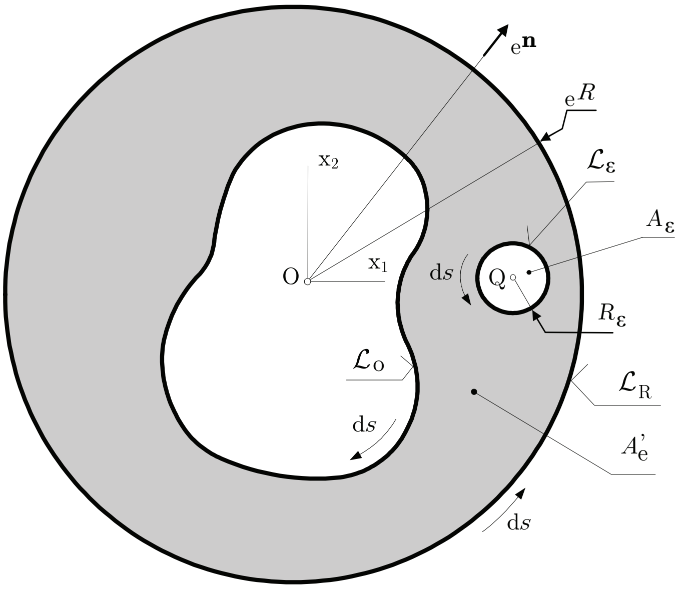

Fig. 1.

Equation

$$\int_{A'_e}\left[u_\lambda(M)\left(\mu \overset{M}{\mathcal{D}}_{\lambda\sigma} g_\sigma(M)\right) - g_\lambda(M)\left(\mu \overset{M}{\mathcal{D}}_{\lambda\sigma} u_\sigma(M)\right)\right] \mathrm{d}A_M =$$
$$\oint_{\mathcal{L}_o}\left[u_\lambda(M_\circ)t_{\lambda\kappa}\left[g_\rho(M_\circ)\right]n_\kappa(M_\circ) - g_\lambda(M_\circ)t_{\lambda\kappa}\left[u_\rho(M_\circ)\right]n_\kappa(M_\circ)\right] \mathrm{d}s_{M_\circ} +$$
$$\oint_{\mathcal{L}_\varepsilon}\left[u_\lambda(M_\circ)t_{\lambda\kappa}\left[g_\rho(M_\circ)\right]n_\kappa(M_\circ) - g_\lambda(M_\circ)t_{\lambda\kappa}\left[u_\rho(M_\circ)\right]n_\kappa(M_\circ)\right] \mathrm{d}s_{M_\circ} +$$
$$\oint_{\mathcal{L}_R}\left[u_\lambda(M_\circ)t_{\lambda\kappa}\left[g_\rho(M_\circ)\right]n_\kappa(M_\circ) - g_\lambda(M_\circ)t_{\lambda\kappa}\left[u_\rho(M_\circ)\right]n_\kappa(M_\circ)\right] \mathrm{d}s_{M_\circ}, \tag{15}$$

in which M over a letter denotes derivatives taken with respect to the point coordinates M and $n_\kappa(M_\circ)$ is the outward normal, is the primal Somigliana identity applied to the triple connected region A'_e.

Let $g_\lambda(Q) = U_{\lambda\kappa}(M, Q)e_\kappa(Q)$, which is a non singular elastic state of the plane in A'_e. We regard $u_\lambda(M)$ as a different elastic state in the region A_e. Further we assume that $u_\lambda(M)$ has the far field pattern (asymptotic behaviour)

$$\tilde{u}_\kappa(M) = c_\kappa + \varepsilon_{3\rho\kappa} x_\rho \omega + e_{\kappa\beta}(\infty)x_\beta \tag{16}$$

as x_β or equivalently M tends to infinity. Here c_κ is a translation, ω is a rotation in finite, $c_\kappa + \varepsilon_{3\rho\kappa}x_\rho\omega$ is the corresponding rigid body motion, $e_{\kappa\beta}(\infty)$ is a constant strain tensor at infinity and $e_{\kappa\beta}(\infty)x_\beta$ is the corresponding displacement field.

The stresses induced by the strains $e_{\kappa\beta}(\infty)$ can be obtained by the Hooke law:

$$\begin{aligned} t_{11}(\infty) &= c_{11}e_{11}(\infty) + c_{12}e_{22}(\infty)\,, \\ t_{22}(\infty) &= c_{12}e_{11}(\infty) + c_{22}e_{22}(\infty)\,, \\ t_{12}(\infty) &= t_{21}(\infty) = 2c_{66}e_{12}(\infty)\,, \end{aligned} \tag{17}$$

Substituting the above quantities into the Somigliana identity, we obtain

$$\int_{A'_e}\left[u_\lambda(M)\left(\mu \overset{M}{\mathcal{D}}_{\lambda\sigma} U_{\sigma\kappa}(M,Q)\right) - \left(\mu \overset{M}{\mathcal{D}}_{\lambda\sigma} u_\sigma(M)\right) U_{\lambda\kappa}(M,Q)\right] \mathrm{d}A_M\, e_\kappa(Q) =$$
$$= \oint_{\mathcal{L}_o}\left[u_\lambda(M_\circ)T_{\lambda\kappa}(M_\circ,Q) - t_\lambda(M_\circ)U_{\lambda\kappa}(M_\circ,Q)\right] \mathrm{d}s_{M_\circ}\, e_\kappa(Q) +$$
$$\oint_{\mathcal{L}_\varepsilon}\left[u_\lambda(M_\circ)T_{\lambda\kappa}(M_\circ,Q) - t_\lambda(M_\circ)U_{\lambda\kappa}(M_\circ,Q)\right] \mathrm{d}s_{M_\circ}\, e_\kappa(Q)$$
$$\oint_{\mathcal{L}_R}\left[u_\lambda(M_\circ)T_{\lambda\kappa}(M_\circ,Q) - t_\lambda(M_\circ)U_{\lambda\kappa}(M_\circ,Q)\right] \mathrm{d}s_{M_\circ}\, e_\kappa(Q) \tag{18}$$

since $t_{\lambda\kappa}\left[u_\rho(M_\circ)\right]n_\kappa(M_\circ) = t_\lambda(M_\circ)$ is the stress on the contour and obviously

$$t_{\lambda\kappa}\left[g_\rho(M_\circ)\right]n_\kappa(M_\circ) = T_{\lambda\kappa}(M_\circ, Q)e_\kappa(Q).$$

In the sequel we shall assume that there are no body forces. This assumption has no effect on the result we will obtain.

It is clear that one can omit $e_\kappa(Q)$. Regarding the equation obtained by omitting $e_\kappa(Q)$, our goal is to compute its limit as $R_\varepsilon \to 0$ and ${}_eR \to \infty$. As is well known, the left hand side vanishes under the conditions detailed above(see [1]), and hence

$$\oint_{\mathcal{L}_o} \cdots + \lim_{R_\varepsilon \to 0} \oint_{\mathcal{L}_\varepsilon} \cdots =$$
$$u_\kappa(Q) + \oint_{\mathcal{L}_o}\left[u_\lambda(\overset{o}{M})T_{\lambda\kappa}(M_\circ,Q) - t_\lambda(M_\circ)U_{\lambda\kappa}(M_\circ,Q)\right] \mathrm{d}s_{M_\circ}\,.$$

Consequently

$$u_\kappa(Q) =$$
$$\lim_{{}_eR\to\infty} \oint_{\mathcal{L}_R}\left[t_\lambda(M_\circ)U_{\lambda\kappa}(M_\circ,Q) - u_\lambda(M_\circ)T_{\lambda\kappa}(M_\circ,Q)\right] \mathrm{d}s_{M_\circ} +$$
$$\oint_{\mathcal{L}_o}\left[t_\lambda(M_\circ)U_{\lambda\kappa}(M_\circ,Q) - u_\lambda(M_\circ)T_{\lambda\kappa}(M_\circ,Q)\right] \mathrm{d}s_{M_\circ}\,. \tag{19}$$

In order to establish the first Somigliana formula for the exterior region, we need to find the limit of the first integral on the right hand side.

4 Somigliana formulas modified for exterior regions

In this section our main objective is to prove that

$$I_\kappa =$$
$$\lim_{{}_eR\to\infty} \oint_{\mathcal{L}_R}\left[t_\lambda(M_\circ)U_{\lambda\kappa}(M_\circ,Q) - u_\lambda(M_\circ)T_{\lambda\kappa}(M_\circ,Q)\right] \mathrm{d}s_{M_\circ} =$$
$$c_\kappa + \varepsilon_{3\rho\kappa}\xi_\rho\omega + e_{\kappa\beta}(\infty)\xi_\beta = \tilde{u}_\kappa(Q)\,. \tag{20}$$

Using

$$x_\beta(\overset{o}{M}) = {}_eR\, n_\beta(\overset{o}{M})$$

we substitute $t_{\lambda\rho}(\infty)n_\rho(M_\circ)$ for $t_\lambda(M_\circ)$ and $c_\lambda + \varepsilon_{3\rho\lambda}x_\rho\omega + e_{\kappa\beta}(\infty)x_\beta$ for $u_\lambda(M_\circ)$. This implies

$$\begin{aligned} I_\kappa = \overset{(1)}{I_\kappa} + \overset{(2)}{I_\kappa} + \overset{(3)}{I_\kappa} + \overset{(4)}{I_\kappa} = \\ -\lim_{_eR\to\infty} \oint_{\mathcal{L}_R} c_\lambda T_{\lambda\kappa}(M_\circ, Q)\,\mathrm{d}s_{M_\circ} - \\ \lim_{_eR\to\infty} \varepsilon_{3\rho\lambda\,e}R\omega \oint_{\mathcal{L}_R} n_\rho(\overset{o}{M})T_{\lambda\kappa}(M_\circ, Q)\,\mathrm{d}s_{M_\circ} + \\ \lim_{_eR\to\infty} t_{\lambda\rho}(\infty) \oint_{\mathcal{L}_R} n_\rho(M_\circ)U_{\lambda\kappa}(M_\circ, Q)\,\mathrm{d}s_{M_\circ} - \\ \lim_{_eR\to\infty} e_{\lambda\beta}(\infty)_e R \oint_{\mathcal{L}_R} n_\beta(\overset{o}{M})T_{\lambda\kappa}(M_\circ, Q)\,\mathrm{d}s_{M_\circ}\,. \end{aligned} \tag{21}$$

Remark 1.: In accordance with Eq. (16), the stresses and strains are taken as constant quantities which are therefore independent of the arc coordinate s.

Since

$$\oint_{\mathcal{L}_R} T_{\lambda\kappa}(M_\circ, Q)\,\mathrm{d}s_{M_\circ} = -\delta_{\kappa\lambda}$$

and

$$\varepsilon_{3\rho\lambda} \oint_{\mathcal{L}_R} r_\rho T_{\lambda\kappa}(M_\circ, Q)\,\mathrm{d}s_{M_\circ} = 0$$

which is the moment about the origin of the stresses due to a unit force applied at Q, one can write

$$\begin{aligned} \overset{(1)}{I_\kappa} + \overset{(2)}{I_\kappa} = c_\kappa - \lim_{_eR\to\infty} \varepsilon_{3\rho\lambda}\omega \oint_{\mathcal{L}_R} (\xi_\rho + r_\rho)T_{\lambda\kappa}(M_\circ, Q)\,\mathrm{d}s_{M_\circ} = \\ c_\kappa - \lim_{_eR\to\infty}\left[\varepsilon_{3\rho\lambda}\omega\xi_\rho \oint_{\mathcal{L}_R} T_{\lambda\kappa}(M_\circ, Q)\,\mathrm{d}s_{M_\circ} + \right. \\ \left. \varepsilon_{3\rho\lambda}\omega \oint_{\mathcal{L}_R} r_\rho T_{\lambda\kappa}(M_\circ, Q)\,\mathrm{d}s_{M_\circ}\right] = c_\kappa + \varepsilon_{3\rho\kappa}\omega\xi_\rho \end{aligned} \tag{22}$$

which clearly shows that (22) reproduces the rigid body motion.

Determining the limits $\overset{(3)}{I_\kappa}$ and $\overset{(4)}{I}_\kappa$ requires long formal transformations. For this reason we confine ourselves to basic argument and the results of the most important steps.

First, we have to expand $U_{\lambda\kappa}$ and $T_{\lambda\kappa}$ into series in terms of $_eR$ to the power $1, 0, -1, -2$ etc. These transformations are based on the relations:

$$n_\alpha(\overset{o}{M}) = n_\alpha \tag{23a}$$

$$r_\alpha = x_\alpha(\overset{o}{M}) - \xi_\alpha(Q) = x_\alpha - \xi_\alpha = {}_eR\left(n_\alpha - \frac{\xi_\alpha}{_eR}\right), \tag{23b}$$

$$\begin{aligned} \rho_\alpha = \sqrt{\lambda_\alpha r_1^2 + r_2^2} \simeq \\ \simeq {}_eR\sqrt{\lambda_\alpha n_1^2 + n_2^2}\left\{1 - \frac{1}{_eR}\frac{\lambda_\alpha n_1\xi_1 + n_2\xi_2}{\lambda_\alpha n_1^2 + n_2^2} + \frac{1}{2_eR^2}\frac{\lambda_\alpha\xi_1^2 + \xi_2^2}{\lambda_\alpha n_1^2 + n_2^2}\right\}, \end{aligned} \tag{23c}$$

$$\begin{aligned} \ln\rho_\alpha \simeq \ln_e R + \frac{1}{2}\ln\left(\lambda_\alpha n_1^2 + n_2^2\right) - \\ \frac{1}{_eR}\frac{\lambda_\alpha n_1\xi_1 + n_2\xi_2}{\lambda_\alpha n_1^2 + n_2^2} + \frac{1}{2_eR^2}\frac{\lambda_\alpha\xi_1^2 + \xi_2^2}{\lambda_\alpha n_1^2 + n_2^2}, \end{aligned} \tag{23d}$$

$$\frac{1}{\rho_\alpha^2} \simeq \frac{1}{_eR^2\left(\lambda_\alpha n_1^2 + n_2^2\right)}\left(1 + \frac{2}{_eR}\frac{\lambda_\alpha n_1\xi_1 + n_2\xi_2}{\lambda_\alpha n_1^2 + n_2^2} - \frac{1}{_eR^2}\frac{\lambda_\alpha\xi_1^2 - \xi_2^2}{\lambda_\alpha n_1^2 + n_2^2}\right) \tag{23e}$$

and

$$\arctan(x + \varepsilon a) = \arctan x + \frac{\varepsilon a}{1 + x^2},$$

where a is constant and ε is a very small quantity.

Consequently

$$\begin{aligned} \arctan\frac{\left(\sqrt{\lambda_1} - \sqrt{\lambda_2}\right)r_1 r_2}{\sqrt{\lambda_1}\sqrt{\lambda_2}r_1^2 + r_2^2} \simeq \arctan\frac{\left(\sqrt{\lambda_1} - \sqrt{\lambda_2}\right)n_1 n_2}{\sqrt{\lambda_1\lambda_2}n_1^2 + n_2^2} + \\ \frac{\left(\sqrt{\lambda_1} - \sqrt{\lambda_2}\right)}{_eR}\frac{\left(\sqrt{\lambda_1\lambda_2}n_1^2 n_2 - n_2^3\right)\xi_1 - \left(\sqrt{\lambda_1\lambda_2}n_1^3 - n_1 n_2^2\right)\xi_2}{n_1^4\lambda_1\lambda_2 + n_2^4 + (\lambda_1 + \lambda_2)n_1^2 n_2^2}\,. \end{aligned} \tag{23f}$$

Using relations (23a),..., (23f) for a sufficiently large $_eR$ we have

$$\begin{aligned} U_{11}(M_\circ, Q) \simeq \\ D\left[\sqrt{\lambda_1}A_2^2\left(\ln_e R + \frac{1}{2}\ln\left(\lambda_1 n_1^2 + n_2^2\right) - \frac{1}{_eR}\frac{\lambda_1 n_1\xi_1 + n_2\xi_2}{\lambda_1 n_1^2 + n_2^2}\right) - \right. \\ \left. \sqrt{\lambda_2}A_1^2\left(\ln_e R + \frac{1}{2}\ln\left(\lambda_2 n_1^2 + n_2^2\right) - \frac{1}{_eR}\frac{\lambda_2 n_1\xi_1 + n_2\xi_2}{\lambda_2 n_1^2 + n_2^2}\right)\right], \end{aligned} \tag{24}$$

$$\begin{aligned} U_{22}(M_\circ, Q) \simeq \\ -D\left[\frac{A_1^2}{\sqrt{\lambda_1}}\left(\ln_e R + \frac{1}{2}\ln\left(\lambda_1 n_1^2 + n_2^2\right) - \frac{1}{_eR}\frac{\lambda_1 n_1\xi_1 + n_2\xi_2}{\lambda_1 n_1^2 + n_2^2}\right) - \right. \\ \left. \frac{A_2^2}{\sqrt{\lambda_2}}\left(\ln_e R + \frac{1}{2}\ln\left(\lambda_\alpha n_1^2 + n_2^2\right) - \frac{1}{_eR}\frac{\lambda_\alpha n_1\xi_1 + n_2\xi_2}{\lambda_\alpha n_1^2 + n_2^2}\right)\right], \end{aligned} \tag{25}$$

$$\begin{aligned} U_{12}(M_\circ, Q) \simeq DA_1A_2\left[\arctan\frac{\left(\sqrt{\lambda_1} - \sqrt{\lambda_2}\right)n_1 n_2}{\sqrt{\lambda_1\lambda_2}n_1^2 + n_2^2} + \frac{\left(\sqrt{\lambda_1} - \sqrt{\lambda_2}\right)}{_eR}\times \right. \\ \left. \frac{\left(\sqrt{\lambda_1\lambda_2}n_1^2 n_2 - n_2^3\right)\xi_1 - \left(\sqrt{\lambda_1\lambda_2}n_1^3 - n_1 n_2^2\right)\xi_2}{n_1^4\lambda_1\lambda_2 + n_2^4 + (\lambda_1 + \lambda_2)n_1^2 n_2^2}\right] \end{aligned} \tag{26}$$

and

$$\begin{aligned} T_{11}(M_\circ, Q) = \\ \frac{D}{_eR}\frac{\sqrt{\lambda_2}A_1}{\lambda_2 n_1^2 + n_2^2}\left[1 + \frac{2}{_eR}\frac{\lambda_2 n_1\xi_1 + n_2\xi_2}{\lambda_2 n_1^2 + n_2^2} - \frac{1}{_eR}(\xi_1 n_1 + \xi_2 n_2)\right] - \\ -\frac{D}{_eR}\frac{\sqrt{\lambda_1}A_2}{\lambda_1 n_1^2 + n_2^2}\left[1 + \frac{2}{_eR}\frac{\lambda_1 n_1\xi_1 + n_2\xi_2}{\lambda_1 n_1^2 + n_2^2} - \frac{1}{_eR}(\xi_1 n_1 + \xi_2 n_2)\right], \end{aligned} \tag{27}$$

$$\begin{aligned} T_{21}(M_\circ, Q) = \frac{D}{_eR}\frac{\sqrt{\lambda_1}A_2}{\lambda_1 n_1^2 + n_2^2}\times \\ \left[(\lambda_1 - 1)n_1 n_2\left(1 + \frac{2}{_eR}\frac{\lambda_1 n_1\xi_1 + n_2\xi_2}{\lambda_1 n_1^2 + n_2^2}\right) - \frac{1}{_eR}(\lambda_1\xi_1 n_2 - \xi_2 n_1)\right] - \\ \frac{D}{_eR}\frac{\sqrt{\lambda_2}A_1}{\lambda_2 n_1^2 + n_2^2}\times \\ \left[(\lambda_2 - 1)n_1 n_2\left(1 + \frac{2}{_eR}\frac{\lambda_2 n_1\xi_1 + n_2\xi_2}{\lambda_2 n_1^2 + n_2^2}\right) - \frac{1}{_eR}(\lambda_2\xi_1 n_2 - \xi_2 n_1)\right], \end{aligned} \tag{28}$$

$$\begin{aligned} T_{12}(M_\circ, Q) = \frac{D}{_eR}\frac{A_1\sqrt{\lambda_1}}{\lambda_1 n_1^2 + n_2^2}\times \\ \left[\left(1 - \frac{1}{\lambda_1}\right)n_1 n_2\left(1 + \frac{2}{_eR}\frac{\lambda_1 n_1\xi_1 + n_2\xi_2}{\lambda_1 n_1^2 + n_2^2}\right) - \frac{1}{_eR}\left(\xi_1 n_2 - \frac{1}{\lambda_1}\xi_2 n_1\right)\right]\times \\ -D\frac{A_2\sqrt{\lambda_2}}{\lambda_2 n_1^2 + n_2^2}\times \\ \left[\left(1 - \frac{1}{\lambda_2}\right)n_1 n_2\left(1 + \frac{2}{_eR}\frac{\lambda_1 n_1\xi_1 + n_2\xi_2}{\lambda_1 n_1^2 + n_2^2}\right) - \frac{1}{_eR}\left(\xi_1 n_2 - \frac{1}{\lambda_2}\xi_2 n_1\right)\right], \end{aligned} \tag{29}$$

$$T_{22}(M_\circ, Q) = \frac{D}{{}_eR}\frac{\sqrt{\lambda_1}A_1}{\lambda_1 n_1^2+n_2^2}\left[1+\frac{2}{{}_eR}\frac{\lambda_1 n_1\xi_1+n_2\xi_2}{\lambda_1 n_1^2+n_2^2}-\frac{1}{{}_eR}\left(\xi_1 n_1-\xi_2 n_2\right)\right] - \frac{D}{{}_eR}\frac{\sqrt{\lambda_2}A_2}{\lambda_2 n_1^2+n_2^2}\left[1+\frac{2}{{}_eR}\frac{\lambda_2 n_1\xi_1+n_2\xi_2}{\lambda_2 n_1^2+n_2^2}-\frac{1}{{}_eR}\left(\xi_1 n_1-\xi_2 n_2\right)\right]. \quad (30)$$

Knowledge of the series of $U_{\lambda\kappa}$ and $T_{\lambda\kappa}$ allows us to calculate the integrals $\overset{(3)}{I}_\kappa$ and $\overset{(4)}{I}_\kappa$. The following observations are used in the calculations:

1 The outward unit normal on $\mathcal{L}_R$ is given by equation

$$n_a = (\sin\varphi, \cos\varphi) \quad (31)$$

where φ is the polar angle.

2 The arc element on $\mathcal{L}_R$ admits the form

$$\mathrm{d}s_{\overset{o}{M}} = {}_eR\,\mathrm{d}\varphi\,. \quad (32)$$

3 As ${}_eR \to \infty$ the coefficient(s) of ${}_eR$ is (are) always an integral (integrals) of zero value. The integrals we have used are all listed in the Appendix.

4 The structure of the coefficients of ${}_eR$ to the power zero is similar; but these terms involve ξ_α and the trigonometric integrals that constitute these coefficients are not necessarily equal to zero.

After performing the integrations for $\overset{(3)}{I}_\kappa$ we have

$$\overset{(3)}{I}_\kappa = \lim_{{}_eR\to\infty} t_{\lambda\rho}(\infty)\oint_{\mathcal{L}_R} n_\rho(M_\circ)U_{\lambda\kappa}(M_\circ, Q)\,\mathrm{d}s_{M_\circ} = t_{11}(\infty)\left(\xi_1\overset{(31)}{I}_{\kappa 1}+\xi_2\overset{(31)}{I}_{\kappa 2}\right)+t_{12}(\infty)\left(\xi_1\overset{(32)}{I}_{\kappa 1}+\xi_2\overset{(32)}{I}_{\kappa 2}\right)+ t_{21}(\infty)\left(\xi_1\overset{(33)}{I}_{\kappa 1}+\xi_2\overset{(33)}{I}_{\kappa 2}\right)+t_{22}(\infty)\left(\xi_1\overset{(34)}{I}_{\kappa 1}+\xi_2\overset{(34)}{I}_{\kappa 2}\right), \quad (33)$$

where

$$\overset{(31)}{I}_{11} = 2\pi D\left\{A_1^2\frac{\lambda_2}{1+\sqrt{\lambda_2}}-A_2^2\frac{\lambda_1}{1+\sqrt{\lambda_1}}\right\}, \qquad \overset{(31)}{I}_{12}=0, \quad (34a)$$

$$\overset{(32)}{I}_{12} = 2\pi D\left\{A_1^2\frac{\sqrt{\lambda_2}}{1+\sqrt{\lambda_2}}-A_2^2\frac{\sqrt{\lambda_1}}{1+\sqrt{\lambda_1}}\right\}, \qquad \overset{(32)}{I}_{11}=0, \quad (34b)$$

$$\overset{(33)}{I}_{12} = -2\pi D A_1A_2\frac{\sqrt{\lambda_1}-\sqrt{\lambda_2}}{\left(\sqrt{\lambda_1}+1\right)\left(\sqrt{\lambda_2}+1\right)}, \qquad \overset{(33)}{I}_{11}=0, \quad (34c)$$

$$\overset{(34)}{I}_{11} = \overset{(33)}{I}_{12}, \qquad \overset{(34)}{I}_{12}=0, \quad (34d)$$

$$\overset{(31)}{I}_{22} = \overset{(33)}{I}_{12}, \qquad \overset{(31)}{I}_{21}=0, \quad (34e)$$

$$\overset{(32)}{I}_{21} = \overset{(34)}{I}_{11}, \qquad \overset{(32)}{I}_{22}=0, \quad (34f)$$

$$\overset{(33)}{I}_{21} = 2\pi D\left\{A_1^2\frac{1}{\sqrt{\lambda_1}+1}-A_2^2\frac{1}{\sqrt{\lambda_2}+1}\right\}, \qquad \overset{(33)}{I}_{22}=0, \quad (34g)$$

$$\overset{(34)}{I}_{22} = 2\pi D\left\{\frac{A_1^2}{\sqrt{\lambda_1}}\frac{1}{\sqrt{\lambda_1}+1}-\frac{A_2^2}{\sqrt{\lambda_2}}\frac{1}{\sqrt{\lambda_2}+1}\right\}, \qquad \overset{(34)}{I}_{21}=0. \quad (34h)$$

In the same way we obtain for $\overset{(4)}{I}_\kappa$ that

$$\overset{(4)}{I}_\kappa = \lim_{{}_eR\to\infty} e_{\lambda\beta}(\infty)\,{}_eR\oint_{\mathcal{L}_R} n_\beta(\overset{o}{M})T_{\lambda\kappa}(M_\circ, Q)\,\mathrm{d}s_{M_\circ} = e_{11}(\infty)\left(\xi_1\overset{(41)}{I}_{\kappa 1}+\xi_2\overset{(41)}{I}_{\kappa 2}\right)+e_{12}(\infty)\left(\xi_1\overset{(42)}{I}_{\kappa 1}+\xi_2\overset{(42)}{I}_{\kappa 2}\right)+ e_{21}(\infty)\left(\xi_1\overset{(43)}{I}_{\kappa 1}+\xi_2\overset{(43)}{I}_{\kappa 2}\right)+e_{22}(\infty)\left(\xi_1\overset{(44)}{I}_{\kappa 1}+\xi_2\overset{(44)}{I}_{\kappa 2}\right), \quad (35)$$

where

$$\overset{(41)}{I}_{11} = 2\pi D\left\{A_1\frac{\sqrt{\lambda_2}}{\sqrt{\lambda_2}+1}-A_2\frac{\sqrt{\lambda_1}}{\sqrt{\lambda_1}+1}\right\}, \qquad \overset{(41)}{I}_{12}=0, \quad (36a)$$

$$\overset{(42)}{I}_{12} = 2\pi D\left\{A_1\frac{\sqrt{\lambda_2}-1}{\lambda_2-1}-A_2\frac{\sqrt{\lambda_1}-1}{\lambda_1-1}\right\}, \qquad \overset{(42)}{I}_{11}=0, \quad (36b)$$

$$\overset{(43)}{I}_{12} = 2\pi D\left\{A_2\frac{\lambda_1-\sqrt{\lambda_1}}{(\lambda_1-1)}-A_1\frac{\lambda_2-\sqrt{\lambda_2}}{(\lambda_2-1)}\right\}, \qquad \overset{(43)}{I}_{11}=0, \quad (36c)$$

$$\overset{(44)}{I}_{11} = -2\pi D\left\{A_2\frac{\lambda_1(\sqrt{\lambda_1}-1)}{(\lambda_1-1)}-A_1\frac{\lambda_2(\sqrt{\lambda_2}-1)}{(\lambda_2-1)}\right\}, \qquad \overset{(44)}{I}_{12}=0, \quad (36d)$$

$$\overset{(41)}{I}_{22} = 2\pi D\left\{A_1\frac{\lambda_1-\sqrt{\lambda_1}}{\lambda_1(\lambda_1-1)}-A_2\frac{\lambda_2-\sqrt{\lambda_2}}{\lambda_2(\lambda_2-1)}\right\}, \qquad \overset{(41)}{I}_{21}=0, \quad (36e)$$

$$\overset{(42)}{I}_{21} = -2\pi D\left\{A_1\frac{\sqrt{\lambda_1}-1}{\lambda_1-1}-A_2\frac{\sqrt{\lambda_2}-1}{\lambda_2-1}\right\}, \qquad \overset{(42)}{I}_{22}=0, \quad (36f)$$

$$\overset{(43)}{I}_{21} = 2\pi D\left\{A_1\frac{\sqrt{\lambda_1}}{\sqrt{\lambda_1}+1}-A_2\frac{\sqrt{\lambda_2}}{\sqrt{\lambda_2}+1}\right\}, \qquad \overset{(43)}{I}_{22}=0, \quad (36g)$$

$$\overset{(44)}{I}_{22} = 2\pi D\left\{A_1\frac{\sqrt{\lambda_1}-1}{\lambda_1-1}-A_2\frac{\sqrt{\lambda_2}-1}{\lambda_2-1}\right\}, \qquad \overset{(44)}{I}_{21}=0. \quad (36h)$$

Substituting the integrals with zero value from Eqs. (34a),....,(34h) and (36a),....,(36h) into Eqs. (33) and (35) then utilizing the Hooke law (17), (3) we obtain

$$\overset{(3)}{I}_1+\overset{(4)}{I}_1 = \frac{s_{22}e_{11}(\infty)-s_{12}e_{22}(\infty)}{s_{11}s_{22}-s_{12}^2}\xi_1\overset{(31)}{I}_{11} + \frac{2}{s_{66}}e_{12}(\infty)\,\xi_2\left(\overset{(32)}{I}_{12}+\overset{(33)}{I}_{12}\right)+ \frac{s_{11}e_{22}(\infty)-s_{21}e_{11}(\infty)}{s_{11}s_{22}-s_{12}^2}\xi_1\overset{(34)}{I}_{11} - e_{11}(\infty)\left(\xi_1\overset{(41)}{I}_{11}+\xi_2\overset{(41)}{I}_{12}\right) - e_{12}(\infty)\left[\xi_1\left(\overset{(42)}{I}_{11}+\overset{(43)}{I}_{11}\right)+\xi_2\left(\overset{(42)}{I}_{12}+\overset{(43)}{I}_{12}\right)\right] - e_{22}(\infty)\left(\xi_1\overset{(44)}{I}_{11}+\xi_2\overset{(44)}{I}_{12}\right) \quad (37a)$$

and

$$\overset{(3)}{I}_2+\overset{(4)}{I}_2 = \frac{s_{22}e_{11}(\infty)-s_{12}e_{22}(\infty)}{s_{11}s_{22}-s_{12}^2}\xi_2\overset{(31)}{I}_{22} + \frac{2}{s_{66}}e_{12}(\infty)\,\xi_1\left(\overset{(32)}{I}_{21}+\overset{(33)}{I}_{21}\right)+ \frac{s_{11}e_{22}(\infty)-s_{21}e_{11}(\infty)}{s_{11}s_{22}-s_{12}^2}\xi_2\overset{(34)}{I}_{22} - e_{11}(\infty)\left(\xi_1\overset{(41)}{I}_{21}+\xi_2\overset{(41)}{I}_{22}\right) - e_{12}(\infty)\left[\xi_1\left(\overset{(42)}{I}_{21}+\overset{(43)}{I}_{21}\right)+\xi_2\left(\overset{(42)}{I}_{22}+\overset{(43)}{I}_{22}\right)\right] - e_{22}(\infty)\left(\xi_1\overset{(44)}{I}_{21}+\xi_2\overset{(44)}{I}_{22}\right). \quad (37b)$$

Using Eqs. (7) to (12), one can prove that the following equations hold (the formal calculations are presented in Appendix B.):

$$-\frac{s_{22}}{s_{12}^2-s_{11}s_{22}}\overset{(31)}{I}_{11}+\frac{s_{21}(\infty)}{s_{12}^2-s_{11}s_{22}}\overset{(34)}{I}_{11}-\overset{(41)}{I}_{11}=1\,,$$

$$\frac{2}{s_{66}}\left(\overset{(32)}{I}_{12}+\overset{(33)}{I}_{12}\right)-\overset{(42)}{I}_{12}-\overset{(43)}{I}_{12}=1\,, \quad (38a)$$

$$\frac{s_{12}}{s_{12}^2-s_{11}s_{22}}\overset{(31)}{I}_{11}-\frac{s_{11}}{s_{12}^2-s_{11}s_{22}}\overset{(34)}{I}_{11}-\overset{(44)}{I}_{11}=0$$

and

$$-\frac{s_{22}}{s_{12}^2-s_{11}s_{22}}\overset{(31)}{I}_{22}+\frac{s_{21}}{s_{12}^2-s_{11}s_{22}}\overset{(34)}{I}_{22}-\overset{(41)}{I}_{22}=0\,,$$
$$\frac{2}{s_{66}}\left(\overset{(32)}{I}_{21}+\overset{(33)}{I}_{21}\right)-\overset{(42)}{I}_{21}-\overset{(43)}{I}_{21}=1\,,\qquad(38b)$$
$$\frac{s_{12}}{s_{12}^2-s_{11}s_{22}}\overset{(31)}{I}_{22}-\frac{s_{11}}{s_{12}^2-s_{11}s_{22}}\overset{(34)}{I}_{22}-\overset{(44)}{I}_{22}=1\,.$$

If we substitute Eqs. (38a,b) into equtions (37a,b) we find that

$$\overset{(3)}{I}_\kappa+\overset{(4)}{I}_\kappa=e_{\kappa\beta}(\infty)\xi_\beta\,.\qquad(39)$$

Neglecting the rigid body motion, i.e., setting $\overset{(1)}{I}_\kappa+\overset{(2)}{I}_\kappa$ to zero and utilizing (39) we obtain

$$I_\kappa=I_\kappa=\overset{(1)}{I}_\kappa+\overset{(2)}{I}_\kappa+\overset{(3)}{I}_\kappa+\overset{(4)}{I}_\kappa=e_{\kappa\beta}(\infty)\xi_\beta\,.$$

Then the first and modified Somigliana formula immediately follow from Eqs. (19) and (20):

$$u_\kappa(Q)=e_{\kappa\beta}(\infty)\xi_\beta(Q)+\oint_{\mathcal{L}_o}[t_\lambda(M_\circ)U_{\lambda\kappa}(M_\circ,Q)-$$
$$u_\lambda(M_\circ)T_{\lambda\kappa}(M_\circ,Q)]\,\mathrm{d}s_{M_\circ}\quad Q\in A_e\qquad(40)$$

If $Q=Q_\circ$ is on $\mathcal{L}_o$ nothing changes concerning the limit of the integral taken on $\mathcal{L}_R$. Consequently

$$C_{\kappa\rho}u_\rho(Q_\circ)=e_{\kappa\beta}(\infty)\xi_\beta(Q_\circ)+$$
$$\oint_{\mathcal{L}_o}[t_\lambda(M_\circ)U_{\lambda\kappa}(M_\circ,Q_\circ)-u_\lambda(M_\circ)T_{\lambda\kappa}(M_\circ,Q_\circ)]\,\mathrm{d}s_{M_\circ}$$
$$Q=Q_\circ\in\mathcal{L}_o\qquad(41)$$

where $C_{\kappa\rho}=\delta_{\kappa\rho}/2$ if the contour is smooth at $Q_\circ$. This integral equation is that of the direct method (or the second Somigliana formula for exterior regions).

If Q is inside the contour $\mathcal{L}_o$ – this region is referred to as A_i – then it is easy to show that

$$0=e_{\kappa\beta}(\infty)\xi_\beta(Q)+$$
$$\oint_{\mathcal{L}_o}[t_\lambda(M_\circ)U_{\lambda\kappa}(M_\circ,Q)-u_\lambda(M_\circ)T_{\lambda\kappa}(M_\circ,Q)]\,\mathrm{d}s_{M_\circ}$$
$$Q=Q_\circ\in A_i\qquad(42)$$

which is the third Somigliana formula for exterior regions.

Using the formulae set up for the strains in Appendix C and the Hooke law (17) we can calculate the stresses:

$$t_{\alpha\beta}(Q)=t_{\alpha\beta}(\infty)+\oint_{\mathcal{L}_o}t_\lambda(M_\circ)\hat{\mathcal{D}}_{\lambda\alpha\beta}(M_\circ,Q)\,\mathrm{d}s_{M_\circ}-$$
$$\oint_{\mathcal{L}_o}u_\lambda(M_\circ)\mathcal{S}_{\lambda\alpha\beta}(M_\circ,Q)\,\mathrm{d}s_{M_\circ}\,Q\in A_e\qquad(43)$$

where

$$\hat{\mathcal{D}}_{\lambda11}=c_{11}\mathcal{D}_{\lambda11}+c_{12}\mathcal{D}_{\lambda22}\,,\qquad\hat{\mathcal{S}}_{\lambda11}=c_{11}\mathcal{S}_{\lambda11}+c_{12}\mathcal{S}_{\lambda22}\,,$$
$$\hat{\mathcal{D}}_{\lambda11}=c_{12}\mathcal{D}_{\lambda11}+c_{22}\mathcal{D}_{\lambda22}\,,\qquad\hat{\mathcal{S}}_{\lambda11}=c_{12}\mathcal{S}_{\lambda11}+c_{22}\mathcal{S}_{\lambda22}\,,$$
$$\hat{\mathcal{D}}_{\lambda12}=2c_{66}\mathcal{D}_{\lambda12}=\hat{\mathcal{D}}_{\lambda21}\,,\qquad\hat{\mathcal{S}}_{\lambda12}=2c_{66}\mathcal{S}_{\lambda12}=\hat{\mathcal{S}}_{\lambda21}\,.$$
$$(44)$$

5 Behaviour at infinity

Our goal in this section is to compute the limit of representation (40) as $Q\to\infty$. This will lead to a characterization of the asymptotic behaviour of $u_\kappa(Q)$. If this behaviour is the same what we have assumed, i.e., if the limit coincides with (16) provided that in the latter the rigid body motion is neglected, then we confirm that the results of the previous section are correct. It is clear from representation (40) that it is sufficient to show that the following relations hold

$$\lim_{Q(\xi_1,\xi_2)\to\infty}\oint_{\mathcal{L}_o}t_\lambda(M_\circ)U_{\lambda\kappa}(M_\circ,Q)\mathrm{d}s_{M_\circ}=0\,,$$
$$\lim_{Q(\xi_1,\xi_2)\to\infty}\oint_{\mathcal{L}_o}u_\lambda(M_\circ)T_{\lambda\kappa}(M_\circ,Q)\mathrm{d}s_{M_\circ}=0\,.\qquad(45)$$

In order to find the limit of the above integrals we have to set up asymptotic relations for the fundamental solutions $U_{\lambda\kappa}(M_\circ,Q)$ and $T_{\lambda\kappa}(M_\circ,Q)$ if $Q\to\infty$.

Using the notations introduced in Fig. 2 as well as Eqs. (6) and (10) we have

$$r_\alpha(\overset{o}{M},Q)=x_\alpha(\overset{o}{M})-\xi_\alpha(Q)=x_\alpha-\xi_\alpha=\qquad(46a)$$
$$-\hat{R}\left(\hat{n}_\alpha-\frac{x_\alpha}{\hat{R}}\right)\approx-\hat{R}\,\hat{n}_\alpha\quad|\hat{n}_\alpha|=1\,,\qquad(46b)$$
$$\rho_\alpha=\sqrt{\lambda_\alpha r_1^2+r_2^2}\approx\hat{R}\sqrt{\lambda_\alpha\hat{n}_1^2+\hat{n}_2^2}\,,\qquad(46c)$$
$$\ln\rho_\alpha\approx\ln\hat{R}+\frac{1}{2}\ln\left(\lambda_\alpha\hat{n}_1^2+\hat{n}_2^2\right)\,.\qquad(46d)$$

Substituting equations (46b,b,c) into (13)and then performing

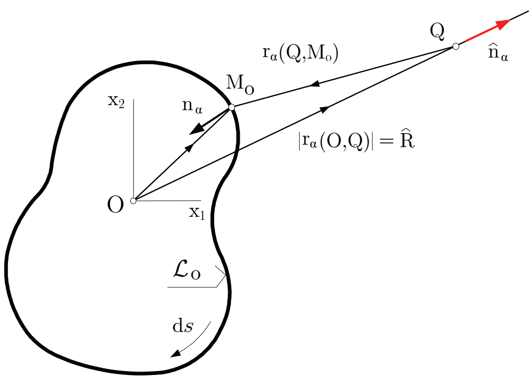

Fig. 2.

some manipulations, we obtain the following asymptotic relations for the fundamental solution of order one:

$$U_{11}(M_\circ,Q)\approx D\left[\sqrt{\lambda_1}A_2^2-\sqrt{\lambda_2}A_1^2\right]\ln\hat{R}\,,\qquad(47a)$$

$$U_{12}(M_\circ,Q)=U_{21}(M_\circ,Q)\approx$$
$$DA_1A_2\arctan\frac{\left(\sqrt{\lambda_1}-\sqrt{\lambda_2}\right)\hat{n}_1\hat{n}_2}{\sqrt{\lambda_1}\sqrt{\lambda_2}\hat{n}_1^2+\hat{n}_2^2}\,,\qquad(47b)$$

$$U_{22}(M_\circ,Q)\approx-D\left[\frac{A_1^2}{\sqrt{\lambda_1}}-\frac{A_2^2}{\sqrt{\lambda_2}}\right]\ln\hat{R}\,.\qquad(47c)$$

It is obvious that asymptotically $U_{\lambda\kappa}(M_\circ,Q)\approx U_{\lambda\kappa}(Q)$, i.e., the kernel in integral $(45)_1$ is independent of $M_\circ$.

Consequently

$$\lim_{Q(\xi_1,\xi_2)\to\infty} \oint_{\mathcal{L}_o} t_\lambda(M_\circ) U_{\lambda\kappa}(M_\circ, Q)\mathrm{d}s_{M_\circ} =$$
$$= \lim_{Q(\xi_1,\xi_2)\to\infty} U_{\lambda\kappa}(Q) \underbrace{\oint_{\mathcal{L}_o} t_\lambda(M_\circ)\, \mathrm{d}s_{M_\circ}}_{\text{resultant}} = 0\,. \quad (48)$$

By repeating the line of thought leading to the asymptotic relations (47a,b,c), for the fundamental solutions of order two we obtain

$$T_{11}(M_\circ, Q) \approx \quad (49a)$$
$$-\frac{D}{\hat{R}}\left(\frac{\sqrt{\lambda_2}A_1}{\lambda_2\hat{n}_1^2+\hat{n}_2^2} - \frac{\sqrt{\lambda_1}A_2}{\lambda_1\hat{n}_1^2+\hat{n}_2^2}\right) = \frac{\tilde{T}_{11}(Q)}{\hat{R}}\,,$$
$$T_{12}(M_\circ, Q) \approx \quad (49b)$$
$$-\frac{D}{\hat{R}}\left[\frac{1-\lambda_2}{\sqrt{\lambda_2}}\frac{A_2}{\lambda_2\hat{n}_1^2+\hat{n}_2^2} - \frac{1-\lambda_1}{\sqrt{\lambda_1}}\frac{A_1}{\lambda_1\hat{n}_1^2+\hat{n}_2^2}\right]\hat{n}_1\hat{n}_2 = \frac{\tilde{T}_{12}(Q)}{\hat{R}}\,,$$
$$T_{21}(M_\circ, Q) \approx \quad (49c)$$
$$-\frac{D}{\hat{R}}\left[\frac{\sqrt{\lambda_1}\,(\lambda_1-1)\,A_2}{\lambda_1\hat{n}_1^2+\hat{n}_2^2} - \frac{\sqrt{\lambda_2}\,(\lambda_2-1)\,A_1}{\lambda_2\hat{n}_1^2+\hat{n}_2^2}\right]\hat{n}_1\hat{n}_2 = \frac{\tilde{T}_{21}(Q)}{\hat{R}}\,,$$
$$T_{22}(M_\circ, Q) \approx \quad (49d)$$
$$-\frac{D}{\hat{R}}\left(\frac{\sqrt{\lambda_2}A_1}{\lambda_1\hat{n}_1^2+\hat{n}_2^2} - \frac{\sqrt{\lambda_1}A_2}{\lambda_2\hat{n}_1^2+\hat{n}_2^2}\right) = \frac{\tilde{T}_{22}(Q)}{\hat{R}}\,,$$

where $\tilde{T}_{\lambda\kappa} = \tilde{T}_{\lambda\kappa}(Q)$ is defined by the above equations. Since for large $\hat{R}$ the kernel $T_{\lambda\kappa}(M_\circ, Q)$ is independent of $M_\circ$ and tends to zero, it follows that

$$\lim_{Q(\xi_1,\xi_2)\to\infty} \oint_{\mathcal{L}_o} u_\lambda(M_\circ) T_{\lambda\kappa}(M_\circ, Q)\mathrm{d}s_{M_\circ} =$$
$$= \lim_{Q(\xi_1,\xi_2)\to\infty} \frac{\tilde{T}_{\lambda\kappa}(Q)}{\hat{R}} \oint_{\mathcal{L}_o} u_\lambda(M_\circ)\,\mathrm{d}s_{M_\circ} = 0\,. \quad (50)$$

This verifies that the asymptotic behaviour of the displacement representation (40) is as expected.

6 Concluding remarks

For the sake of completeness, note that Constanda [3] gives an asymptotic expansion for the displacements at infinity which ensures the validity of the Betti formula for exterior regions and isotropic bodies. Under this condition the total strain energy stored in the region is bounded. In addition uniqueness and existence proofs are easy to give.

We have modified the Somigliana formulas for exterior regions of orthotropic bodies by assuming that the strains are constants and accordingly the displacements are linear at infinity. Under this condition the strain energy density is bounded (although the strain energy is not), and there is no need to replace the exterior region by a finite one if a constant stress condition is prescribed at infinity. This can be an advantage if one considers an infinite plane with holes or cracks subjected to constant stresses at infinity, and an attempt is made to determine the stresses in finite. Existing codes can be modified easily to perform computations.

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A Trigonometric integrals

When determining the integrals $\overset{(3)}{I}_\kappa + \overset{(4)}{I}_\kappa$ we have used the following trigonometric integrals:

$$\frac{1}{_eR}\oint_{\mathcal{L}_R} n_1 \ln_e R\,\mathrm{d}s = \ln\,_eR\int_0^{2\pi}\cos\psi\,\mathrm{d}\psi = 0, \quad \frac{1}{_eR}\oint_{\mathcal{L}_R} n_2 \ln\,_eR\,\mathrm{d}s = \ln_e R\int_0^{2\pi}\sin\psi\,\mathrm{d}\psi = 0\,, \tag{51}$$

$$\frac{1}{_eR}\oint_{\mathcal{L}_R} n_1 \ln\left[\lambda_\alpha n_1^2 + n_2^2\right]\mathrm{d}s = \int_0^{2\pi}(\cos\psi)\ln\left[\lambda_\alpha\cos^2\psi + \sin^2\psi\right]\mathrm{d}\psi = 0\,, \tag{52}$$

$$\frac{1}{_eR}\oint_{\mathcal{L}_R} n_2 \ln\left[\lambda_\alpha n_1^2 + n_2^2\right]\mathrm{d}s = \int_0^{2\pi}(\sin\psi)\ln\left[\lambda_\alpha\cos^2\psi + \sin^2\psi\right]\mathrm{d}\psi = 0\,, \tag{53}$$

$$\frac{1}{_eR}\oint_{\mathcal{L}_R} \frac{n_1^2}{\lambda_\alpha n_1^2 + n_2^2}\mathrm{d}s = \int_0^{2\pi}\frac{\cos^2\psi}{\lambda_\alpha\cos^2\psi + \sin^2\psi}\mathrm{d}\psi = \frac{2\pi}{\lambda_\alpha + \sqrt{\lambda_\alpha}}\,, \tag{54}$$

$$\frac{1}{_eR}\oint_{\mathcal{L}_R} \frac{n_1 n_2}{\lambda_\alpha n_1^2 + n_2^2}\mathrm{d}s = \int_0^{2\pi}\frac{\cos\psi\sin\psi}{\lambda_\alpha\cos^2\psi + \sin^2\psi}\mathrm{d}\psi = 0\,, \tag{55}$$

$$\frac{1}{_eR}\oint_{\mathcal{L}_R} \frac{n_2^2}{\lambda_\alpha n_1^2 + n_2^2}\mathrm{d}s = \int_0^{2\pi}\frac{\sin^2\psi}{\lambda_\alpha\cos^2\psi + \sin^2\psi}\mathrm{d}\psi = \frac{2\pi}{\sqrt{\lambda_a} + 1}\,, \tag{56}$$

$$\frac{1}{_eR}\oint_{\mathcal{L}_R} \frac{n_1^3 n_2}{\left(\lambda_1 n_1^2 + n_2^2\right)\left(\lambda_1 n_1^2 + n_2^2\right)}\mathrm{d}s = \int_0^{2\pi}\frac{\cos^3\psi\,\sin\psi}{\left(\lambda_1\cos^2\psi + \sin^2\psi\right)\left(\lambda_2\cos^2\psi + \sin^2\psi\right)}\mathrm{d}\psi = 0\,, \tag{57}$$

$$\frac{1}{_eR}\oint_{\mathcal{L}_R} \frac{n_1 n_2^3}{\left(\lambda_1 n_1^2 + n_2^2\right)\left(\lambda_1 n_1^2 + n_2^2\right)}\mathrm{d}s = \int_0^{2\pi}\frac{\cos\psi\,\sin^3\psi}{\left(\lambda_1\cos^2\psi + \sin^2\psi\right)\left(\lambda_2\cos^2\psi + \sin^2\psi\right)}\mathrm{d}\psi = 0\,, \tag{58}$$

$$\begin{aligned}&\frac{1}{_eR}\oint_{\mathcal{L}_R} \frac{n_1^4}{\left(\lambda_1 n_1^2 + n_2^2\right)\left(\lambda_2 n_1^2 + n_2^2\right)}\mathrm{d}s = \\ &-\int_0^{2\pi}\frac{\cos^4\psi}{\left(\lambda_1\cos^2\psi + \sin^2\psi\right)\left(\lambda_2\cos^2\psi + \sin^2\psi\right)}\mathrm{d}\psi = 2\,\pi\,\frac{\sqrt{\lambda_1} + \sqrt{\lambda_2} + 1}{\sqrt{\lambda_1\lambda_2}\left(\sqrt{\lambda_1} + \sqrt{\lambda_2}\right)\left(\sqrt{\lambda_1} + 1\right)\left(\sqrt{\lambda_2} + 1\right)}\,,\end{aligned} \tag{59}$$

$$\begin{aligned}&\frac{1}{_eR}\oint_{\mathcal{L}_R} \frac{n_1^2 n_2^2}{\left(\lambda_1 n_1^2 + n_2^2\right)\left(\lambda_2 n_1^2 + n_2^2\right)}\mathrm{d}s = \\ &\int_0^{2\pi}\frac{\cos^2\psi\,\sin^2\psi}{\left(\lambda_1\cos^2\psi + \sin^2\psi\right)\left(\lambda_2\cos^2\psi + \sin^2\psi\right)}\mathrm{d}\psi = 2\pi\,\frac{1}{\left(\sqrt{\lambda_1} + \sqrt{\lambda_2}\right)\left(\sqrt{\lambda_1} + 1\right)\left(\sqrt{\lambda_2} + 1\right)}\,,\end{aligned} \tag{60}$$

$$\begin{aligned}&\frac{1}{_eR}\oint_{\mathcal{L}_R} \frac{n_2^4}{n_1^4\lambda_1\lambda_2 + n_2^4 + (\lambda_1 + \lambda_2)\,n_1^2 n_2^2}\mathrm{d}s = \\ &\int_0^{2\pi}\frac{\sin^4\psi}{\left(\lambda_1\cos^2\psi + \sin^2\psi\right)\left(\lambda_2\cos^2\psi + \sin^2\psi\right)}\mathrm{d}\psi = 2\,\pi\,\frac{\sqrt{\lambda_1} + \sqrt{\lambda_2} + \sqrt{\lambda_1}\sqrt{\lambda_2}}{\left(\sqrt{\lambda_1} + \sqrt{\lambda_2}\right)\left(\sqrt{\lambda_1} + 1\right)\left(\sqrt{\lambda_2} + 1\right)}\,.\end{aligned} \tag{61}$$

The integrals detailed above can be checked either by using Maple 9.5 or the Table of Integrals [5, 6] by Gradstein and Ryzhik.

B Proof of equation (38a)

B.1. First consider equation $(38\text{a})_1$. Substituting the integrals $\overset{(31)}{I}_{11}$, $\overset{(34)}{I}_{11}$ and $\overset{(41)}{I}_{11}$ – see equations (34a), (34d) and (36a) – we have

$$\begin{aligned}&-\frac{2\pi}{s_{12}^2 - s_{11}s_{22}}\left\{s_{22}D\left[\frac{A_1^2\lambda_2}{1 + \sqrt{\lambda_2}} - \frac{A_2^2\lambda_1}{1 + \sqrt{\lambda_1}}\right] + s_{21}DA_1A_2\frac{\sqrt{\lambda_1} - \sqrt{\lambda_2}}{\left(\sqrt{\lambda_1} + 1\right)\left(\sqrt{\lambda_2} + 1\right)}\right\} - \\ &\qquad - 2\pi\,DA_1\frac{\sqrt{\lambda_2}}{\sqrt{\lambda_2} + 1} + 2\pi\,DA_2\frac{\sqrt{\lambda_1}}{\sqrt{\lambda_1} + 1} - 1 = 0\,.\end{aligned} \tag{62}$$

After multiplying throughout by the common denominator $(s_{12}^2 - s_{11}s_{22})(\lambda_1 - 1)(\lambda_2 - 1)$ and some further manipulations we obtain

$$\Big\{s_{22}\Big[A_1^2\lambda_2\Big(\sqrt{\lambda_1}+1\Big) - A_2^2\lambda_1\Big(\sqrt{\lambda_2}+1\Big)\Big] + s_{21}\,A_1A_2\Big(\sqrt{\lambda_1}-\sqrt{\lambda_2}\Big) + \Big(s_{12}^2 - s_{11}\,s_{22}\Big)\Big[A_1\sqrt{\lambda_2}\Big(\sqrt{\lambda_1}+1\Big) - A_2\sqrt{\lambda_1}\Big(\sqrt{\lambda_2}+1\Big)\Big]\Big\}\Big(\sqrt{\lambda_1}-1\Big)\Big(\sqrt{\lambda_2}-1\Big) + (\lambda_1-\lambda_2)\,s_{22}\Big(s_{12}^2 - s_{11}\,s_{22}\Big)(\lambda_1-1)(\lambda_2-1) = 0\,. \quad (63)$$

If we divide throughout by $\big(\sqrt{\lambda_1}-1\big)\big(\sqrt{\lambda_2}-1\big)$ and substitute (9) then we find

$$-(s_{12}-\lambda_1 s_{22})\sqrt{\lambda_2}\Big(\sqrt{\lambda_1}+1\Big)\Big[(s_{12}-\lambda_1 s_{22})s_{22}\sqrt{\lambda_2} + \Big(s_{12}^2 - s_{11}s_{22}\Big)\Big] + (s_{12}-\lambda_2 s_{22})\sqrt{\lambda_1}\Big(\sqrt{\lambda_2}+1\Big)\Big[(s_{12}-\lambda_2 s_{22})s_{22}\sqrt{\lambda_1} + \Big(s_{12}^2 - s_{11}s_{22}\Big)\Big] - (\lambda_1-\lambda_2)\,s_{22}\Big(s_{12}^2 - s_{11}\,s_{22}\Big)\Big(\sqrt{\lambda_1}+1\Big)\Big(\sqrt{\lambda_2}+1\Big) - s_{21}(s_{12}-\lambda_1 s_{22})(s_{12}-\lambda_2 s_{22})\Big(\sqrt{\lambda_1}-\sqrt{\lambda_2}\Big) = 0\,. \quad (64)$$

Here we can again divide throughout by $\sqrt{\lambda_1}-\sqrt{\lambda_2}$. Hence

$$-\lambda_1^{\frac{3}{2}}\lambda_2^{\frac{3}{2}}s_{22}^3 + \lambda_1 s_{11}s_{22}^2 + \lambda_2 s_{11}s_{22}^2 - \lambda_1\lambda_2^2 s_{22}^3 - \lambda_1^2\lambda_2 s_{22}^3 - \lambda_1\lambda_2^{\frac{3}{2}}s_{22}^3 - \lambda_1^{\frac{3}{2}}\lambda_2 s_{22}^3 + \sqrt{\lambda_1}s_{11}s_{22}^2 + \sqrt{\lambda_2}s_{11}s_{22}^2 - s_{11}s_{12}s_{22} + \sqrt{\lambda_1}\sqrt{\lambda_2}s_{11}s_{22}^2 + 2\lambda_1\lambda_2 s_{12}s_{22}^2 - \lambda_1\lambda_2 s_{21}s_{22}^2 = 0 \quad (65)$$

where

$$\lambda_1 s_{11}s_{22}^2 + \lambda_2 s_{11}s_{22}^2 = s_{22}^2 s_{11}(\lambda_1+\lambda_2) = s_{22}^2 s_{11}\frac{2s_{12}+s_{66}}{s_{22}} = (2s_{12}+s_{66})\,s_{22}s_{11}$$

and

$$-\lambda_1\lambda_2^2 s_{22}^3 - \lambda_1^2\lambda_2 s_{22}^3 = -\lambda_2\lambda_1 s_{22}^3(\lambda_1+\lambda_2) = -\frac{s_{11}}{s_{22}}\,s_{22}^3\,\frac{2s_{12}+s_{66}}{s_{22}} = -(2s_{12}+s_{66})\,s_{22}s_{11}\,.$$

Consequently

$$-\lambda_1^{\frac{3}{2}}\lambda_2^{\frac{3}{2}}s_{22}^3 - \lambda_1\lambda_2^{\frac{3}{2}}s_{22}^3 - \lambda_1^{\frac{3}{2}}\lambda_2 s_{22}^3 + \sqrt{\lambda_1}s_{11}s_{22}^2 + \sqrt{\lambda_2}s_{11}s_{22}^2 - s_{11}s_{12}s_{22} + \sqrt{\lambda_1}\sqrt{\lambda_2}s_{11}s_{22}^2 + \lambda_1\lambda_2 s_{12}s_{22}^2 = 0\,.$$

Here

$$-\lambda_1\lambda_2^{\frac{3}{2}}s_{22}^3 - \lambda_1^{\frac{3}{2}}\lambda_2 s_{22}^3 = -s_{22}^3\Big(\lambda_1\lambda_2^{\frac{3}{2}} + \lambda_1^{\frac{3}{2}}\lambda_2\Big) = -s_{22}^3\lambda_1\lambda_2(\sqrt{\lambda_1}+\sqrt{\lambda_2}) = -s_{22}^3\frac{s_{11}}{s_{22}}(\sqrt{\lambda_1}+\sqrt{\lambda_2}) = -s_{22}^2 s_{11}\Big(\sqrt{\lambda_1}+\sqrt{\lambda_2}\Big)$$

and

$$\sqrt{\lambda_1}s_{11}s_{22}^2 + \sqrt{\lambda_2}s_{11}s_{22}^2 = s_{22}^2 s_{11}\Big(\sqrt{\lambda_1}+\sqrt{\lambda_2}\Big)\,.$$

Therefore

$$-\lambda_1^{\frac{3}{2}}\lambda_2^{\frac{3}{2}}s_{22}^3 - s_{11}s_{12}s_{22} + \sqrt{\lambda_1}\sqrt{\lambda_2}s_{11}s_{22}^2 + \lambda_1\lambda_2 s_{12}s_{22}^2 = -\lambda_1^{\frac{3}{2}}\lambda_2^{\frac{3}{2}}s_{22}^3 + s_{22}^2\Big(s_{12}\lambda_1\lambda_2 - s_{22}\lambda_1^{\frac{3}{2}}\lambda_2^{\frac{3}{2}}\Big) + \lambda_1\lambda_2 s_{12}s_{22}^2 =$$
$$s_{22}s_{11}\Big(s_{22}\sqrt{\lambda_1}\sqrt{\lambda_2} - s_{12}\Big) - \lambda_1^{\frac{3}{2}}\lambda_2^{\frac{3}{2}}s_{22}^3 + \lambda_1\lambda_2 s_{12}s_{22}^2 = s_{22}s_{11}\Big(s_{22}\sqrt{\lambda_1}\sqrt{\lambda_2} - s_{12}\Big) - s_{22}s_{11}\Big(s_{22}\sqrt{\lambda_1}\sqrt{\lambda_2} - s_{12}\Big) = 0\,.$$

B.2. Consider equation $(38a)_2$. If we substitute the integrals $\overset{(32)}{I}_{12}$, $\overset{(33)}{I}_{12}$, $\overset{(42)}{I}_{12}$ and $\overset{(43)}{I}_{12}$ – see equations (34b), (34c) (36b) and (36c) – and perform some manipulations we obtain

$$\frac{4\pi D}{s_{66}}\left[A_1^2\frac{\sqrt{\lambda_2}}{\sqrt{\lambda_2}+1} - A_2^2\frac{\sqrt{\lambda_1}}{\sqrt{\lambda_1}+1} - A_1A_2\frac{\sqrt{\lambda_1}-\sqrt{\lambda_2}}{\big(\sqrt{\lambda_1}+1\big)\big(\sqrt{\lambda_2}+1\big)} + A_1\frac{\sqrt{\lambda_2}-1}{\sqrt{\lambda_2}+1} - A_2\frac{\sqrt{\lambda_1}-1}{\sqrt{\lambda_1}+1}\right] - 1 = 0\,. \quad (66)$$

After substituting D and multiplying throughout by the common denominator

$$s_{22}s_{66}(\lambda_1-\lambda_2)(\sqrt{\lambda_1}+1)(\sqrt{\lambda_2}+1)$$

we have

$$2\Big[A_1^2\sqrt{\lambda_2}\Big(\sqrt{\lambda_1}+1\Big) - A_2^2\sqrt{\lambda_1}\Big(\sqrt{\lambda_2}+1\Big) - A_1A_2\Big(\sqrt{\lambda_1}-\sqrt{\lambda_2}\Big)\Big] - s_{66}\Big[A_1(\sqrt{\lambda_1}+1)\Big(1-\sqrt{\lambda_2}\Big) - A_2(\sqrt{\lambda_2}+1)(1-\sqrt{\lambda_1})\Big] - s_{22}s_{66}(\lambda_1-\lambda_2)(\sqrt{\lambda_1}+1)(\sqrt{\lambda_2}+1) = 0\,. \quad (67)$$

If we make use of equation (9) and perform some transformations we find

$$(s_{12}-\lambda_1 s_{22})\left(\sqrt{\lambda_1}+1\right)\left[2(s_{12}-\lambda_1 s_{22})\sqrt{\lambda_2}-s_{66}\left(1-\sqrt{\lambda_2}\right)\right]-$$
$$(s_{12}-\lambda_2 s_{22})\left(\sqrt{\lambda_2}+1\right)\left[2(s_{12}-\lambda_2 s_{22})\sqrt{\lambda_1}+s_{66}\left(\sqrt{\lambda_1}-1\right)\right]+2(s_{12}-\lambda_1 s_{22})(s_{12}-\lambda_2 s_{22})\left(\sqrt{\lambda_1}-\sqrt{\lambda_2}\right)-$$
$$s_{22}s_{66}\,(\lambda_1-\lambda_2)\,(\sqrt{\lambda_1}+1)(\sqrt{\lambda_2}+1)=0\,.$$

If we factor the above expression we have

$$\left(\sqrt{\lambda_1}-\sqrt{\lambda_2}\right)\left[s_{12}+s_{22}\sqrt{\lambda_1}\sqrt{\lambda_2}\left(1+\sqrt{\lambda_1}+\sqrt{\lambda_2}\right)\right][s_{22}\,(\lambda_1+\lambda_2)-2s_{12}-s_{66}]=0\,.$$

The first factor is obviously different from zero. Hence we should investigate the other two. We shall start with the last one:

$$s_{22}(\lambda_1+\lambda_2)-2s_{12}-s_{66}=-2s_{12}-s_{66}+s_{22}\frac{(2s_{12}+s_{66})}{s_{22}}=-2s_{12}-s_{66}+2s_{12}+s_{66}=0\,.$$

Since the above expression is zero equation $(38a)_2$ is truly satisfied.

B.3. The third equation to consider is $(38a)_3$. If we substitute the integrals $\overset{(31)}{I_{11}}$, $\overset{(34)}{I_{11}}$ and $\overset{(41)}{I_{11}}$ - see equations (34a), (34d) and (36d) – we obtain

$$\frac{2\pi D}{s_{12}^2-s_{11}s_{22}}\left[s_{12}\left(A_1^2\frac{\lambda_2}{\sqrt{\lambda_2}+1}-A_2^2\frac{\lambda_1}{\sqrt{\lambda_1}+1}\right)+s_{11}A_1A_2\frac{\sqrt{\lambda_1}-\sqrt{\lambda_2}}{\left(\sqrt{\lambda_1}+1\right)\left(\sqrt{\lambda_2}+1\right)}\right]+$$
$$2\pi D\left(\frac{\lambda_1 A_2}{\left(\sqrt{\lambda_1}+1\right)}-\frac{\lambda_2 A_1}{\left(\sqrt{\lambda_2}+1\right)}\right)=0\,. \quad (68)$$

Let us multiply throughout by the common denominator $(s_{12}^2-s_{11}s_{22})(\sqrt{\lambda_2}+1)(\sqrt{\lambda_1}+1)$ and substitute D. We have

$$s_{12}\left[A_1^2\lambda_2(\sqrt{\lambda_1}+1)-A_2^2\lambda_1(\sqrt{\lambda_2}+1)\right]+s_{11}A_1A_2(\sqrt{\lambda_1}-\sqrt{\lambda_2})+$$
$$(s_{12}^2-s_{11}s_{22})\left[A_2\lambda_1\left(\sqrt{\lambda_2}+1\right)-A_1\lambda_2\left(\sqrt{\lambda_1}+1\right)\right]=0\,. \quad (69)$$

If we make use of equations (9) we find

$$s_{12}(s_{12}-\lambda_1 s_{22})^2\lambda_2(\sqrt{\lambda_1}+1)-s_{12}(s_{12}-\lambda_2 s_{22})^2\lambda_1(\sqrt{\lambda_2}+1)+$$
$$s_{11}(s_{12}-\lambda_1 s_{22})(s_{12}-\lambda_2 s_{22})(\sqrt{\lambda_1}-\sqrt{\lambda_2})+(s_{12}^2-s_{11}s_{22})(s_{12}-\lambda_2 s_{22})\lambda_1\left(\sqrt{\lambda_2}+1\right)-$$
$$(s_{12}^2-s_{11}s_{22})(s_{12}-\lambda_1 s_{22})\lambda_2\left(\sqrt{\lambda_1}+1\right)=0\,. \quad (70)$$

After factoring the above expression we obtain the product:

$$\left(\sqrt{\lambda_1}-\sqrt{\lambda_2}\right)\left[-s_{12}+s_{22}\left(\lambda_1+\lambda_2+\sqrt{\lambda_1}+\sqrt{\lambda_2}+\sqrt{\lambda_1}\sqrt{\lambda_2}\right)\right](\lambda_1\lambda_2 s_{22}-s_{11})=0\,.$$

The first factor is apparently not equal to zero. Therefore we have to consider the other two. The last one clearly vanishes:

$$(-s_{11}+\lambda_1\lambda_2 s_{22})=\left(-s_{11}+\frac{s_{11}}{s_{22}}s_{22}\right)=-s_{11}+s_{11}=0\,.$$

Consequently equation $(38a)_3$ is also satisfied.

C Proof of equation (38b)

C.1. Third, consider equation $(38b)_1$. Upon substitution of the integrals $\overset{(31)}{I_{22}}$, $\overset{(34)}{I_{22}}$ and $\overset{(41)}{I_{22}}$ – see equations (34e), (34h) and (36e) – we have

$$\frac{2\pi D}{s_{12}^2-s_{11}s_{22}}\left[\frac{s_{22}A_1A_2\left(\sqrt{\lambda_1}-\sqrt{\lambda_2}\right)}{\left(\sqrt{\lambda_1}+1\right)\left(\sqrt{\lambda_2}+1\right)}+s_{21}\left(\frac{A_1^2}{\sqrt{\lambda_1}}\frac{1}{\sqrt{\lambda_1}+1}-\frac{A_2^2}{\sqrt{\lambda_2}}\frac{1}{\sqrt{\lambda_2}+1}\right)\right]-$$
$$2\pi D\left(\frac{A_1}{\sqrt{\lambda_1}\left(\sqrt{\lambda_1}+1\right)}-\frac{A_2}{\sqrt{\lambda_2}\left(\sqrt{\lambda_2}+1\right)}\right)=0\,. \quad (71)$$

After multiplying throughout by the common denominator

$$(s_{12}^2 - s_{11}s_{22})\sqrt{\lambda_1}\sqrt{\lambda_2}\left(\sqrt{\lambda_1}+1\right)\left(\sqrt{\lambda_2}+1\right)$$

and making use of equations (9) we obtain

$$\begin{aligned}\left(\sqrt{\lambda_1}-\sqrt{\lambda_2}\right)\Big[s_{12}^3\left(\sqrt{\lambda_1}+\sqrt{\lambda_2}\right) - s_{12}^2 s_{21} + s_{12}^3 + \lambda_1^{\frac{3}{2}}\lambda_2^{\frac{3}{2}}s_{22}^3 - \sqrt{\lambda_1}s_{12}^2 s_{21} - \\ - \sqrt{\lambda_2}s_{12}^2 s_{21} - -s_{11}s_{12}s_{22} - \sqrt{\lambda_1}\sqrt{\lambda_2}s_{11}s_{22}^2 + \underbrace{\sqrt{\lambda_1}\lambda_2^{\frac{3}{2}}s_{21}s_{22}^2 - \sqrt{\lambda_1}\lambda_2^{\frac{3}{2}}s_{12}s_{22}^2} + \\ \underbrace{\lambda_1^{\frac{3}{2}}\sqrt{\lambda_2}s_{21}s_{22}^2 - \lambda_1^{\frac{3}{2}}\sqrt{\lambda_2}s_{12}s_{22}^2} + \lambda_1\lambda_2 s_{21}s_{22}^2 - \sqrt{\lambda_1}s_{11}s_{12}s_{22} - \sqrt{\lambda_2}s_{11}s_{12}s_{22} + \\ + \lambda_1\lambda_2^{\frac{3}{2}}s_{21}s_{22}^2 + \lambda_1^{\frac{3}{2}}\lambda_2 s_{21}s_{22}^2 + \underbrace{2\sqrt{\lambda_1}\sqrt{\lambda_2}s_{12}^2 s_{22} - 2\sqrt{\lambda_1}\sqrt{\lambda_2}s_{12}^2 s_{22}}\Big] = 0\,.\end{aligned}$$

Let us divide throughout by $\sqrt{\lambda_1} - \sqrt{\lambda_2}$ and cancel the terms braced. If, in addition, we take into account that

$$-s_{12}^2 s_{21} + s_{12}^3 = -s_{12}^3 + s_{21}^3 = 0$$

we shall have

$$\begin{aligned}\Big(\sqrt{\lambda_1}s_{12}^3 + \sqrt{\lambda_2}s_{12}^3 + \lambda_1^{\frac{3}{2}}\lambda_2^{\frac{3}{2}}s_{22}^3 - \sqrt{\lambda_1}s_{12}^2 s_{21} - \\ - \sqrt{\lambda_2}s_{12}^2 s_{21} - s_{11}s_{12}s_{22} - \sqrt{\lambda_1}\sqrt{\lambda_2}s_{11}s_{22}^2 + 2\sqrt{\lambda_1}\sqrt{\lambda_2}s_{12}^2 s_{22} - \sqrt{\lambda_1}\lambda_2^{\frac{3}{2}}s_{12}s_{22}^2 + \\ + \sqrt{\lambda_1}\lambda_2^{\frac{3}{2}}s_{21}s_{22}^2 - \lambda_1^{\frac{3}{2}}\sqrt{\lambda_2}s_{12}s_{22}^2 + \lambda_1^{\frac{3}{2}}\sqrt{\lambda_2}s_{21}s_{22}^2 + \lambda_1\lambda_2 s_{21}s_{22}^2 - \sqrt{\lambda_1}s_{11}s_{12}s_{22} - \\ - \sqrt{\lambda_2}s_{11}s_{12}s_{22} + \lambda_1\lambda_2^{\frac{3}{2}}s_{21}s_{22}^2 + \lambda_1^{\frac{3}{2}}\lambda_2 s_{21}s_{22}^2 - 2\sqrt{\lambda_1}\sqrt{\lambda_2}s_{12}s_{21}s_{22}\Big) = 0\end{aligned}$$

in which

$$\lambda_1^{\frac{3}{2}}\lambda_2^{\frac{3}{2}}s_{22}^3 - s_{11}s_{12}s_{22} - \sqrt{\lambda_1}\sqrt{\lambda_2}s_{11}s_{22}^2 + \lambda_1\lambda_2 s_{21}s_{22}^2 - \sqrt{\lambda_1}s_{11}s_{12}s_{22} - \sqrt{\lambda_2}s_{11}s_{12}s_{22} + \lambda_1\lambda_2^{\frac{3}{2}}s_{21}s_{22}^2 + \lambda_1^{\frac{3}{2}}\lambda_2 s_{21}s_{22}^2 = 0$$

where

$$\begin{aligned}\lambda_1\lambda_2^{\frac{3}{2}}s_{21}s_{22}^2 + \lambda_1^{\frac{3}{2}}\lambda_2 s_{21}s_{22}^2 - \sqrt{\lambda_1}s_{11}s_{12}s_{22} - \sqrt{\lambda_2}s_{11}s_{12}s_{22} = \\ = s_{21}s_{22}^2\lambda_1\lambda_2(\sqrt{\lambda_1}+\sqrt{\lambda_2}) - s_{11}s_{12}s_{22}(\sqrt{\lambda_1}+\sqrt{\lambda_2}) = s_{21}s_{22}^2\frac{s_{11}}{s_{22}}(\sqrt{\lambda_1}+\sqrt{\lambda_2}) - s_{11}s_{12}s_{22}(\sqrt{\lambda_1}+\sqrt{\lambda_2}) = \\ = s_{11}s_{12}s_{22}(\sqrt{\lambda_1}+\sqrt{\lambda_2}) - s_{11}s_{12}s_{22}(\sqrt{\lambda_1}+\sqrt{\lambda_2}) = 0\,.\end{aligned}$$

Therefore

$$\begin{aligned}\lambda_1^{\frac{3}{2}}\lambda_2^{\frac{3}{2}}s_{22}^3 + \lambda_1\lambda_2 s_{21}s_{22}^2 - s_{11}s_{12}s_{22} - \sqrt{\lambda_1}\sqrt{\lambda_2}s_{11}s_{22}^2 = \\ = s_{22}^2\lambda_1\lambda_2(s_{22}\sqrt{\lambda_1}\sqrt{\lambda_2}+s_{21}) - s_{12}s_{22}(s_{11}+s_{22}\sqrt{\lambda_1}\sqrt{\lambda_2}) = s_{22}^2\frac{s_{11}}{s_{22}}(s_{22}\sqrt{\lambda_1}\sqrt{\lambda_2}+s_{21} - s_{12}s_{22}(s_{11}+s_{22}\sqrt{\lambda_1}\sqrt{\lambda_2}) = \\ = s_{11}s_{22}^2\sqrt{\lambda_1}\sqrt{\lambda_2} + s_{11}s_{12}s_{22} - s_{11}s_{12}s_{22} - s_{11}s_{22}^2\sqrt{\lambda_1}\sqrt{\lambda_2} = 0\,.\end{aligned}$$

C.2. Consider equation $(38b)_2$. If we substitute the integrals $\overset{(32)}{I}_{21}$, $\overset{(33)}{I}_{21}$, $\overset{(42)}{I}_{21}$ and $\overset{(43)}{I}_{21}$ – see equations (34b), (34c) (36b) and (36c) – and perform some manipulations we obtain

$$\begin{aligned}2\pi D\frac{2}{s_{66}}\left[A_1^2\frac{1}{\sqrt{\lambda_1}+1} - A_2^2\frac{1}{\sqrt{\lambda_2}+1} - A_1A_2\frac{\sqrt{\lambda_1}-\sqrt{\lambda_2}}{(\sqrt{\lambda_1}+1)(\sqrt{\lambda_2}+1)}\right] - \\ - 2\pi D\left(\frac{A_2}{(\sqrt{\lambda_2}+1)} - \frac{A_1}{(\sqrt{\lambda_1}+1)}\right) - 2\pi D\left(A_1\frac{\sqrt{\lambda_1}}{\sqrt{\lambda_1}+1} - A_2\frac{\sqrt{\lambda_2}}{\sqrt{\lambda_2}+1}\right) = 1\,. \quad (72)\end{aligned}$$

Let us multiply throughout by the common denominator $s_{66}(\sqrt{\lambda_1}+1)(\sqrt{\lambda_2}+1)$ and make some manipulations. In this way we have

$$2\left[A_1^2\,(\sqrt{\lambda_2}+1)-A_2^2(\sqrt{\lambda_1}+1)\right]-2A_1A_2(\sqrt{\lambda_1}-\sqrt{\lambda_2})-$$
$$-\,s_{66}\left[A_2(\sqrt{\lambda_1}+1)-A_1(\sqrt{\lambda_2}+1)\right]-s_{66}\left[A_1\sqrt{\lambda_1}(\sqrt{\lambda_2}+1)-A_2\sqrt{\lambda_2}(\sqrt{\lambda_1}+1)\right]-$$
$$-\,(\lambda_1-\lambda_2)s_{22}s_{66}(\sqrt{\lambda_1}+1)(\sqrt{\lambda_2}+1)=0\,. \quad (73)$$

If simplify (68) then we have

$$2(s_{12}-\lambda_1 s_{22})^2\,(\sqrt{\lambda_2}+1)-2(s_{12}-\lambda_2 s_{22})^2(\sqrt{\lambda_1}+1)-$$
$$-\,2(s_{12}-\lambda_1 s_{22})(s_{12}-\lambda_2 s_{22})(\sqrt{\lambda_1}-\sqrt{\lambda_2})+s_{66}(s_{12}-\lambda_1 s_{22})(\sqrt{\lambda_2}+1)\left(1-\sqrt{\lambda_1}\right)+$$
$$+\,s_{66}(s_{12}-\lambda_2 s_{22})(\sqrt{\lambda_1}+1)\left(\sqrt{\lambda_2}-1\right)-(\lambda_1-\lambda_2)s_{22}s_{66}(\sqrt{\lambda_1}+1)(\sqrt{\lambda_2}+1)=0\,. \quad (74)$$

If we factoring this expression we shall get

$$\left(\sqrt{\lambda_1}-\sqrt{\lambda_2}\right)\left(s_{12}+\sqrt{\lambda_1}s_{22}+\sqrt{\lambda_2}s_{22}+\sqrt{\lambda_1}\sqrt{\lambda_2}s_{22}\right)(-2s_{12}-s_{66}+\lambda_1 s_{22}+\lambda_2 s_{22})=0\,.$$

The first factor is evidently different from zero. Hence we have to investigate the other two. We shall start with the last one

$$(-2s_{12}-s_{66}+\lambda_1 s_{22}+\lambda_2 s_{22})=-2s_{12}-s_{66}+s_{22}(\lambda_1+\lambda_2)=-2s_{12}-s_{66}+s_{22}\frac{2s_{12}+s_{66}}{s_{22}}=-2s_{12}-s_{66}+2s_{12}+s_{66}=0\,.$$

C.3. Third consider equation $(38b)_3$.Upon substitution of the integrals $\overset{(31)}{I}_{22}$, $\overset{(34)}{I}_{22}$ and $\overset{(41)}{I}_{22}$ – see equations (34e), (34h) and (36e) – we have

$$-\frac{2\pi D}{s_{12}^2-s_{11}s_{22}}\left[\frac{s_{12}A_1A_2\left(\sqrt{\lambda_1}-\sqrt{\lambda_2}\right)}{\left(\sqrt{\lambda_1}+1\right)\left(\sqrt{\lambda_2}+1\right)}+\left(\frac{A_1^2}{\sqrt{\lambda_1}(\sqrt{\lambda_1}+1)}-\frac{A_2^2}{\sqrt{\lambda_2}(\sqrt{\lambda_2}+1)}\right)\right]-$$
$$2\pi D\left(\frac{A_1}{\sqrt{\lambda_1}+1}-\frac{A_2}{\sqrt{\lambda_2}+1}\right)=1\,. \quad (75)$$

Then multiplying throughout by the common denominator

$$(s_{12}^2-s_{11}s_{22})\sqrt{\lambda_1}\sqrt{\lambda_2}\left(\sqrt{\lambda_1}+1\right)\left(\sqrt{\lambda_2}+1\right)$$

and making some manipulations we obtain

$$-\,s_{12}A_1A_2\sqrt{\lambda_1\lambda_2}(\sqrt{\lambda_1}-\sqrt{\lambda_2})-s_{11}\left[A_1^2\sqrt{\lambda_2}(\sqrt{\lambda_2}+1)-A_2^2\sqrt{\lambda_1}(\sqrt{\lambda_1}+1)\right]-$$
$$-\,(s_{12}^2-s_{11}s_{22})\sqrt{\lambda_1\lambda_2}\left[A_1\left(\sqrt{\lambda_2}+1\right)-A_2\left(\sqrt{\lambda_1}+1\right)\right]-$$
$$-\,(\lambda_1-\lambda_2)\,s_{22}(s_{12}^2-s_{11}s_{22})\sqrt{\lambda_1\lambda_2}\left(\sqrt{\lambda_1}+1\right)\left(\sqrt{\lambda_2}+1\right)=0\,. \quad (76)$$

If we substitute equation (9) we have

$$s_{11}s_{12}\lambda_1-s_{11}s_{12}\lambda_2+s_{12}s_{22}\lambda_1\lambda_2^2-s_{12}s_{22}\lambda_1^2\lambda_2+s_{11}s_{12}\sqrt{\lambda_1}-s_{11}s_{12}\sqrt{\lambda_2}+$$
$$+\,s_{11}s_{22}\lambda_1\sqrt{\lambda_2}-s_{11}s_{22}\sqrt{\lambda_1}\lambda_2+s_{12}s_{22}\lambda_1\lambda_2^{\frac{3}{2}}-s_{12}s_{22}\lambda_1^{\frac{3}{2}}\lambda_2-s_{22}^2\lambda_1^2\lambda_2^{\frac{3}{2}}+s_{22}^2\lambda_1^{\frac{3}{2}}\lambda_2^2=0\,. \quad (77)$$

After factoring this equations we shall find

$$\left(\sqrt{\lambda_1}-\sqrt{\lambda_2}\right)\left(s_{12}+\sqrt{\lambda_1}s_{12}+\sqrt{\lambda_2}s_{12}+\sqrt{\lambda_1}\sqrt{\lambda_2}s_{22}\right)(s_{11}-\lambda_1\lambda_2 s_{22})=0\,.$$

One can check easily that the third factor is zero:

$$(s_{11}-\lambda_1\lambda_2 s_{22})=\left(s_{11}-\frac{s_{11}}{s_{22}}s_{22}\right)=s_{11}-s_{11}=0\,.$$

Consequently $(38b)_3$ is fulfilled.

D Formulae for strains

Making use of equations (13), (14) and (40) from the kinematic equations (1) we obtain

$$e_{\alpha\beta} = \oint_{\mathcal{L}_o} t_\lambda(M_\circ)\mathcal{D}_{\lambda\alpha\beta}(M_\circ, Q)\,\mathrm{d}s_{M_\circ} - \oint_{\mathcal{L}_o} u_\lambda(M_\circ)\mathcal{S}_{\lambda\alpha\beta}(M_\circ, Q)\,\mathrm{d}s_{M_\circ} \qquad Q \in A_e \tag{78}$$

where

$$\begin{aligned} &\mathcal{D}_{\lambda 11} = U_{\lambda 1}\,\partial_1\,, && \mathcal{S}_{\lambda 11} = T_{\lambda 1}\,\partial_1\,,\\ &\mathcal{D}_{\lambda 12} = \frac{1}{2}\,(U_{\lambda 2}\,\partial_1 + U_{\lambda 1}\,\partial_2) = \mathcal{D}_{\lambda 21}\,, && \mathcal{S}_{\lambda 12} = \frac{1}{2}\,(T_{\lambda 2}\,\partial_1 + T_{\lambda 1}\,\partial_2) = \mathcal{D}_{\lambda 21}\,,\\ &\mathcal{D}_{\lambda 22} = U_{\lambda 2}\,\partial_2\,, && \mathcal{S}_{\lambda 22} = T_{\lambda 2}\,\partial_2\,. \end{aligned} \tag{79}$$

If we introduce the notations

$$\begin{aligned} &b_1 = \frac{\sqrt{\lambda_1}}{\rho_1^2}\,, \quad b_2 = \frac{\sqrt{\lambda_2}}{\rho_2^2}\,, \quad c = \sqrt{\lambda_1}\,\sqrt{\lambda_2}r_1^2 + r_2^2\,, \quad d = (\sqrt{\lambda_1} - \sqrt{\lambda_2})r_2\,,\\ &k_1 = \frac{\lambda_1^{\frac{3}{2}} A_1}{\rho_1^2}\,, \quad k_2 = \frac{\lambda_2^{\frac{3}{2}} A_2}{\rho_2^2}\,, \quad f_1 = \frac{A_1}{(\lambda_1^{\frac{3}{2}} r_1^2 + r_2^2)^2}\,, \quad f_2 = \frac{A_2}{(\lambda_2^{\frac{3}{2}} r_1^2 + r_2^2)^2} \end{aligned} \tag{80}$$

for the derivatives in equation (79) we have

$$\begin{aligned} \partial_1 U_{11} &= D\left(b_2\lambda_2 A_1^2 r_1 - b_1\lambda_1 A_2^2 r_1\right)\,,\\ \partial_1 U_{12} &= \frac{D A_1 A_2}{1 + d^2 r_1^2}\left(2\sqrt{\lambda_1}\,\sqrt{\lambda_2} d r_1^2 - cd\right) = \partial_1 U_{21}\,,\\ \partial_1 U_{22} &= D\left(b_1 A_1^2 r_1 - b_2 A_2^2 r_1\right)\,,\\ \partial_2 U_{11} &= D\left(\,b_2 A_1^2 r_2 - b_1 A_2^2 r_2\right)\,,\\ \partial_2 U_{12} &= \frac{D A_1 A_2}{1 + d^2 r_1^2}\left[2 d r_1 r_2 - c(\sqrt{\lambda_1} - \sqrt{\lambda_2}) r_1\right] = \partial_2 U_{21}\,,\\ \partial_2 U_{22} &= -D\left(\frac{A_2^2 r_2}{\rho_2^2\sqrt{\lambda_2}} - \frac{A_1^2 r_2}{\rho_1^2\sqrt{\lambda_1}}\right) \end{aligned} \tag{81}$$

and

$$\begin{aligned} \partial_1 T_{11} &= D\left[\frac{2b_2\lambda_2 A_1 r_1}{\rho_2^2} - \frac{2b_1\lambda_1 A_2 r_1}{\rho_1^2}\right](r_1 n_1 + r_2 n_2) - D\,[b_2 A_1 - b_1 A_2]\,n_1\,,\\ \partial_1 T_{12} &= D\left[\frac{2b_1\lambda_1 A_1 r_1}{\rho_1^2} - \frac{2b_2\lambda_2 A_2 r_1}{\rho_2^2}\right] r_1 n_2 - D\,[b_1 A_1 - b_2 A_2]\,n_2 - D\,[2 f_1\lambda_1 r_1 - 2 f_2\lambda_2 r_1]\,r_2 n_1\,,\\ \partial_1 T_{21} &= D\left[\frac{2b_1\lambda_1^2 A_2 r_1}{\rho_1^2} - \frac{2b_1\lambda_2^2 A_1 r_1}{\rho_2^2}\right] r_1 n_2 - D\,[b_1\lambda_1 A_2 - b_2\lambda_2 A_1]\,n_2 - D\left[\frac{2b_1\lambda_1 A_2 r_1}{\rho_1^2} - \frac{2b_2\lambda_2 A_1 r_1}{\rho_2^2}\right] r_2 n_1\,,\\ \partial_1 T_{22} &= D\left[\frac{2b_1\lambda_1 A_1 r_1}{\rho_1^2} - \frac{2b_2\lambda_2 A_2 r_1}{\rho_2^2}\right](r_1 n_1 + r_2 n_2) - D\,[b_1 A_1 - b_2 A_2]\,n_1\,, \end{aligned} \tag{82a}$$

$$\begin{aligned} \partial_2 T_{11} &= D\left[\frac{b_2 A_1 2 r_2}{\rho_2^2} - \frac{b_1 A_2 2 r_2}{\rho_1^2}\right](r_1 n_1 + r_2 n_2) - D\,[\,b_2 A_1 - \,b_1 A_2]\,n_2\,,\\ \partial_2 T_{12} &= D\left[\frac{b_1 A_1 2 r_2}{\rho_1^2} - \frac{b_2 A_2 2 r_2}{\rho_2^2}\right] r_1 n_2 - D\,[2 f_1 r_2 - 2 f_2 r_2]\,r_2 n_1 + D\,[f_1 - f_2]\,n_1\,,\\ \partial_2 T_{21} &= D\left[\frac{b_1\lambda_1 A_2 2 r_2}{\rho_1^2} - \frac{b_2\lambda_2 A_1 2 r_2}{\rho_2^2}\right] r_1 n_2 - D\left[\frac{b_1 A_2 2 r_2}{\rho_1^2} - \frac{b_2 A_1 2 r_2}{\rho_2^2}\right] r_2 n_1 + D\,[b_1 A_2 - b_2 A_1]\,n_1\,,\\ \partial_2 T_{22} &= D\left[\frac{b_1 A_1 2 r_2}{\rho_1^2} - \frac{b_2 A_2 2 r_2}{\rho_2^2}\right](r_1 n_1 + r_2 n_2) - D\,[b_1 A_1 - b_2 A_2]\,n_2\,. \end{aligned} \tag{82b}$$