PERIODICA POLYTECHNICA SER. CIV. ENG. VOL. 50, NO. 2, PP. 149–160 (2006)

NEW ASPECT OF THE ANALYSIS OF ELASTO-PLASTIC STRUCTURES WITH TIME-DEPENDENT LOADING

Anna VÁSÁRHELYI

Budapest University of Technology and Economics
H–1521 Budapest, Hungary,
Phone: 36 1 213 9897
e-mail: anna.vasarhelyi@freemail.hu

Received: Nov. 6, 2006

Abstract

In case of time-dependent loading the state variables of a structure depend on space and time and in a significant part of calculations time is the independent parameter. By use of the Hamiltonian principle one can overcome the difficulties if the time is an independent parameter. In this paper a new approach is presented where the solution of the boundary value problem is approximated by an infinite function series of the state variables in time and is obtained by mathematical programming. The space of the structure is discretized at the usual way (finite domain), but the state variables, ordered to the nodes, are continuous functions of time. The problem is solved by mathematical programming in the function space ℓ^2 and in spite of direct solution technique of the mathematical programming, the time-dependent structural response for the time-dependent loading can be followed and the energy dissipation is considered. A comparison of the regular and the new method is discussed, as well. Finally, a numerical example is presented.

Keywords: mathematical programming, elasto-plastic structures, time dependent loading.

1. Introduction

The plastic deformations are caused by the change of material micro structure without cracking and the process is irreversible [13]. Due to the plastic deformations some kinds of energy is dissipated always. Generally, the theories used in structural mechanics for describing elasto-plastic state change with time-independent material flow are formulated on the basis of macroscopic observation of the material behaviour [14] and the dissipation is not considered. The ability of the classical theories is generally accepted (MAIER [10]) to describe realistically the plastic deformation, however, certain recommendations and modifications are necessary for their use in modelling with different phenomena. In the 70-s a collection of studies on elastic-plastic theories by mathematical programming is edited by MAIER and COHN [5]. The mathematical programming methods as helping tools play an important part in the structural plasticity [1, 4, 6, 14, 15] and give the direct solution techniques. It is very common to simplifying the computational model that the authors apply internal variables with different physical meaning (rotation of the plastic hinge, plastic yield stress in shearing) [4, 7, 8, 9, 19].

In the last decades several papers were published in structural plasticity with time-dependent loading. The inelastic structural behaviour with a given load history can be solved by step-by-step methods what are connected to time integration techniques and a variety of space discretization methods. Unfortunately, each of them has significant difficulties during the time stepping analysis due to the lack of information in the load history, constitutive equation problems of describing inelastic behaviour the structures, high computational cost. To decrease the computational work and simplify the analysis several global (or direct) methods were elaborated. The common is in these methods that they provide useful statements about the inelastic behaviour of structures which avoid the complete computation of the history of stress and strain. The need of decreasing computational work and cost on result such an information (or computational results), on the other side where the time dependent structural response can not be followed, the final state is examined only. All these critical thresholds of the direct methods are characterized by the missing bounds in time of the space-cumulative energy dissipation due to the plastic yielding processes. To provide information on quantities which are local aspects of the inelastic structural response to external actions and this information may be upper bounds or lower bounds which are approximations of the history-dependent quantities or only they estimate the required quantities.

In case of time-dependent loading the state variables of a structure depend on space and time and in a significant part of calculations the time is independent parameter. By use of the Hamiltonian principle [12] one can overcome the difficulties if the time is the independent parameter. In this paper a new approach is presented where the solution of the boundary value problem is approximated by an infinite function series of the state variables in time and is obtained by mathematical programming. The space of the structure is discretized at the usual way (finite domain), but the state variables, ordered to the nodes, are continuous functions of time. The problem is solved by mathematical programming in the function space ℓ^2 and in spite of direct solution technique of the mathematical programming, the time-dependent structural response for the time-dependent loading can be followed and the energy dissipation is considered. This mathematical method is a suitable tool in structural plasticity [7, 8]. The presented models include the theoretical results of the earlier studies [16, 17, 18]. Furthermore we will point to the limitation of the classical mechanical models [5]. A comparison of the regular and the new method is discussed as well.

In Chapter 2 the mathematical background is discussed shortly, the model of nonholonom system of structures is presented in Chapter 3, the classical approach can be found in Chapter 4 and Chapter 5 contains the comparison of them. The numerical example is solved by the proposed method in the Chapter 6.

2. Mathematical Background

A short interpretation of the applied mathematical procedure is presented in this chapter. The detailed method can be found in earlier communications [16, 17]. The structure is given in global co-ordinate system and it is discretized in space E^3. Local co-ordinate systems are ordered to each element. By the use of the Hamiltonian principle (the time is independent parameter) the state variables ($\mathbf{u}(x,t)$) are approximated by:

$$\mathbf{u}(x,t) \approx \sum_{i=1}^{n} \mathbf{x}(t) \cdot N(\xi) \tag{1}$$

where $N(\xi)$ notes the shape functions which depend on the position vector (ξ) and the time-dependent coefficients $\mathbf{x}$(t). Within the context of the small displacement theory the position vector does not depend on time. The state variables are unknown vector-vector functions, whose elements depend on time. These elements are described in a function space which is determined on the local co-ordinate axes. This means that the state variables can be given as a vector with function elements on the local co-ordinate axes in a function subspace. The state variables ($\mathbf{x}$(t)) can be expressed by the generalized Fourier series according to the basis of the function space:

$$\mathbf{x}^{\ell}(t) = \sum_{j=1}^{s} \left[\sum_{i=1}^{\infty} \alpha_{ij} P_i(t) \right] \mathbf{e}_j^{\ell},\ \alpha_{ij} \in \Re,\ P_i(t) \in L^2,\ t \in [t_1, t_2] \tag{2}$$

where $\mathbf{e}_j^{\ell}$ (j=1,...,s): j-th unit vector of the local coordinate system ordered to the ℓ-th node, s: number of degrees of freedom at the nodes, P_i(t): i-th element of basis of the function space (orthonormal polynomial system on the interval $[t_1, t_2]$).

The state variables are described on every node in the $F = L_1^2 \times L_2^2 \times ... \times L_s^2$ space. The structure has n nodes, the space of the whole structure is:

$$F^n = (L_2^2 \times L_3^2 \times ... \times L_s^2)^n. \tag{3}$$

In the following the Wolfe's duality theorem is used. Originally the theorem is proved in finite Euclidian space. The primal problem is:

$$\left\{ \min f(x) \;\middle|\; h_i(x) = 0,\ i = 1, \ldots, m,\ g_j(x) \le 0,\ j = 1, \ldots, k,\ x \in E^n \right\},$$

where $f(x)$ denotes the objective function, $h_i(x)$ is the i-th equality and $g_j(x)$ are the j-th inequality constraints, respectively.

The Wolfe's dual problem is:

$$\left\{ \max f(x) + \sum_{i}^{m} v_i h_i(x) + \sum_{j}^{k} u_j g_j(x) \;\middle|\; \nabla f(x) + v_i \nabla h_i(x) + u_j \nabla g_j(x) = 0, \right.$$
$$\left. u_j \ge 0,\ v \in E^m,\ u \in E^k,\ x \in E^n \right\}$$

where v_i, u_j are the dual variables ordered to h_i, and g_j, respectively.

According to the strong duality theorem the minimum value of primal problem is equal to the maximum value of the dual problem. It has been proved [16] that the Wolfe's duality [2] is valid in space ℓ^2. To describe the elasto-plastic process a nonlinear mathematical programming problem is created in space F^n. Transforming the problem into space ℓ^2 [17] the variables of the nonlinear mathematical programming problem are the Fourier coefficients α_{ij}. The result is mapping back into the space F^n. The result of the nonlinear programming problem is a stationer curve. The limitations of the presented model are:

- at least one continuos component has to be assumed,
- the small displacement theory is valid,
- stability problems are neglected,
- inertia forces are not taken into consideration.

The presented mathematical method was successfully used for the optimal limit design of structures by LÓGÓ [7, 8, 9].

3. Nonholonom System of the Elasto-plastic State

To facilitate an interpretation of the problem it is necessary to describe, what type of the energies are taken into consideration. The model is created on the energy minimum theorem. The state variables are defined on the Gaussian points of the discretized elements.

The construction of the *primal problem* of the elasto-plastic state change is the following. The primal state variables of the objective function and the constraints concern to the physically intensive (force type) variables.

The meaning of the objective function is the energy minimum theorem; it contains the sum of the complementary strain and the dissipative energies using their additive properties. The constraints contain the equilibrium equations, the plastic yield conditions, the yield conditions of the dissipation capacity. The force-type boundary conditions are written as an equality or inequality constraints directly by the primal state variables, in the displacement-type boundary conditions the displacements are expressed by the primal state variables. The switch equations are described later.

During the process the plastic nodes are in non-equilibrium state [11] in space F^n. The course of the dissipation [13] is described by the help of force-type internal variables. The state space F^n is extended by the space of internal variables acting on the Gaussian points. In extended space using internal variables the non-equilibrium state becomes an equilibrium one.

The primal problem is in the F^n space:

$$[B]^* \sigma(t) \langle \rho \rangle + [B]^* r(t) + p(t) = 0, \tag{4.a}$$

$$\text{Boundary conditions,} \tag{4.b}$$

$$f_i(\sigma_k(t),\ k = 1, \ldots, z) \leq 0,\ i = 1, \ldots, G \tag{4.c}$$

$$\varphi_i(r_k(t),\ k = 1, \ldots, z) \leq 0,\ i = 1, \ldots, G, \tag{4.d}$$

$$f_i(\sigma_k(t),\ k = 1, \ldots, z) r_i(t) = 0,\ i = 1, \ldots, G,\ k = 1, \ldots, z, \tag{4.e}$$

$$\frac{1}{2}\sigma(t)^* [F]\, \sigma(t)\, \langle\rho\rangle + \frac{1}{2} r(t)^* [A]\, r(t) \to \min,\ \forall t, t \in [t_1, t_2]\,, \tag{4.f}$$

where n is number of the nodes, s is the freedom of nodal displacements, G is total number of the Gaussian points, z is the freedom of the stresses, $[B]^*$ is the transfer matrix of the structure $[n.s, G.z]$, $r(t)$ is the force type internal variable $[n.s]$, $< \rho >$ is a matrix of the Gaussian weights $[G.z, G.z]$ because numerical integral is used, σ(t) is the vector of the stresses $[G.z]$, $p(t)$ is the vector of the external loads acting on the nodes $[n.s]$, $[F]$ is the flexibility matrix $[G.z, G.z]$, $[A]$ is the matrix of the dissipative property of the phenomena $[G.z, G.z]$.

The mechanical interpretations of the problem (4.a-f) are: (4.a)are the equilibrium equations concern to time-functions of stresses, force type internal variables and external forces (the number of the equation is n.s). The boundary conditions concerning to the forces and displacements are expressed by equality and/or inequality constraints. The plastic yield conditions on the Gaussian points of the structure are the inequalities (4.c). The inequalities (4.d) yield the size of the force-type internal variables on the Gaussian points. The equalities (4.e) take a switch role. The objective function (4.f) is the sum of the complementary strain and the dissipation energies.

The model (4.a-f) is operated in the following way: A point is in elastic state, if the plastic yield condition holds the inequality sign in a given point. The value of the force-type internal variables is zero owing to the switch conditions (4.d). In the equilibrium equations the force-type internal variables do not appear. The objective function contains the strain co-energy function only. This is the computational model of time-dependent elastic state.

If the plastic yield condition becomes equality in a given point the structure is in elasto-plastic state. According to the switch equations the force-type internal variable can be zero or not zero. If a force-type internal variable is zero, the classical model of elasto-plastic state-change is formed. If a force-type internal variable is not zero, it is appeared in equations (4.a). That is the equations (4.a) become inequalities in the space F^n. The force-type internal variables are not zero, they appeared in the equilibrium equations and in the extended space the equality relation is valid. This expresses the fact that the point being in plastic state is in non-equilibrium state. The equations (4.a) are equalities in the extended function space

by the force-type internal variables. If the constraint (4.b) is an equality the energy dissipation capacity of the material reaches its maximum in a given Gaussian point, the load caring capacity of the point is lost. If the constraint (4.c) is an inequality the given Gaussian point of the structure is in either elastic state or plastic state and the material is able to dissipate.

The matrix [**A**] contains material constants which characterize the energy dissipation ability of the phenomena. The material constants of matrix [**A**] should be determined with experiments in laboratory.

The *dual problem* is formed by Wolfe's procedure [9] in the space $\ell^2$and it is mapped back to the extended space F^n. The results are: the compatibility equations which are expressed in terms of the physically extensive variables, the inequalities express the direction of the process, the boundary conditions and the sum of the different types of energies are obtained in the objective function.

In this way there is a possibility to check the primal model, because the dual problem has to be clear from a physical point of view.

The Wolfe's dual problem of (4.a-e) is formed as follows (the dual variables $u(t), \lambda(t), \Psi(t)$ and $x(t)$ belong to equalities (4.a), inequalities (4.c), (4.d) and equalities (4.e), respectively, and the dual vectors have function elements):

$$[B]\,u(t) + [F]\,\sigma(t)\,\langle\rho\rangle + \lambda(t)^* \frac{\partial f(\sigma(t))}{\partial \sigma(t)} + x(t)^* r(t) \frac{\partial f(\sigma(t))}{\partial \sigma(t)} = 0, \tag{5.a}$$

$$[B]\,u(t) + \psi(t)^* \frac{\partial \varphi(r(t))}{\partial r(t)} + x(t)^* f(\sigma(t)) + r(t)^*\,[A] = 0, \tag{5.b}$$

$$\text{Boundary conditions,} \tag{5.c}$$

$$\lambda(t) \geq 0, \quad \lambda(t)_i f_i(\sigma_k(t),\ k = 1, \ldots, z) = 0,\ i = 1, \ldots, G, \tag{5.d}$$

$$\psi(t) \geq 0, \quad \psi(t)_i \varphi_i(r_k(t),\ k = 1, \ldots, z) = 0,\ i = 1, \ldots, G, \tag{5.e}$$

$$\tfrac{1}{2}\,\langle\rho\rangle\,\sigma(t)^*\,[F]\,\sigma(t) + \tfrac{1}{2} r(t)^*\,[A]\,r(t) + u(t)^*\,[B]\,\sigma(t)\,\langle\rho\rangle + u(t)^* r(t) + \\ + u(t)^* p(t) + \lambda(t)^* f(\sigma(t)) + \psi(t)^* \varphi(r(t)) + x(t)^* r(t) f(\sigma(t)) \to \max \tag{5.f}$$

$$\forall t, t \in\ t_1, t_2\ ,$$

where $u(t)$ is the vector of the displacements, $\lambda(t)$ and Ψ(t) are the plastic and dissipative multipliers, respectively. The Stieltes derivative is used at the derivation according to a function.

The mechanical interpretations of the problem (5.a-e) are:

(5.a) expresses the compatibility between the displacements and the strains (the number of the equations is G.z). The 1^{st} member means the nodal displacements of the element, the 2^{nd} member gives the displacements due to the elastic stresses and the 3^{rd} member gives the displacements due to the plastic strains in the direction of the gradient of the plastic yield conditions. The measure of these displacements origins from two parts: one is the plastic multiplier and the other is the dissipation in the plastic zone. (5.b) equations express the compatibility due to the dissipation (G.z). In elastic state the equation system (5.b) does not exist since the force-type internal variables are zero. In plastic state the equation system (5.b) consists of the following parts: 1^{st} member means the nodal displacements of the element, 2^{nd} member contains the displacements on the direction of the gradients of the dissipation conditions, 3^{rd} member is zero in plastic state since the function value of the plastic yield conditions become zero, 4^{th} member expresses the displacement type internal variables. The boundary conditions are saved in original form (5.c). The inequalities (5.d) and (5.e) give the sign constrain of the plastic and the dissipation multiplier, respectively. They show the direction of the process, because of the definition of the gradient vector, which expresses an associated flow. The interpretation of the objective function (5.f) can be obtained after the following rearranging: multiplying the equations (5.a) and (5.b) by their dual variables and substituting them into the objective function and rewriting the maximum problem into a minimum one the following function is obtained:

$$\begin{aligned}
&\frac{1}{2}\langle\rho\rangle\,\sigma(t)^*[F]\,\sigma(t) + \frac{1}{2}r(t)^*[A]\,r(t) - u(t)^*p(t) - \\
&- \lambda(t)^* f(\sigma(t)) - \psi(t)^*\varphi(r(t)) - x(t)^* r(t) f(\sigma(t)) + \\
&+ x(t)^* r(t) f(\sigma(t)) + \lambda(t)^* \frac{\partial f(\sigma(t))}{\partial \sigma(t)}\sigma(t) + \\
&+ x(t)^* r(t) \frac{\partial f(\sigma(t))}{\partial \sigma(t)}\sigma(t) + \psi(t)^* \frac{\partial \varphi(r(t))}{\partial r(t)} r(t) \to \min
\end{aligned} \tag{6.a}$$

The 1^{st} member expresses the strain energy, the 2^{nd} member is the dissipative energy and the 3^{rd} member gives the energy due to the external loads. The 4^{th} and the 5^{th} members are zero, if the inequalities (4.c) are less then zero, the correspondent dual variables are zero due to the constraints (5.d) and (5.e). If the inequalities become equalities the value of the yield functions are zero. The sum of the 6^{th} and the 7^{th} members is zero. The 8^{th}, the 9^{th} and the 10^{th} members are zero if functions f and φ are the homogenous functions of the stresses and the force-type internal variables [18]. The dual of the problem (4.a-e) is after the above mentioned modifications:

$$-\frac{1}{2}\langle\rho\rangle\,\sigma(t)^*[F]\,\sigma(t) - \frac{1}{2}r(t)^*[A]\,r(t) + u(t)^*p(t) \to \min \tag{6.b}$$

The objective function expresses the principle of energy conservation.

4. Connection between the Classical and the Presented Models

In classical history analysis [10] the time interval is divided into time steps. The path-dependent nature of strains is allowed within each step and the external loads are taken as a constant during each step. It is supposed that at every step the structure is in local equilibrium state which instead of that is the velocities of the strains satisfy the governing constitutive laws. Keeping the earlier notations the primal problem can be formed by the use of the velocities of the stresses:

$$[B]^* \dot{\sigma}(t) \langle \rho \rangle + [B]^* r(t) + p(t) = 0, \tag{7.a}$$

$$\text{Boundary conditions,} \tag{7.b}$$

$$f_i(\sigma_k(t), k = 1, \ldots, z) \leq 0,\ i = 1, \ldots, G \tag{7.c}$$

$$\frac{1}{2} \dot{\sigma}(t)^* [F] \dot{\sigma}(t) \langle \rho \rangle \to \min,\ \forall t,\ t \in [t_1, t_2]\,. \tag{7.d}$$

Using Wolfian process, the dual problem is:

$$[B] \dot{u}(t_1) + [F] \dot{\sigma}(t_1) \langle \rho \rangle + \dot{\lambda}(t_1)^* \frac{\partial f(\dot{\sigma}(t_1))}{\partial \dot{\sigma}(t_1)} = 0, \tag{8.a}$$

$$\text{Boundary conditions,} \tag{8.b}$$

$$\lambda(t) \geq 0, \quad \lambda(t)_i f_i(\sigma_k(t),\ k = 1, \ldots, z) = 0,\ i = 1, \ldots, G, \tag{8.c}$$

$$-\frac{1}{2} \langle \rho \rangle \dot{\sigma}(t)^* [F] \dot{\sigma}(t) - \frac{1}{2} r(t)^* [A] r(t) + \dot{u}(t)^* \dot{p}(t) \to \min\,. \tag{8.d}$$

In the classical model the dissipation is neglected, so the force-type internal variables do not appear.

It has to be examined what kind of assumptions are necessary to obtain the classical model [5, 14] from problem (4.a-e).

The derivatives of the equilibrium conditions (4.a) are formed and the objective function (4.e) according to time at a given point t_0 can be written as follows:

$$[B]^* \dot{\sigma}(t_0) \langle \rho \rangle + \dot{p}(t_0) = 0\,, \tag{9.a}$$

$$\text{Boundary conditions,} \tag{9.b}$$

$$f_i(\sigma_k(t_0),\ k = 1, \ldots, z) \leq 0,\ i = 1, \ldots, G, \tag{9.c}$$

$$\left(\frac{1}{2}\langle\rho\rangle\,\sigma(t)^*\,[F]\,\sigma(t)\right)^{\cdot}\Big|_{t=t_0} = \langle\rho\rangle\,\dot{\sigma}(t)^*\,[F]\,\sigma(t)|_{t=t_0} \to \min \tag{9.d}$$

Let us approximate the objective function (9.d) with Taylor series at the point t_0:

$$\begin{aligned}\dot{\sigma}(t)^*\,[F]\,\sigma(t) \approx \dot{\sigma}(t_0)^*\,[F]\,\sigma(t_0) + \dot{\sigma}(t_0)^*\,[F]\,\dot{\sigma}(t_0)(t-t_0) + \\ + \ddot{\sigma}(t_0)^*\,[F]\,\sigma(t_0)(t-t_0)\end{aligned} \tag{10}$$

The classical model is based on a step-by-step method, the differences between the steps are examined, so the stresses at point t_0 can be taken zero and the stresses at point t_1 contain the increment. The final form of expression (10) is:

$$\dot{\sigma}(t_1)^*\,[F]\,\sigma(t_1) \approx \dot{\sigma}(t_0)^*\,[F]\,\dot{\sigma}(t_0)(t_1 - t_0).$$

It can be seen that the minimum value of the above objective function is influenced by the time difference $(t_1 - t_0)$, but the location of the minimum is not. The classical model is obtained and it is a quadratic programming problem if the stresses are linear functions in time. This means that the 'minimum principle of the velocities' is valid in this case only.

Summarizing the statements above one can obtain: the problem pair (4) and (5) of the elastic-plastic state change include the classical problem pair [5], at a given time point if

- the dissipation is neglected,
- displacements are described by rectilinear uniform motion,
- the discretized time points are independent from each other.

This kind of assumption is not allowed in many cases. The problem pair (4) and (5) can be used for a wider class of structural material. The classical and the presented models indicate different methods for numerical solution. In case of classical problem pair a step-by-step – a path following – method is reasonable. In case of the problem pair (4) and (5) the functions of the state variables are approximated on a given time interval and the method described above is suggested.

5. Numerical Example

As an illustration of the proposed computational methods, let us consider a three-supported beam with time-dependent loading. The data of the structure can be seen in *Fig. 1.* The loads act on the nodes. The time-dependent function of the external loads is given by the following formula:

$$F(t) = 9.75 + 0.25t + 0.75t^2 + 1.25t^3.$$

The loads are approximated by the Legendre polynomial system up to four members. The structure is divided into 6 members with 7 nodes. The unknowns are the

moments and the shear forces of the members. The rigidity (EI) of the beam is 200 kNm^2.

The material constant of the dissipation is considered by a diagonal matrix **[A]** and $\langle a_i \rangle = \langle 0.09 \rangle$.

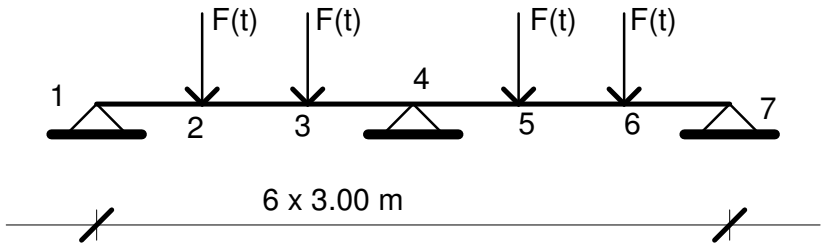

Fig. 1. Example

On the basis of the general form, the statically admissible internal forces are determined by the equilibrium equations and force boundary conditions. The objective function contains the complementary potential energy and the dissipative energy. The displacement boundary conditions are taken into consideration in the objective function.

After the transformation to solve this problem a sequential quadratic programming system is used. The results (the evolution of the time-dependent bending moments at the nodes without the residual moments) can be seen in *Fig. 2* and at *Table 1*. If there are some restrictions for the internal forces they give inequalities in the mathematical programming problem. In our example, the Huber-Mises-Hencky yield condition is used. In *Fig. 2* the formation of the plastic hinges can be followed during the process

Table 1. Time-dependent bending moments at the nodes

	time				
node	0.	0. 01.	0.1	0.5	0.6
1	0	0	0	0	0
2	−19,9947	−20,0	−20,0	−20,0	−20,0
3	−13,3195	−15,6351	−20,0	−20,0	−20,0
4	20,0	16,6418	8,845	2,6318	0
5	−13,3195	−15,6351	−20,0	−20,0	−20,0
6	−19,9947	−20,0	−20,0	−20,0	−20,0
7	0	0	0	0	0

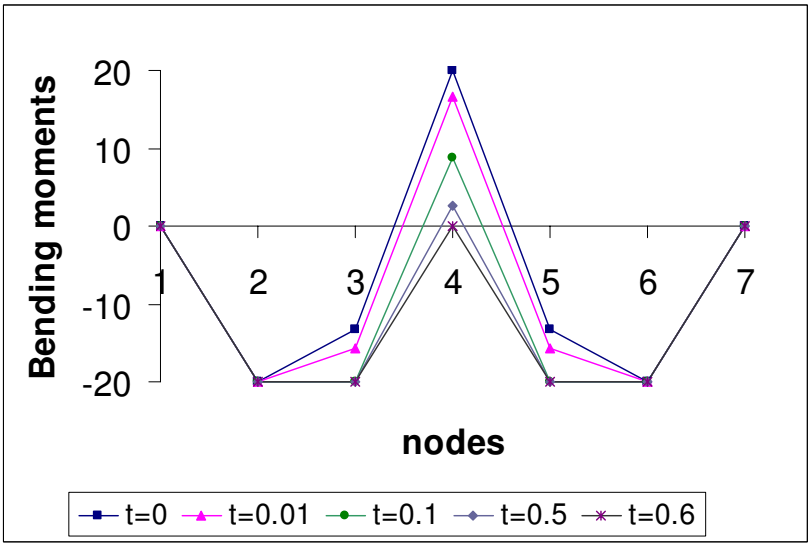

Fig. 2. Evolution of bending moments in time

To control this new type of solution procedure and the obtained results, the above problem was solved by the use of a commercial program which was based on the classical theory. At time point zero one can obtain the same bending moments as it can be seen in *Fig. 2*. All the other cases are not comparable because the classical model neglects the effect of the energy dissipation.

6. Conclusion

A time-dependent static problem is solved in function space where the skeleton of the structure is made of elasto-plastic material and the energy dissipation is taken into consideration. Generalized Fourier series is used to approximate the function of the state variables. For the sake of simplicity the stability of the elements is not considered because in that case the convexity does not exist. The elaborated method can be extended to problems dealing with the analysis and design of complex engineering structures (containment, concrete under fire load, hydro-thermo-chemo-mechanical processes).

Acknowledgements

The present study was supported by The Hungarian National Scientific and Research Foundation (OTKA # K 62555).

References

[1] BAUER, J., Algorithms of Nondifferentiable Optimization in Discrete Optimum Structural Design, *ZAMM Z. angew Math. Mech.* **72** (1992) 6, pp. 563–566.

[2] BAZARRA, M.S. – SHETTY, C.M., Nonlinear Programming, Theory and Algorithms, John Wiley & Sons , New York, 1979.

[3] BHAKTA, P.C. – ROYCHANDHURI, S., Optimization in Banach Spaces, Journal of Math. Analysis and Applications, No.134, pp.460–470, 1988.

[4] CORIGLIANO, A. – MAIER, G. – PYCKO, S., Dynamic shakedown analysis and bounds for elastoplastic structures with nonassociative, internal variable constitutive laws. *Int. J. Sol. Struct.*, **32**, pp. 3145–3166, 1995.

[5] COHN, M.Z. – MAIER, G. (Eds.), Engineering Plasticity by Mathematical Programming, Pergamon Press, New York, 1979.

[6] KALISZKY, S., Plasticity, Theory and Engineering Application, Elsevier Science Publishers, Amsterdam, 1989.

[7] LÓGÓ, J., Optimal Limit Design of Elasto-Plastic Structures for Time-dependent Loading, Journal of Structural and Multidisciplinary Optimization. DOI 10.1007/s00158-006-0053-7, 2006.

[8] LÓGÓ, J., Optimal Design of Skeletal Structures in Time-Dependent Elasto-Plastic State, In: Engineering Design Optimization, Product and Process Improvement, ed. by O.M. Querin, J. Sienz, P.D. Gosling and V.V. Toropov, University of Leeds Press, LS2 9JT, UK, CD paper, ISBN 0-85316-246-8, 2005.

[9] LÓGÓ, J., Layout Optimisation Elasto-plastic Structures Subjected to Time-Dependent Loading, In: The 6th World Congress of Structural and Multidisciplinary Optimization, eds. Herskovits, J., Mazorche, S., Canelas, A. Rio de Janeiro, Brasil, CD paper, ISBN 85–285–0070-5, 2005.

[10] MAIER, G., A Matrix Structural Theory of Piecewise Linear Elastoplasticity with Interacting Yield Planes, *Meccanica*, **5**, 1970.

[11] MUSCHIK, S., Non-equilibrium Thermodynamics with Applications to Solids, Springer-Verlag, Wien, New York, 1993.

[12] RUBINSTEIN, M. F., Structural Systems- Statics, Dynamics and Stability, Prentice-Hall, Inc., 1970.

[13] SAWCZUK, A., Mechanics and Plasticity of Structures, Ellis Horwood Ltd., Chichester, 1989.

[14] SAWCZUK, A. – BIANCHI, G.(EDS.), Plasticity Today, Elsevier Science Publishers, London, 1985.

[15] LLOYD SMITH, D., Mathematical Programming Methods in Structural Plasticity, Springer Verlag, Wien, New York, 1990.

[16] VÁSÁRHELYI, A. – LÓGÓ, J., Mathematical Models of Processes in Mechanics. *J. Struct. Opt.*, **8**, pp. 138–145, 1994.

[17] VÁSÁRHELYI, A., Mathematical Models for Processes of Structural Analysis. Computer Assisted Mechanics and Engineering Sciences, 3, 1996, pp.297-316.

[18] VÁSÁRHELYI,A. – LÓGÓ,J., Dynamic Elasto-Plastic Analysis of Structures by Mathematical programming, *Acta Technica Acad. Sci. Hung.*, **107**, (1-2),. pp.137–148, 1995-96.

[19] VERHÁS, J., Thermodynamics and Reology. -in Hungarian- Műszaki Könykiadó Publisher, Budapest, 1985.