PERIODICA POLYTECHNICA SER. CIV. ENG. VOL. 50, NO. 2, PP. 139–147 (2006)

A PARAMETRIC SURVEY OF THE INFLUENCE OF THE SEMI-RIGID CONNECTIONS ON THE SHAKEDOWN OF ELASTO-PLASTIC FRAMES

János LÓGÓ*, Sándor KALISZKY* and Mohammed HJIAJ**

* Department of Structural Mechanics,
Budapest University of Technology and Economics,
H–1521, Budapest, HUNGARY
**LGCGM, INSA de Rennes, 20 avenue des Buttes de Coësmes,
35043 Rennes cedex - FRANCE

Received: Nov. 6, 2006

Abstract

In the past it was generally assumed that the connection of the beams and columns of the steel-framed multistorey structures are either rigid or pinned. In the reality, however, they are semi-rigid. This circumstance influences significantly the behaviour of the entire structure, therefore, it has to be taken into consideration in the analysis and design. In this paper a parametric study is presented to analyse the influence of the semi-rigid connections on the shakedown of elasto-plastic steel framed structures under multi-parameter static loading. To control the plastic behaviour of the structure bound on the complementary strain energy of the residual forces is also applied. The semi-rigid behaviour is modelled by appropriate internal springs at the beam column-connections. The formulation of the problem yields to nonlinear mathematical programming which is solved by the use of an iterative procedure. The parametric study is illustrated by the solution of an example.

Keywords: semi-rigid connection, shakedown analysis, mathematical programming, nonlinear programming.

1. Introduction

The plastic analysis and design methods are widely used in structural engineering practice since they provide information about the post yield behaviour and load carrying capacity of structures and by utilizing the plastic reserve they generally lead to significant saving in material. The "cost" of these advantages, however, is that the plastic deformations accumulated during the loading history might exceed the plastic deformation capacities of the structural elements, and the structure might become unserviceable.

Hence, it is evident that the determination or at least the assessment of plastic deformations and residual displacements and their consideration in plastic analysis and design of structures, especially in case of variable and repeated loads, is an important requirement.

The main structural elements of the steel-framed multi-storey structures are the columns, the beams and their connections. The assumption that the stiffnesses of the connection is either rigid or pinned has been widely applied in the past.

However, the real behaviour of the connections is never as ideal as it was assumed, but they behave somewhere in between these limits, such they are semi-rigid. This circumstance which can influence significantly the behaviour of the structure has to be taken into account in the analysis and design. To model these types of connections there are different methods which are presented in national standards, as well. (e.g. EUROCODE 3) [23].

A typical moment-rotation relationship of semi-rigid connections is illustrated in *Fig. 1.a*. The diagram can be approximated by bilinear or linear relationships shown in *Fig. 1.b* and *Fig. 1.c*, respectively. In the latter case r_0 denotes the stiffness and M_0 is the design moment of the connection. According to the recommendations and surveys based on linear approximation it is suggested to assume M_0 to $2/3$ of the ultimate moment M_u of the connection [1, 5, 23]. In this paper the linear approximation will be used. A more detailed study is under preparation in which bilinear relationships will be applied.

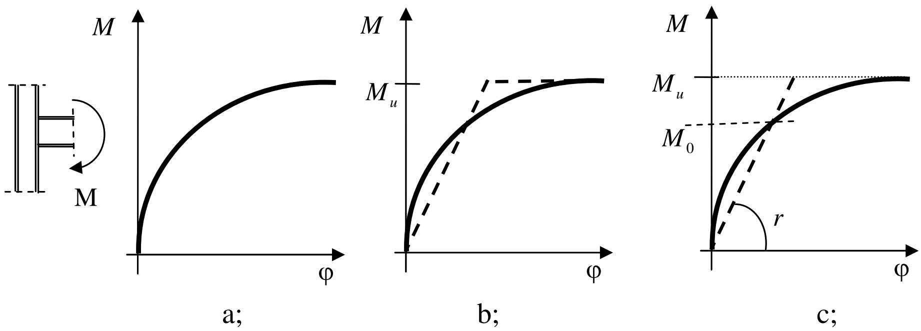

Fig. 1. a-c. Moment-rotation relations

The aim of this paper is to present a method for the shakedown analysis of elasto-plastic frames with semi-rigid connections subjected to quasi-static multiple loading and to conduct a parametric study of the influence of the semi-rigid connections on the shakedown multiplier of the structure. In the present method the plastic behaviour of the frame is controlled by

- the use of the static theorem of shakedown and
- the application of bounds on plastic deformations.

The effect of the semi-rigid connections are taken into consideration by an appropriately chosen spring coefficient in the elementary stiffness matrix of the beam elements. The parametric study is illustrated by a numerical example where the effect of the change of the spring coefficient is investigated.

2. Notations

In the paper the finite element method with *i=1,2,…,n* elements will be applied and the following notations will be used:

$\mathbf{P}_h$; $(h = 1, 2, ..., s)$:	prescribed combinations of the external static loads,
m:	common shakedown multiplier of the loads $\mathbf{P}_h$,
$\mathbf{M}^{eh}$:	fictitious elastic internal moments calculated from the loads $\mathbf{P}_h$; $(h = 1, 2, ..., s)$ assuming that the beam is purely elastic,
$\mathbf{M}^r$:	residual internal moments,
$\mathbf{M}^h = \mathbf{M}^{eh} + \mathbf{M}^r$:	actual internal moments,
σ_y	yield stress,
r_j	spring coefficient at the j-th (*j=1,2,…,k*) beam-column connection,
$\mathbf{F}(\mathbf{r})$, $\mathbf{K}(\mathbf{r})$, $\mathbf{G}$, $\mathbf{G}^*$	flexibility, stiffness, geometrical and equilibrium matrices,
E:	Young's Modulus,
A_i, I_i, S_{0i} and ℓ_i:	area, moment of inertia, static moment of the cross-sections and length of the beam and column elements (*i=1,2,…,n*), respectively,
W_p:	complementary strain energy of the residual forces,
W_{p0} :	allowable complementary strain energy of the residual forces,
$\bar{M}_j$, $\bar{M}_{0j}$:	moment and design moment of the j-th semi-rigid connection, respectively.

3. Control of Plastic Deformations

The exact determination of the plastic deformations and residual displacements of elasto-plastic structures under variable static and dynamic loading requires a complete load history analysis. The problem is, however, that there are only special cases (e.g. single loading) when the load history is known and load history analysis can be conducted. For this purpose several methods and solution techniques based on incremental step-by-step analysis have been developed their application, however, even to simple problems requires large computational work. To overcome these difficulties the shakedown analysis and a variety of bounding theorems have been proposed for the control and estimation of plastic deformations and residual displacements of structures under single and multiparameter loading.

3.1. Shakedown Analysis

The most important tool for controlling the plastic behaviour of structures is the application of static and kinematic theorems of shakedown proposed by MELAN [17] and KOITER [12]. Using these theorems the accumulation of unrestricted plastic deformations can be prevented, no information can be obtained, however, about the magnitudes of the plastic deformations and residual displacements developed at the shakedown of the structure. In this paper the static theorem of shakedown analysis will be used [6, 13].

3.2. Bounds on Plastic Deformations

In the past decades the shakedown analysis has been supplemented by other methods which provide further information in terms of bounds on the plastic deformations and residual displacements. Bounding theorems were elaborated for the analysis of elastoplastic bodies and structures under static loading (see e.g. KOITER [12], MAIER [14, 15], POLIZOTTO [18], PONTER [20], TIN-LOI [21], KALISZKY [7], KALISZKY and LÓGÓ [8] CAPURSO et. al. [3], DOROSZ [4]). Later, similarly to the shakedown theorems, these theorems have been extended to topics concerning dynamic loading, large deformations, strain hardening and nonassociated constitutive laws and have been applied to the solution of several problems (see e.g. CAPURSO [2, 3], MAIER and VITELLO [16], POLIZOTTO [19], LANGE-HANSEN [14], TIN-LOI [21, 22]).

On the basis of the upper bound theorems developed by CAPURSO [2], CAPURSO, CORRADI and MAIER [3], it was suggested by KALISZKY and LÓGÓ [9, 10, 11] that the complementary strain energy of the residual forces (W_p) could be considered an overall measure of the plastic performance of structures. Using this idea the plastic deformations can be controlled by the following constraint:

$$\frac{1}{2}\sum_{i=1}^{n} \mathbf{Q}_i^r \mathbf{F}_i \mathbf{Q}_i^r - W_{p0} \leq 0. \tag{1}$$

Here W_{p0} is the allowable residual complementary strain energy. In the lack of other data it may be assumed, for example, that $W_{p0} = \alpha W_e$, where W_e is the complementary elastic strain energy calculated from the largest load acting on the structure and α is an appropriately chosen multiplier (say $1 \leq \alpha \leq 3$).

The above equation becomes much simpler if the residual moments (M_i^R) are used to compute the residual complementary strain energy, the residual normal and shear forces are taken into consideration at the calculation of the residual moments only and the torsion moments of the beams are neglected:

$$W_p = \frac{1}{2E}\sum_{i=1}^{n} \frac{1}{I_i} \int_0^{\ell_i} M_i^R(s)^2 ds. \tag{2}$$

If the beam and column elements are loaded only at the nodes the residual bending moment function $M_i^R(s)$ is linear along the elements and omitting the details *Eq.* (2) can be written as follows:

$$W_p = \frac{1}{6E}\sum_{i=1}^{n}\frac{\ell_i}{I_i}\left[\left(M_{i1}^R\right)^2 + \left(M_{i1}^R\right)\left(M_{i2}^R\right) + \left(M_{i2}^R\right)^2\right]. \tag{3}$$

Here M_{i1}^R and M_{i2}^R are the residual bending moments at the ends of the i-th element.

4. Formulation of the Problem

The problem to be solved is to determine the maximum load multiplier $m_{\max}$ under the conditions that the frame is strong enough to carry the loads $m_{\max}\mathbf{P}_h$, shakes down, and satisfies the constraint on plastic deformations given by *Eq.* (3). Hence,

$$\text{maximize } m \tag{4}$$

subject to

$$\mathbf{G}^*\mathbf{M}^r = \mathbf{0}; \tag{5}$$

$$\mathbf{M}^{eh} = \mathbf{F}^{-1}(\mathbf{r})\,\mathbf{G}\mathbf{K}^{-1}(\mathbf{r})\,\mathbf{P}_h,\ (h = 1, 2, ..., s); \tag{6}$$

$$-2S_{0i}\sigma_{yí} \le \left(mM_i^{eh} + M_i^r\right) \le 2S_{0i}\sigma_{yí},\ (h = 1, 2, ..., s),\ (i = 1, 2, ..., n); \tag{7}$$

$$-\bar{M}_{0j} \le m\bar{M}_j \le \bar{M}_{0j},\quad (h = 1, 2, ..., s),\quad (j = 1, 2, ..., k); \tag{8}$$

$$\frac{1}{6E}\sum_{i=1}^{n}\frac{\ell_i}{I_i}\left[\left(M_{i1}^R\right)^2 + \left(M_{i1}^R\right)\left(M_{i2}^R\right) + \left(M_{i2}^R\right)^2\right] - W_{p0} \le 0; \tag{9}$$

Here m and M_i^r are unknown. The flexibility $\mathbf{F}(\mathbf{r})$ and stiffness $\mathbf{K}(\mathbf{r})$ matrices of the frame depends on the spring coefficient values of the beam-column connections.

5. Example

The application of the method is illustrated by an example shown in *Fig. 2*. For the shakedown analysis the standard finite element procedure with two node beam elements was applied. The mathematical programming solution is based on a sequential quadratic programming concept.

In the following the influence on the shakedown parameter (m) of the allowable residual strain energy (W_{p0}) and the stiffness of the semi-rigid connections characterized by the spring coefficient (r) will be used.

All the necessary data of the problems are given in *Fig. 2*. The external loads are F_1 =1kN and F_2 =4kN. The yield stress and the Young's Modulus are σ_y =21kN/cm^2 and $E = 2.07 \cdot 10^6$kN/cm^2. The cross-sectional data of the beam are: A_B =28.5cm^2, $I_B = 1943$cm^4, $S_{B0} = 130.0$cm^3, while for the columns $A_c = 39.0$cm^2, $I_c = 3891.6$cm^4, $S_{c0} = 210.0$cm^3. At nodes 2 and 4 the beam-column connections can be fixed, semi-rigid with variable internal spring values **r** and pinned. If the connection is semi-rigid, the behaviour is represented by a spring which value is varied from 10 kNcm/rad (≈pinned connection) to 1.00E+21 kNcm/rad (≈ fixed connection). At that time the 2/3 reduction rule was not applied at the semi-rigid connections.

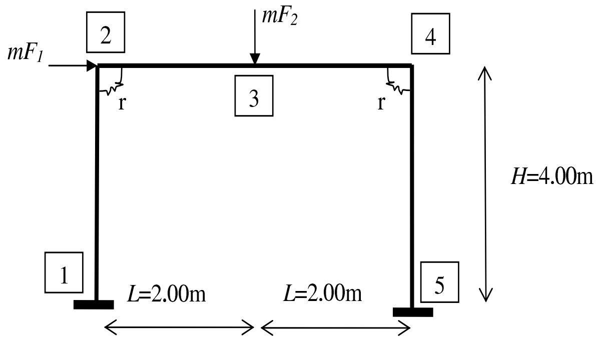

Fig. 2. Test problem

The variation of shakedown parameter in terms of the allowable residual strain energy can be seen in *Fig. 3*. Here the beam-column connections are pinned, semi-rigid and rigid. One can see that as it can be expected the rigid beam-column connection provides the highest shakedown parameter while the pinned connection gives the lowest one. If $W_{p0} = 0$ then plastic deformations do not develop. This case corresponds to the elastic solution and the shakedown of the structure is senseless. If $W_{p0} > 4.00$ kNm then the constraint on the residual moments become inactive. This case corresponds to the shakedown solution where the magnitudes of the plastic deformations are not limited. Considering these two limit cases the differences between the corresponding shakedown parameters related to the values of the rigid connections are 33.85% and 14.62%, respectively. Hence, at this special problem in an average situation the influence of the semi-rigid connection on the shakedown parameter is significant. The variation of the shakedown parameter in terms of the stiffness of the semi-rigid connections represented by the spring coefficients can be seen in *Fig. 4*, where the logarithmic scale is used for the spring values. Three cases were investigated where the allowable residual strain energies are given values ($W_{0p1} = 4.00$ kNm, $W_{0p2} = 2.00$ kNm, *elastic* $W_{0p3} = 0.001$ kNm).

Considering the intermediate case when $W_{0p2} = 2.00$ kNm the difference between the shakedown parameters belonging to the pinned and rigid connections and related to the latter one is $\sim 20\%$.

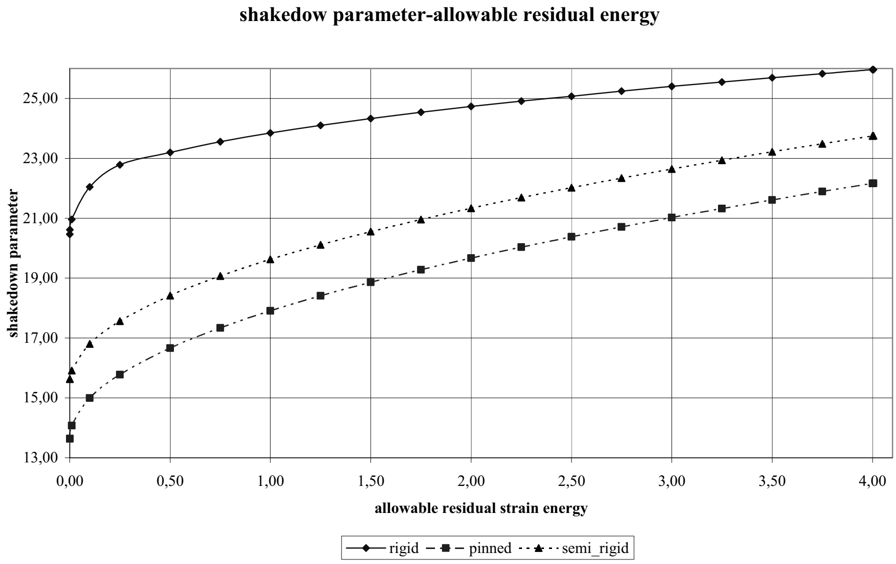

Fig. 3. Variation of the shakedown parameter in terms of the allowable residual energy

The figures show that depending on the magnitude of W_{p0} the variation of the stiffness of the semi-rigid connections influences significantly the shakedown parameter. Between the two limit cases (i.e. the pinned connection and the rigid one) the variation of the shakedown parameter might reach $\sim 35\%$. The above results show the importance of the effect of the semi-rigid connections on the shakedown of structures, but from this special example, general statements cannot be made.

6. Conclusion

In the paper a parametric study is presented to analyse the influence of the semi-rigid connections on the shakedown of elasto-plastic steel-framed structures under multi-parameter static loading. In addition to the shakedown analysis, to control the plastic behaviour of the structure, bound on the complementary strain energy of the residual forces is also applied. The semi-rigid behaviour is modelled by appropriate internal springs at the beam-column connections. The formulation of the problem yields to nonlinear mathematical programming which is solved by the use of a sequential quadratic algorithm. The applied numerical method is relatively simple and can be conducted easily.

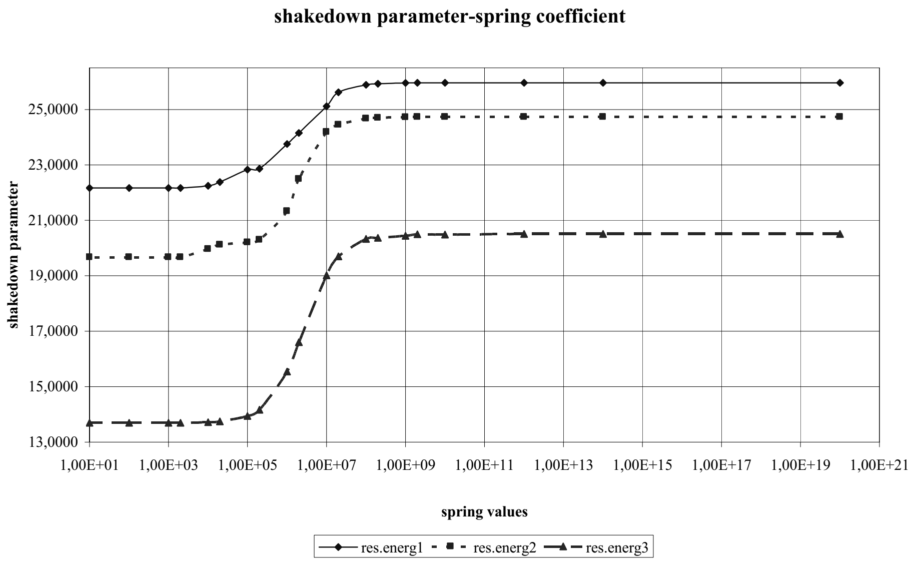

Fig. 4. Variation of the shakedown parameter in terms of spring coefficient

It is worthwhile to mention that in special cases the method yields to two extreme solutions. Namely, if $W_{p0} = 0$, then the elastic solution can be obtained, where the shakedown is senseless, while if W_{p0} is satisfactorily large then the corresponding constraints become inactive and the method provides the shakedown solution where the magnitude of the plastic deformations are not limited (see the diagrams of *Fig. 3*). The results of the numerical calculation are in agreement with the expectation, that the stiffness of the semi-rigid connections influences significantly the magnitude of the shakedown parameter. This effect depends very much on the number of the semi-rigid connections, the arrangement of the structure and the loads, the ratio of the lengths of the beams and columns and several other circumstances. The results of this simple example draws the attention to the importance of the problem but further investigations are necessary to make more general statements. This research is in progress and will be published later.

Acknowledgements

The present study was supported by The Hungarian National Scientific and Research Foundation (OTKA) (grant T042993) and the Ministry of Education (TET grant F-44/05).

References

[1] ARIBERT J. M. – LACHAL. A. – DINGA. O. N., Modélisation du comportement d'assemblages métalliques semi-rigides de type poutre-poteau boulonnés par platine d'extrémité, Revue Construction Métallique n°1, 1999, 25–46.

[2] CAPURSO, M., Displacement Bounding Principle in Dynamic Shakedown of Structures Subjected to Cyclic Loads . *Int. J. Solids Struct.* 1974, **10**, pp. 77–92.

[3] CAPURSO, M.– CORRADI, L.– MAIER, G., Bounds on Deformations and Displacements in Shakedown Theory. In *Proc. Materiaux et Structures sous Chargement Cyclique. Palaiseau.* France, 1978, pp. 231–244.

[4] DOROSZ, S., An Improved Upperbound to Maximum Deflections of Elasto-Plastic Structures at Shakedown. *J. Struct. Mech.* 1978, **6**, pp. 267–287.

[5] IVÁNYI, M., Semi-rigid Connections in Steel Frames, Part I. In M. Iványi, C.C. Baniotopoulos, editors, Semi-Rigid Connections in Structural Steelwork, CISM Courses and Lectures No. 419. Springer-Wien New York, 2000, pp. 1–101.

[6] KALISZKY, S., *Plasticity. Theory and Application.* Elsevier Publishers Amsterdam. 1989.

[7] KALISZKY, S. – LÓGÓ, J., Elasto-Plastic Analysis and Optimal Design with Limited Plastic Deformations and Displacements, In: Structural and Multidisciplinary Optimization, eds. by N. Olhoff, G.I.N. Rozvany, Pergamon Press, (1995), 465-470.

[8] KALISZKY, S., Elasto-plastic Analysis with Limited Plastic Deformations and Displacements, *Mech. Struct. and Mach.* 1996, **24** (1), pp. 39–50.

[9] KALISZKY, S. – LÓGÓ, J., Optimal Plastic Limit and Shakedown Design of Bar Structures with Constraints on Plastic Deformation, *Engineering Structures*, 1997, **19**, No.1., pp. 19–27.

[10] KALISZKY, S. – LÓGÓ, J., Plastic Behaviour and Stability Constraints in the Shakedown Analysis and Optimal Design of Trusses, *Journal of Structural and Multidisciplinary Optimization*, 2002, **24** No. 2., pp. 118–124.

[11] KALISZKY, S.– LÓGÓ, J.; Layout and Shape Optimization of Elastoplastic Disks with Bounds on Deformation and Displacement. *Journal of Mechanics of Structures and Machines*, 2002, 30(2), pp. 177–191.

[12] KOITER, W. T., General Theorems for Elasto-plastic Solids. In Progress in Soil Mechanics, (eds. I.N. Sneddon, R. Hills) North-Holland, Amsterdam, 1960, pp. 165–221.

[13] KÖNIG, J. A., Shakedown of Elastic-Plastic Structures. Elsevier, Amsterdam. 1987.

[14] LANGE-HANSEN, P., Comparative Study of Upper Bound Methods for the Calculation of Residual Deformation After Shakedown. Dept. of Struct. Engineering and Materials. Technical University of Denmark, Series R No. 49/1998.

[15] MAIER, G., Complementary Plastic Work Theorems in Piecewise-linear Elasto-plasticity. *Int. J. Solids Struct.* 1969, **5**, 261.

[16] MAIER, G.– VITELLO, E., Bounds on Plastic Strains and Displacements in Dynamic Shakedown of Workhardening Structures. *J. Appl. Mech.* 1974, **41**, pp. 443–440.

[17] MELAN, E, Der Spannungszustand eines Mieses-Henckischen Kontinuums bei veränderlichen Belastung. Sitz. Berg Akad. Wis. Wien Abt. 11/a ., 1938, pp. 147–173.

[18] POLIZZOTTO, C. A., Unified Treatment of Shakedown Theory and Related Bounding Techniques. S.M. Arch. 1982, **7**, pp. 19–75.

[19] POLIZZOTTO, C. A., Convergent Bounding Principle for a Class of Elasto-Plastic Strain-Hardening Solids. Int. J. Plasticity. 1986, 2, 359-370.

[20] PONTER, A. R. S., An Upperbound on the Small Displacements of Elastic, Perfectly Plastic Structures, *Journal of Applied Mechanics* 1972, 39, 959-963.

[21] TIN-LOI, F., Deflection Bounding at Shakedown. *J. Struct. Div. Proc. ASCE* 1980, **106**, pp. 1209–1215.

[22] TIN-LOI, F., Optimum Shakedown Design under Residual Displacement Constraints. *Struct. and Multidisc. Optimization,* 2000, 19. 130-139.

[23] EUROCODE 3, Design of Steel Structures, Commission of the European Communities, 1999.