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THE STRESSED SHAPE OF AXISYMMETRIC HEXAGONAL NETS

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Abstract

The paper presents the theoretical background to a numerical method for calculating the meridian of axisymmetric stressed shapes of infinitesimal hexagonal grids. Two types of stressing are assumed, one with a pure tension and another when tension is combined with twist. Also some results of an illustrative example are given.

Keywords: hexagonal nets, underconstrained structures, membrane analogy.

1. Introduction

In a previous paper ([1]) , the author presented an analysis of axisymmetric shapes of stressed infinitesimal Chebyshev nets. Chebyshev nets consist of two sets of continuous, flexible, inextensional fibres which are connected to each other at equidistant points. In original state, the net has a square network. It can be deformed like woven tissues. Most analyses assume the distance of points of connections infinitesimally small (infinitesimal net), and replace the net by a membrane with incomplete rigidities. Both finite and infinitesimal Chebyshev nets are typical underconstrained structures [3] which can develop large displacements without any change in length in fibre directions.

The shape of stressed membranes of incomplete rigidities is determined using the consideration that inextensional deformations modify the initial shape until the surface gets rigid, that is, the missing rigidities are not necessary to develop stresses resisting further deformation. Equilibrium conditions for the possible membrane forces can be used to determine the assumed shape.

For axisymmetric stressed shapes the equilibrium of membrane components requires

$$\frac{n_a}{n_\vartheta} = -\frac{r_\alpha}{r_\vartheta}. \qquad (1)$$

n_α, and n_θ in *Eq* (1) are membrane forces, r_α, and n_θ are radii of curvatures in meridian, and in annular directions, respectively.

If initial (unstressed) configuration of the net is a cylinder of the radius a with given directions of the fibres, then the stressed shape is a surface of revolution with the meridian $r(z)$. Ratio r/a determines the changes in directions of the intersecting fibres, directions of the fibres determine the ratio of n_α, and n_θ . On the other hand, ratio of r_α, and n_θ can be expressed using derivatives of $r(z)$ as

$$-\frac{r_\alpha}{r_\theta} = \frac{1 + \left(r'\right)^2}{rr''}, \tag{2}$$

in this way an equation can be constructed which only contains $r(z)$, $r(z)$', and $r(z)$' as unknown functions. Since $r(z)$" can be expressed from this equation, the rearranged equation permits us a numeric integration of $r(z)$ after taking appropriate initial values for $r(z_0)$, and $r(z_0)$'. Moreover, for specific initial values, analytic function for the meridian of the stretched axisymmetric Chebysev net can be obtained.

Hexagonal nets are fundamental patterns of assemblies in nature as well as in the theory of nets. Unlike Chebyshev nets, they consist of non-continuous fibres (*Fig. 1*). In initial state, fibres of hexagonal nets form a regular hexagonal network. Finite and infinitesimal hexagonal nets are also underconstrained surfaces, their deformability is more or less the same as that of knitted fabrics or crochets.

In the next chapter, homogeneous deformations of finite and infinitesimal plane hexagonal nets are analysed. The results are used in Chapter 3 to determine the meridian of axisymmetric stressed shapes of infinitesimal hexagonal nets.

2. Homogeneous deformations of plane hexagonal nets

Homogeneous deformation means in our case that all hexagons of the net deform in the same way. That is only possible if parallel fibres of the net stay parallel after the deformation. Two types of homogeneous deformation will be assumed: symmetric, and asymmetric deformation. Symmetric means that the deformation does not upset the symmetry of the net.

First-order theory of deformations assumes that powers of small changes in length can be neglected. In first-order theory, regular hexagonal nets are discrete models for surfaces that can only perform deformations which preserve the area of the surface meanwhile they develop a hydrostatic state of stress against stretching. These properties can be checked by analysing the deformations of the elementary cell of the net shown in *Fig. 1* and by checking the equilibrium of nodal forces.

Actually, if fibres of the cell are assumed absolutely rigid and powers of small changes in length are neglected, then small deformations of the cell preserve the area

$$A = 3\sqrt{3}l^2. \tag{3}$$

The analysis of the neglected terms shows that $A = 3\sqrt{3}l^2$ is the maximum area of the elementary cell.

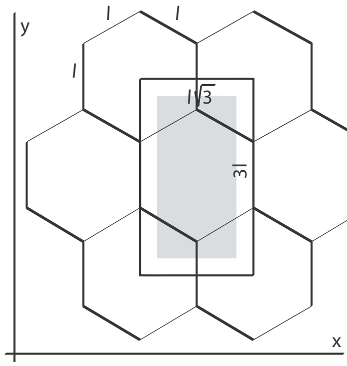

Fig. 1. Elementary cell of the hexagonal net in initial state

Analogous joints of the net displace as if they were points of a continuous surface subjected to a homogeneous state of strain with principal strains

$$\varepsilon_1 = -\varepsilon_2. \tag{4}$$

This surface will be referred to as the replacement membrane of the net.

Forces acting at a common point of application at lines of action in the directions α, $\alpha + \pi/3$, and $\alpha + 2\pi/3$ must be the same. That means, in initial state of the net, forces acting at the fibres are either zero or equal. This force distribution is analogous to hydrostatic states of stress of continuous membranes. If the value of the fibre forces is N, then principal membrane stresses of the replacement membrane are

$$n_1 = n_2 = \frac{\sqrt{3}N}{3l}. \tag{5}$$

Strains developed in stressing the net, cannot be assumed small. Consequently, first-order theory has to be replaced by a geometrically exact theory of deformation.

2.1. *Geometrically exact symmetric homogeneous deformation of the net*

Let us first analyse finite symmetric deformations shown in *Fig. 2*. The changes in the geometry and in the equilibrium conditions of nodal forces will be expressed using α as geometric parameter.

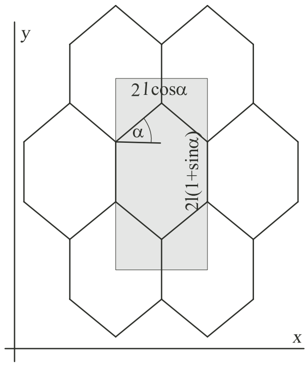

Fig. 2. Elementary cell of the symmetrically deformed hexagonal net

Finite strains in directions x and y can be expressed matching side lengths of the elementary cell in initial and deformed states of the net. Ratios of the lengths are

$$(1+\varepsilon_x) = \frac{2\sqrt{3}\cos\alpha}{3}, \tag{6}$$

$$\left(1+\varepsilon_y\right) = \frac{2\,(1+\sin\alpha)}{3}. \tag{7}$$

Assuming N the fibre force at the bars in direction y, equilibrium of joint forces can be ensured by forces $N/(2\sin\alpha)$ at the inclined bars. Principal directions of stresses of the replacement membrane are directions x and y, the replacement membrane forces are

$$n_x = \frac{N\cos\alpha}{2l\sin\alpha\,(1+\sin\alpha)}, \tag{8}$$

$$n_y = \frac{N}{2l\cos\alpha}, \tag{9}$$

$$n_{xy} = 0. \tag{10}$$

Limiting values of α are 0, and $\pi/2$. If $\alpha = 0$, that is, the inclined fibres get parallel to the parallel circles,

$$1+\varepsilon_x = \frac{2\sqrt{3}}{3}, \quad 1+\varepsilon_y = \frac{1}{2}, \quad \frac{n_y}{n_x} = 0,$$

if $\alpha = \pi/2$, that is, if inclined fibres get parallel to the axis of rotation,

$$1+\varepsilon_x = 0, \quad 1+\varepsilon_y = 2, \quad \frac{n_y}{n_x} = \infty.$$

2.2. Asymmetric homogeneous deformations of the net

Asymmetric homogeneous deformations are analysed using two angles as geometric parameters. One is α, the same as used in the previous section, another is γ which is the angle of inclination of fibres 1 in *Fig. 3*.

General description of homogeneous deformation needs a third geometric parameter, e.g. the angle of the rigid body-like rotation of the whole net, but our analysis does not need that generalization.

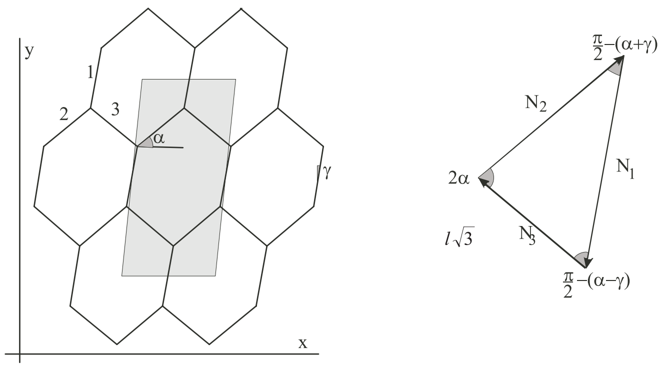

Fig. 3. Elementary cell, and vector diagram of equilibrium forces in the state of skew finite deformation

Finite deformations in directions x and y are

$$(1+\varepsilon_x) = \frac{2\sqrt{3}\cos\alpha}{3}, \tag{11}$$

$$\left(1+\varepsilon_y\right) = \frac{2\,(\cos\gamma + \sin\alpha)}{3}, \tag{12}$$

tangent of the finite angle of distortion γ_0 is

$$\tan\gamma_0 = \frac{\sin\gamma}{\sin\alpha + \cos\gamma} \quad . \tag{13}$$

Ratios N_2/N_1, andN_3/N_1 of bar forces acting at fibres 1, 2, and 3 can be expressed from the vector diagram of equilibrium forces (*Fig. 3*.) as

$$N_2/N_1 = \frac{\cos(\alpha-\gamma)}{\sin 2\alpha}, \quad N_3/N_1 = \frac{\cos(\alpha+\gamma)}{\sin 2\alpha}. \tag{14}$$

Replacement membrane forces can be obtained in two steps. First, oblique components membrane forces can be calculated, which are parallel with the sides of the deformed elementary cell, then components can be expressed in usual orthogonal resolution. Neglecting details, the lengthy procedure results in

$$n_x = \frac{N_1 \left(2\cos^2\alpha\cos\gamma + \sin\alpha\cos^2\gamma\right)}{2l\sin 2\alpha\,(\cos\gamma + \sin\alpha)}, \tag{15}$$

$$n_y = \frac{N_1\cos\gamma}{2l\cos\alpha}, \tag{16}$$

$$n_{xy} = \frac{N_1\sin\gamma}{2l\cos\alpha}. \tag{17}$$

Limiting values of γ are $\pm\,\pi/2$.

3. Stressing cylinders of hexagonal net

A rotationally symmetric stressed hexagonal net is produced in steps as follows. First, a regular plane net is bent and welded into a cylinder of the radius and height a, H, respectively. Then, cylindrical edges are fixed to rigid boundary rings. Finally, the distance of the boundaries is increased until the net gets stressed.

Bending will be assumed in two ways. One when generators of the cylinder are parallel with x (Case 1), and another when they are parallel with y (Case 2).

Also stretching will be assumed in two ways. One when the distance of the boundaries is increased with parallel moving in direction of the axis of rotation, and another when the parallel moving is combined with a rotation of the boundary circles. Both types of stressing preserve the axisymmetric nature of the cylinder.

3.1. Analysis using first-order infinitesimal theory

In first-order theory of infinitesimal deformations, invariance of area of the elementary cell can be assumed. On the basis of that invariance, the problem of determining stressed shape of the cylinder can be directly converted into a minimal surface problem: what is the meridian of the axisymmetric minimal surface of surface area $2\pi Ha$ and of boundary radii a. Cases 1 and 2 are the same, and rotating of the boundary circles has no effect on the solution.

In [2], a detailed analysis of this problem is presented in a different context. It is proven, that the problem can only be solved for limited values of H/a, and also a dual solution exists. Meridians are catenary lines

$$r = C\cosh\frac{z}{C} \quad . \tag{18}$$

Parameter C and the changed height h of the net emerge as solutions of *Eqs*(19a-b):

$$C\cosh\frac{h}{2C}=a, \tag{19.a}$$

$$2\pi Ch+\pi C^2\sinh\frac{h}{C}=2\pi Ha. \tag{19.b}$$

3.2. *Analysis using geometrically exact infinitesimal theory*

Since first order theory yields reliable results only in a drastically limited range of parameters, a geometrically exact infinitesimal theory is applied. ‘Infinitesimal’ means that a replacement membrane is used, ‘geometrically exact’ means that geometric relations at infinitesimal neighbourhood of points of the membrane are analysed as if they were homogeneous plane deformations in the tangent plane, that is, in the cases of symmetric deformations *Eqs.* (4)-(10), in the case of asymmetric deformation *Eqs.* (11)-(17) can be used.

3.2.1. *Stressing with pure tension*

Case 1

In Case 1 meridian of the replacement membrane coincides with direction x (see *Fig.* 2). Consequently, $n_\alpha = n_x, n_\theta = n_{y,}\ \varepsilon_\alpha = \varepsilon_x, \varepsilon_\theta = \varepsilon_y$,

$$\frac{n_a}{n_\theta}=\frac{\cos^2\alpha}{\sin\alpha\,(1+\sin\alpha)}, \tag{20}$$

and α varies within the limits $\pi/6$, and zero.

According to the procedure outlined in Section 1, $r(z)$” can be expressed on the basis of *Eqs.* (1)-(2), and *Eqs.* (6)-(7) as

$$r''=\frac{1+\left(r'\right)^2}{r}\frac{\sin\alpha\,(1+\sin\alpha)}{\cos^2\alpha}. \tag{21}$$

Connection between r, and α can be obtained using

$$\frac{r}{a}=(1+\varepsilon_\theta)=\frac{2\,(1+\sin\alpha)}{3}. \tag{22}$$

Let the length of an infinitesimal section of the generator of the lattice cylinder be ds_0. The change in length of ds_0 due to stressing is

$$ds_\alpha=(1+\varepsilon_\alpha)\,ds_0=\frac{2\sqrt{3}\cos\alpha}{3}ds_0. \tag{23}$$

Differentials dz, and dr belonging to ds_0 are

$$dz = \frac{ds_\alpha}{\sqrt{1+(r')^2}}, \quad dr = \frac{ds_\alpha r'}{\sqrt{1+(r')^2}}. \tag{24}$$

Eqs. (21)-(26) enable us to construct a step-by-step method for computing coordinates z, and r of meridians belonging to initial values of z, r>2a/3, and r'.

Case 2

In Case 2 directions x and y change, consequently,

$$r'' = \frac{1+(r')^2}{r} \frac{\cos^2\alpha}{\sin\alpha\,(1+\sin\alpha)} \tag{25}$$

and the connection between r and α can be calculated from

$$\frac{r}{a} = (1+\varepsilon_\theta) = \frac{2\sqrt{3}\cos\alpha}{3}, \tag{26}$$

where α varies within the limits $\pi/6$, and $\pi/2$. The equation shows that r can take arbitrary small values.

$$ds_\alpha = (1+\varepsilon_\alpha)\, ds_0 = \frac{2\,(1+\sin\alpha)}{3} ds_0 \tag{27}$$

On the basis of *Eqs* (22)-(27) a similar step-by-step method for computing coordinates z, and r of meridians can be constructed as used in Case 1.

3.2.2. Stressing with a normal force and a torque

Equilibrium conditions for axisymmetric membrane forces of a shell of rotation can also be met by assuming membrane force $n_{\alpha\theta}$ provided the equation

$$2r^2\pi n_{\alpha\theta} = T \tag{28}$$

applies to each parallel circle of the shell. T has a direct mechanical meaning in *Eq.* (28). It is a torque. A similarly constant value is

$$2r\pi \frac{n_\alpha}{\sqrt{1+(r')^2}} = F, \tag{29}$$

which is a tensile force acting at the axis of rotation. The ratio of T and Fa is also constant.

In Case 1, the only effect of torque T is that forces at fibres inclined to the right and left get different. This change does not modify the stressed equilibrium shape, provided the tensile force is sufficiently large to keep inclined fibres of both directions in tension.

In Case 2, fibres of meridian direction cannot transfer shear. Fibres 1 also have to get inclined and the deformation becomes asymmetric. The angle of inclination is determined by the equation

$$\frac{T}{F} = r\sqrt{1 + (r')^2}\tan\gamma. \tag{30}$$

Eq. (30) shows that γ varies along the meridian.

Inclination of fibres 1 changes the equilibrium shape of the stressed net. *Eqs.* (1)-(2), and *Eqs.* (12)-(13) yield for the second derivative of r

$$r'' = \frac{1 + \left(r'\right)^2}{r}\frac{2\cos^2\alpha + \sin\alpha\cos\gamma}{2\sin\alpha\left(\cos\gamma + \sin\alpha\right)} \quad . \tag{31}$$

Connection of α and r is the same as in Case 2 in Section 3.2.2.

$$\frac{r}{a} = (1 + \varepsilon_\theta) = \frac{2\sqrt{3}\cos\alpha}{3}, \tag{32}$$

but the changed length of ds_0 is different:

$$ds_\alpha = (1 + \varepsilon_\alpha)\, ds_0 = \frac{2\left(\cos\gamma + \sin\alpha\right)}{3} ds_0. \tag{33}$$

The step-by-step method of the solution can be the same as used before.

4. Illustrative example

In *Fig. 4*, the results of an illustrative example are presented. The half of the meridian of a stressed infinitesimal net is plotted for Case 1, Case 2, and for Case 2 assuming also a torque $T = 0.4aF$. Ratio H/a is 4, the meridian of the unstressed net is drawn with dotted line.

Only geometrically exact theory has been used because $H/a = 4$ does not permit any mathematical solution of the minimal surface problem.

Meridian belonging to Case 1 obeys the limitation $r \geq 2a/3$ and its first section is close to a straight line indicating the limit value of r. Increasing ratio H/a results in small changes in the last section, only the length of the almost straight section increases.

In Case 2, the meridian looks different, because the limiting value of r is zero and the meridian gets the closer to zero the higher the ratio H/a is.

The third meridian shows that torque reduces both the stressed height and the minimum radius. For increasing values of the torque the difference of twisted and not twisted nets rises, beyond a limiting value of T/aF the problem has no solution.

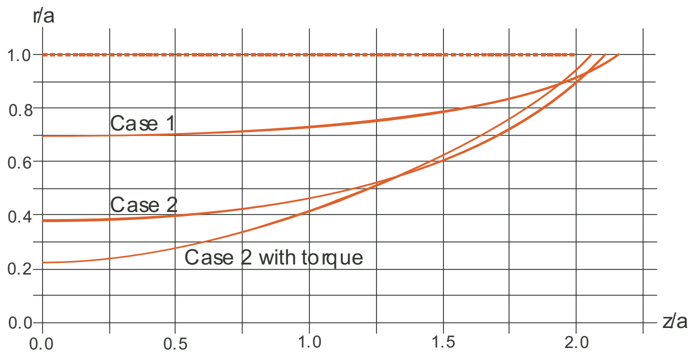

Fig. 4. Meridians of a stressed hexagonal net

5. Conclusions

The results have shown that the analysis of axisymmetric shapes of stressed infinitesimal hexagonal nets requires the use of a geometrically exact infinitesimal theory of deformation. Meridians of the stressed net essentially depend on the direction of the non-continuous fibres. Also the effect of twist is different if the orientation of the fibres is different.

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