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FREE VIBRATIONS OF A PARABOLOID SHELL

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Abstract

The paper presents an application of the method of the generator function. The method is used in the analytic investigation of the free vibrations of a parabolic antenna dish, which is considered as a thin shell. For solving the differential equation system of the vibration problem, the method of the generator function is applied. This method is based on the generalization of determinants and cofactors of quadratic matrices.

In an illustrative example the natural frequencies of the paraboloid are compared with those of a circular plate having the same radius and material properties as the paraboloid, and also with those of the circular plate resting on a fictitious elastic foundation.

Keywords: vibration, surface structure, analytic solution, modal analysis, generator function.

1. Introduction

The paper presents the analytic vibration analysis of a levitating parabolic antenna dish. The solution was obtained by using the method of the generator function. The antenna is assumed as a thin, flat shell with the shape-function

$$z = \frac{f r^2}{a^2} \tag{1}$$

in an r, ϑ, z cylindrical co-ordinate system (*Fig 1.)*. In Eq(1) a is the boundary radius and f is the height of the dish. This paraboloid can also be assumed as a replacement of a flat calotte cut out of a sphere of the radius

$$R = \frac{a^2}{2f}$$

Uniform mass distribution and constant thickness are assumed, the material is homogeneous and isotropic with the elastic constants E and ν. Zero damping and no co-vibrating masses are considered.

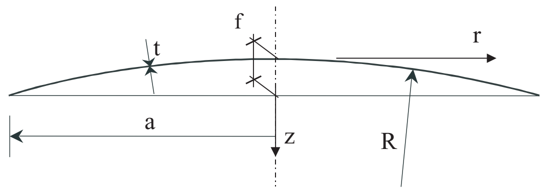

Fig. 1. Geometric data of the shell

The governing equations of the analysis are those of Marguerre's differential equation system of bent shallow shells [1, 2]

$$B_1(w) - P(z, F) = p,$$
$$B_2(F) + P(z, w) = 0$$

in which w is the normal displacement of the middle surface, F is the stress function of membrane forces, and p is the function of external loads. Operators B_1, and B_2 are fourth order linear partial differential operators; for homogeneous isotropic shallow shells of constant thickness, $B_1 = K\Delta\Delta$, and $B_2 = \frac{1}{Et}\Delta\Delta$, where Δ is the two-dimensional Laplace operator and $K = \frac{Et^3}{12(1-\nu^2)}$.

P is called Kármán's shell operator. It is a bilinear partial differential operator of second order for both operands. For a given surface function z, Kármán's shell operator assumes the form of a linear partial differential operator of varying coefficients. For paraboloid surfaces, its coefficients are constant in Cartesian co-ordinate system.

Similarly to Kirchoff's plate theory, Marguerre's equations neglect the out of plane shear deformations.

2. The Differential Equation System and Boundary Conditions of the Problem

In vibration problems, loads are inertia forces expressed using the second time derivative of w. When appropriate derivatives of z and time derivative of w are substituted into Marguerre's differential equation system, the following equations emerge:

$$K\Delta\Delta w - \frac{1}{R}\Delta F = -\rho t\frac{\partial^2 w}{\partial \tau^2}, \tag{2}$$

$$\frac{1}{Et}\Delta\Delta F + \frac{1}{R}\Delta w = 0, \tag{3}$$

in which ρ is the density of the material of the shell, τ is the time variable. Since inertia forces are not associated in Eq(2) with F, vibration modes with zero transversal displacement are excluded from the investigation.

The boundary conditions of the problem have to be stated at the free boundary circle $r = a$. Conditions for vanishing the boundary values of internal forces and moments are:

$$N_r = 0, \qquad N_{r\vartheta} = 0, \qquad M_r = 0, \qquad M_{r\vartheta} = 0, \qquad Q_r = 0\,. \tag{4}$$

3. Analysis of Vibration as Eigenvalue Problem of the Differential Equation System

3.1. Introduction of the Generator Function

For solving the vibration problem, the method of the generator function is applied. The method of the generator function has been worked out for solving systems of ordinary differential equations with constant coefficients. It is based on the generalization of determinants and co-factors of quadratic matrices.

The basic principle of the method is as follows:

Differential operations, involved in linear differential equation systems with constant coefficients are permutable, like scalar multiplications. That means, multiple differential operations can be performed in arbitrary sequences.

Homogeneous linear differential equation systems for unknowns y_1, ..., y_n can be written in a vectorial form as

$$\Theta \mathbf{y} = \mathbf{0}. \tag{5}$$

In this equation Θ is an n-th order quadratic matrix

$$\Theta = \left[\theta_{ij}\right], \qquad i, j = 1, ...n,$$

the elements of which are permutable operators. The formal permutativity allows us to produce the operator determinant $det(\Theta)$, as if Θ were a common quadratic matrix. As the determinant of a matrix consists of products of its elements, $\det(\Theta)$ consists of products of permutable differential operators; that is, $\det(\Theta)$ is a higher order differential operator. The order of $\det(\Theta)$ is the order of the differential equation system. Co-factors (the signed minors of Θ)

$$(-1)^{i+j} \det(\Theta)_{ij} \qquad i, j = 1, ...n$$

and the cofactor matrix of Θ

$$\mathbf{Cof}(\Theta) = \left[(-1)^{i+j} \det(\Theta)_{ij}\right] \qquad i, j = 1, ...n.$$

can also be produced.

According to the Lagrangian expansion theorem of determinants, for any values of i

$$\sum_{j=1}^{n} (-1)^{i+j}\, \theta_{ij} \det(\Theta)_{ij} = \det(\Theta)\,, \tag{6}$$

and for each couples of values $i \neq k$, equations

$$\sum_{j=1}^{n} (-1)^{k+j}\, \theta_{ij} \det(\Theta)_{kj} = 0 \tag{7}$$

identically hold. These equations also apply to operator matrices with permutable operators. If function H meets the homogeneous differential equation

$$\det(\Theta)\, H = 0, \tag{8}$$

then equations

$$\sum_{j=1}^{n} \theta_{ij} \left[(-1)^{k+j}_{kj} \det(\Theta)\, H \right] = 0 \tag{9}$$

hold for each couples of values, inclusively $i = k$, hence, equation

$$[\theta_{ij}] \begin{bmatrix} (-1)^{k+1} \det(\Theta)_{k1}\, H \\ (-1)^{k+2}_{k2} \det(\Theta)_{k2}\, H \\ \dots \\ (-1)^{k+n}_{kn} \det(\Theta)_{kn}\, H \end{bmatrix} = 0 \tag{10}$$

also holds.

Eq. (10) shows that

$$\mathbf{y}^{(k)} = \begin{bmatrix} (-1)^{k+1} \det(\Theta)_{k1}\, H \\ (-1)^{k+2} \det(\Theta)_{k2}\, H \\ \dots \\ (-1)^{k+n} \det(\Theta)_{kn}\, H \end{bmatrix}$$

can be assumed a solution vector for Eq(5). In this way solutions H of Eq(8) can be used for generating solution vectors of the differential equation system. Introducing H into a row k of $\mathbf{Cof}\,(\Theta)$ yields the transpose of a function vector which is a solution vector of the differential equation system Eq(5). Eq(8) is called the characteristic differential equation of the differential equation system Eq(5) and function H is a generator function of the solution vectors.

If H is the general solution of the characteristic differential equation Eq(8) and $\mathbf{y}^{(k)}$ contains all the free parameters of H, then this vector is the general solution of the differential equation system. If $\mathbf{y}^{(k)}$ does not contain all the free parameters of H, the general solution of Eq(5) can be obtained as a combination of solution vectors $\mathbf{y}^{(k_1)}$, ..., $\mathbf{y}^{(k_m)}$, provided each free parameter of H appears at least in one solution.

In some cases, one or more common operator terms can be factored out from all elements of the co-factor matrix, hence, the same terms can also be factored out from $det(\Theta)$. In such so called reducible cases the order of the differential equation system can be reduced by cancelling off the common terms both in the determinant and in the elements of the co-factor matrix.

The main advantage of using generator functions is, that after solving the characteristic differential equation, all elements of the solution vector can be obtained by derivation, in this way the solution is free of redundant constants of integration. The boundary conditions cannot be directly stated for the generator function, however, conditions stated for the elements of $\mathbf{y}$ can be transferred to the derivatives of H.

The method can be generalized for the solution of inhomogeneous differential equation systems as well.

3.2. *Application of the Generator Function in the Vibration Problem*

The Laplace operator Δ in cylindrical co-ordinate system, takes the form

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \vartheta^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \vartheta^2}. \tag{11}$$

This operator has varying coefficients that might confront with using a method which assumes ordinary differential equations and constant coefficients. Nevertheless, the method can also be used for solving partial differential equation systems if the problem can be reduced to the solution of a series of ordinary differential equation systems and differential operators with varying coefficients will only be used in the final step of the analysis.

Boundary conditions of our problem do not depend on variables τ, and ϑ. That allows us a successive separation of variables. Functions w, and ϑ can be assumed as $w = w(r, \vartheta)_m \sin\omega_m\tau$, and $F = F(r, \vartheta)_m \sin\omega_m\tau$, respectively. Having introduced these functions into *Eqs* (2) and (3) the common multiplier $\sin\omega_m\tau$ can be dropped out. In this way differential equations

$$K\Delta\Delta w_m - \left(\rho t\omega_m^2\right) w_m - \frac{1}{R}\Delta F_m = 0, \tag{12}$$

$$\frac{1}{R}\Delta w_m + \frac{1}{Et}\Delta\Delta F_m = 0. \tag{13}$$

and the boundary conditions define an eigenvalue problem in which eigenvalues ω_m, $m = 0, 1, 2, ...\infty$ are the natural frequencies of the dish.

In a vectorial form, homogeneous linear differential equations (12), and (13) emerge as

$$\Theta_m \left[\begin{array}{c} w_m \\ F_m \end{array} \right] = 0, \tag{14}$$

with the operator matrix

$$\Theta_m = \left[\begin{array}{cc} K\Delta\Delta - \rho t \omega_m^2 & -\frac{1}{R}\Delta \\ \frac{1}{R}\Delta & \frac{1}{Et}\Delta\Delta \end{array} \right]. \tag{15}$$

The operator determinant and the cofactor matrix of Θ_m are

$$\det\left(\Theta_m\right) = \frac{K}{Et}\Delta\Delta\Delta\Delta + \left(\frac{1}{R^2} - \frac{\rho\omega_m^2}{E}\right)\Delta\Delta, \tag{16}$$

$$\mathbf{Cof}\left(\Theta_m\right) = \left[\begin{array}{cc} \frac{1}{Et}\Delta\Delta & -\frac{1}{R}\Delta \\ \frac{1}{R}\Delta & K\Delta\Delta w - \left(\rho t \omega_m^2\right) \end{array} \right]. \tag{17}$$

It can be seen that the cofactor matrix is irreducible. The generator function H_m has to be obtained using the characteristic equation

$$\frac{K}{Et}\Delta\Delta\Delta\Delta H_m + \left(\frac{1}{R^2} - \frac{\rho\omega_k^2}{E}\right)\Delta\Delta H_m = 0. \tag{18}$$

A substantial simplification can be achieved by factorizing the eighth order differential operator of Eq. (18) as follows:

$$\det(\Theta_m) = \frac{K}{Et}\left[\Delta\Delta\right]\left[\Delta\Delta - \left(\frac{\rho t \omega_m^2}{K} - \frac{Et}{KR^2}\right)\right]. \tag{19}$$

The first operator factor in Eq(19) is that of the differential equation defining planar biharmonic functions, the second is formally equivalent with the operator of the differential equation for the deflections of an isotropic plate resting on a fictitious Winkler-type elastic foundation. The contribution of the geometric and dynamic properties in that fictitious elastic support can be visualized by introducing characteristic lengths L_{stat} and L_{ω_m}. Characteristic length L_{stat} is a constant, the analogue of that used in the analysis of elastically supported plates, assuming a Winkler coefficient

$$C = \frac{Et}{R^2}, \tag{20}$$

that is,

$$L_{\text{stat}} = \sqrt[4]{\frac{K}{C}}. = \sqrt[4]{\frac{KR^2}{Et}}, \tag{21}$$

while L_{ω_m} is the dynamic characteristic length which is the analogue of that used in the vibration analysis of unsupported plates:

$$L_{\omega_m} = \sqrt[4]{\frac{K}{\rho t \omega_m^2}}. \tag{22}$$

In this way, after a further factorization, *Eq* (19) takes the form

$$\det(\Theta_m) = \frac{L_{\text{stat}}^4}{R^2}[\Delta\Delta]\left[\Delta + \sqrt{\frac{1}{L_\omega^4} - \frac{1}{L_{\text{stat}}^4}}\right]\left[\Delta - \sqrt{\frac{1}{L_\omega^4} - \frac{1}{L_{\text{stat}}^4}}\right]. \tag{23}$$

Eq(23) shows that the solution of the eighth order characteristic differential equation can be reduced to those of a forth order and two second order differential equations as follows:

$$\Delta\Delta H_m^{(1)} = 0, \tag{24}$$

$$\left[\Delta + \sqrt{\frac{1}{L_\omega^4} - \frac{1}{L_{\text{stat}}^4}}\right] H_m^{(2)} = 0, \tag{25}$$

$$\left[\Delta - \sqrt{\frac{1}{L_\omega^4} - \frac{1}{L_{\text{stat}}^4}}\right] H_m^{(3)} = 0. \tag{26}$$

Solutions of Eq(24) are biharmonic functions, Eqs(25), and (26) can be solved in polar co-ordinate system using Bessel's method of separating variables r, and θ.

3.3. *The Solution of the Characteristic Differential Equation*

In the following steps a double index k, l will be generated instead of the previously used single index m. Separation of variables r, and θ can be achieved by assuming

$$H_k = A_k(r)\cos k\vartheta. \tag{27}$$

To build up $A_k(r)$, Eq(27) is introduced into Eqs(24), (25), and (26) and the their solutions $A_k^{(1)}(r)$, $A_k^{(2)}(r)$, and $A_k^{(3)}(r)$ are summed as:

$$A_k(r) = A_k^{(1)}(r) + A_k^{(2)}(r) + A_k^{(3)}(r). \tag{28}$$

In non-degenerate cases, four solutions for $A_k^{(1)}(r)$ and two solutions for $A_k^{(2)}(r)$, and $A_k^{(3)}(r)$ respectively, are linearly independent and the sum $A_k(r)$ is really the general solution of the unfactorized eighth-order differential equation.

The characteristic length of the vibration modes of the shell is

$$L_k = \left(\frac{1}{L_\omega^4} - \frac{1}{L_{\text{stat}}^4}\right)^{-\frac{1}{4}}. \tag{29}$$

Both L_{stat}, and L_{ω_m} in Eq(29) are of real values, however, the difference under the square root in Eqs(25), and (26) may also take negative value. In this case, characteristic length L_k gets complex. That induces no difficulties if the software of the numerical analysis permits the use of functions with complex argument, because the natural frequency remains real and both the real, and imaginary parts of the conjugate complex solutions can be used as real solutions of the characteristic differential equation [3].

Using a dimensionless radial co-ordinate

$$\xi_k = \frac{r}{L_k}, \tag{30}$$

the solution of *Eq* (18) can be constructed as

$$H_k = \left[A_k^{(1)}(r) + C_5 J_k(\xi_k) + C_6 I_k(\xi_k) + C_7 N_k(\xi_k) + C_8 K_k(\xi_k)\right] \cos k\vartheta, \tag{31}$$

in which

$$A_k^{(1)}(r) \begin{array}{ll} = C_1 + C_2 r^2 + C_3 \ln r + C_4 r^2 \ln r & \text{if} \quad k = 0, \\ = C_1 r + C_2 r^3 + C_3 r^{-1} + C_4 r \ln r, & \text{if} \quad k = 1, \\ = C_1 r^k + C_2 r^{k+2} + C_3 r^{-k} + C_4 r^{-k+2} & \text{if} \quad k > 1, \end{array}$$

and J_k, N_k, and I_k, K_k are k-th order Bessel functions and k-th order modified Bessel functions of the first and second kind, respectively. Properties of the Bessel functions are discussed e.g. in [3].

Functions in *Eq* (31) which are singular at point $r = 0$ have to be disregarded in case of complete paraboloid or spherical cup. Consequently, coefficients C_3, C_4, C_7, and C_8 must vanish and H_k consists of only four components:

$$H_k = \left[C_{1k}\xi_k^k + C_{2k}\xi_k^{k+2} + C_{5k} J_k(\xi_k) + C_6 I_k(\xi_k)\right] \cos k\vartheta \tag{32}$$

Generating the solution vector using the second row of the cofactor matrix (17) results in

$$w_k = \frac{1}{R}\Delta\{H_k\} = \frac{1}{RL_k^2}\left[4C_2(k+1)\xi_k^k - C_5 J_k(\xi_k) + C_6 I_k(\xi_k)\right] \cos k\vartheta, \tag{33}$$

$$\begin{aligned} F_k &= K\left[\Delta\Delta - \frac{1}{L_\omega^4}\right]\{H_k\} = \qquad (34) \\ &= -K\left[\frac{1}{L_\omega^4}\left(C_1\xi_k^k + C_2\xi_k^{k+2}\right) + \frac{1}{L_{\text{stat}}^4}\left(C_5 J_k(\xi_k) - C_6 I_k(\xi_k)\right)\right] \cos k\vartheta. \end{aligned}$$

On the basis of *Eqs* (4), five independent boundary conditions can be stated at $r = a$ for functions w_k and F_k, respectively. These are as follows
for the radial membrane and shear forces:

$$\frac{1}{r}\frac{\partial F_k}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F_k}{\partial \theta^2} = 0, \tag{35}$$

$$-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F_k}{\partial \theta}\right) = 0, \tag{36}$$

for the radial bending and the torsional moments:

$$-K\left[\frac{\partial^2 w_k}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial w_k}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_k}{\partial \theta^2}\right)\right] = 0, \tag{37}$$

$$-K(1-\nu)\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w_k}{\partial \theta}\right)\right] = 0, \tag{38}$$

and for the radial shear force:

$$-K\frac{\partial}{\partial r}(\Delta w_k) = 0. \tag{39}$$

The four parameters in $A_k(r)$ are not adequate to meet the boundary conditions Eqs(35)-(39). To overcome this difficulty, (see Eqs 5.35a-b in [1]), replacement boundary conditions have to be used by omitting Eq(38) and building in the effects of torsional moments into conditions Eq(36) and Eq(39):

$$N_{r\vartheta} - \frac{M_{r\vartheta}}{R} = 0, \tag{40}$$

and

$$Q_r + \frac{1}{r}\frac{\partial M_{r\vartheta}}{\partial \vartheta} = 0. \tag{41}$$

This replacement is the analogue of using replacement shear forces at the free edges in Kirchhoff's plate theory. By expressing $N_{r\vartheta}$, $M_{r\vartheta}$,and Q_r in terms of F, and w, the following two boundary conditions can be stated at $r = a$:

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F_k}{\partial \theta}\right) + \frac{K(1-\nu)}{R}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w_k}{\partial \theta}\right) = 0, \tag{42}$$

$$-K\frac{\partial}{\partial r}(\Delta w_k) - \frac{K(1-\nu)}{r}\frac{\partial^2}{\partial r\partial \theta}\left(\frac{1}{r}\frac{\partial w_k}{\partial \theta}\right) = 0. \tag{43}$$

The next task is converting the boundary conditions for w, and F to conditions for their generator function H. For that purpose function H has to be substituted into Eqs(33), and (34), then, expressions obtained in this way have to be substituted into the Eqs(35), (37), (42), and (43).

After this procedure, for each k a linear system of algebraic equations

$$\begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_5 \\ C_6 \end{bmatrix} = 0 \tag{44}$$

is obtained for coefficients C_1, C_2, C_5, and C_6 . Entries in the coefficient matrix of Eq(44) are as follows:

$$D_{11} = \frac{K}{L_{\omega}^4}\left(k^2 - k\right)\alpha_k^{k-2}, \tag{45}$$

$$D_{12} = \frac{K}{L_{\omega}^4}\left(k^2 - k - 2\right)\alpha_k^{k}, \tag{46}$$

$$D_{13} = \frac{K}{L_{\text{stat}}^4}\left[-\frac{\left(k - k^2\right)}{\alpha_k^2} J_k(\alpha_k) + \frac{1}{\alpha_k} J_{k+1}(\alpha_k)\right], \tag{47}$$

$$D_{14} = -\frac{K}{L_{\text{stat}}^4}\left[\frac{\left(k^2 - k\right)}{\alpha_k^2} I_k(\alpha_k) + \frac{1}{\alpha_k} I_{k+1}(\alpha_k)\right], \tag{48}$$

$$D_{21} = \frac{K}{L_{\omega}^4}\left(k^2 - k\right)\alpha_k^{k-2}, \tag{49}$$

$$D_{22} = K\left[\frac{1}{L_{\omega}^4}\left(k^2 + k\right) - \frac{4\,(1-\nu)}{R^2 a^2}\left(k^3 - k\right)\right]\alpha_k^{k}, \tag{50}$$

$$D_{23} = K\left(\frac{1}{L_{\text{stat}}^4} + \frac{(1-\nu)}{R^2 L_k^2}\right)\left[\frac{k^2 - k}{\alpha_k^2} J_k(\alpha_k) - \frac{k}{\alpha_k} J_{k+1}(\alpha_k)\right], \tag{51}$$

$$D_{24} = K\left(\frac{1}{L_{\text{stat}}^4} - \frac{(1-\nu)}{R^2 L_k^2}\right)\left[\frac{k^2 - k}{\alpha_k^2} I_k(\alpha_k) + \frac{k}{\alpha_k} I_{k+1}(\alpha_k)\right], \tag{52}$$

$$D_{31} = 0, \tag{53}$$

$$D_{32} = \frac{4\,(1-\nu)}{Ra^2}\left(k^3 - k\right)\alpha_k^{k}, \tag{54}$$

$$D_{33} = -\frac{1}{Ra^2}\left\{\left[(1-\nu)\left(k^2 - k\right) - \alpha_k^2\right] J_k\left(\alpha_k\right) + (1-\nu)\,\alpha_k J_{k+1}\left(\alpha_k\right)\right\}, \tag{55}$$

$$D_{34} = \frac{1}{Ra^2}\left\{\left[(1-\nu)\left(k^2 - k\right) + \alpha_k^2\right] I_k\left(\alpha_k\right) - (1-\nu)\,\alpha_k I_{k+1}\left(\alpha_k\right)\right\},$$

$$D_{41} = 0, \tag{56}$$

$$D_{42} = -\frac{4K(1-\nu)}{Ra^2}\left(k^4 - k^2\right)\alpha_k^{k-1}, \tag{57}$$

$$D_{43} = \frac{1}{Ra^2}\left\{\left[\frac{(1-\nu)(k^3-k^2)}{\alpha_k} + k\alpha_k\right] J_k(\alpha_k) - \left[(1-\nu)k^2 + \alpha_k^2\right] J_{k+1}(\alpha_k)\right\},$$

$$D_{44} = \frac{1}{Ra^2}\left\{-\left[\frac{(1-\nu)(k^3-k^2)}{\alpha_k} - k\alpha_k\right] I_k(\alpha_k) - \left[(1-\nu)k^2 - \alpha_k^2\right] I_{k+1}(\alpha_k)\right\},$$

where

$$\alpha_k = \frac{a}{L_k}. \tag{58}$$

Solution of the homogeneous algebraic equation system (44) requires

$$\det\left[\mathbf{D}_k\left(\alpha_k\right)\right] = 0. \tag{59}$$

$\det\left[\mathbf{D}_k\left(\alpha\right)\right]$ is a function having infinite zeroes for $\alpha > 0$. Eq(59) assigns for each k an infinite series of parameter α_{kl}, $l = 1, 2, \ldots$ To these values an infinite series of natural frequencies ω_{kl} can be calculated using

$$\omega_{kl} = \sqrt{\frac{E}{\rho R^2}}\sqrt{\alpha_{kl}^4 \frac{t^2 R^2}{12(1-\nu^2)a^4} + 1}. \tag{60}$$

To each values of α_{kl} a set of C_1, C_2, C_5, and C_6 can also be calculated which makes the generator function and the vibration mode definite.

4. Illustrative Example

Practical use of the presented analytical method needs an efficient code for finding zero values of *Eq* (59). For that purpose MATLAB has been used due to the applicability of complex arguments of elementary and special functions and due to efficient built-in functions for calculating determinants and finding zeroes of transcendent functions.

Input data of the illustrative example are: $R = 25$ m, $a = 2.5$ m, $t = 0.001$ m, $E_s = 100$ kN/mm^2, $\nu = 1/3$, $\rho_s = 2500$ kg/m^3.

Some solutions α_{kl} of *Eq* (59) are indicated in *Table 1* natural frequencies ω_{kl} calculated using these solutions are seen in *Table 2*.

Table 1. Solutions α_{kl} of Eq(59)

	k=0	k=1	k=2	k=3	k=4	k=5	k=10	k=15
l=0	–	–	20.2062+ 20.2062i	20.2050+ 20.2050i	20.2018+ 20.2018i	20.1953+ 20.1953i	20.0436+ 20.0436i	19.4527+ 19.4527i
l=1	3.0125	4.5296	5.9338	7.2622	8.5379	9.7757	15.6514	21.2907
l=2	6.2059	7.7372	9.1855	10.5767	11.9252	13.2403	19.4938	25.4516
l=3	9.3712	10.9093	12.3826	13.8081	15.1961	16.5539	23.0333	29.1997
l=4	12.5254	14.0688	15.5586	17.0068	18.4215	19.8083	26.4464	32.7676

Table 2. Natural frequencies ω_{kl} [1/s] of the paraboloid

	k=0	k=1	k=2	k=3	k=4	k=5	k=10	k=15
l=0	–	–	1.7376	4.2123	7.6057	11.8838	45.1419	95.0181
l=1	252.9978	253.0620	253.2173	253.5093	253.9882	254.7087	264.1200	289.3452
l=2	253.2634	253.6611	254.3290	255.3450	256.7898	258.7461	279.0334	322.9145
l=3	254.4409	255.6549	257.4032	259.7864	262.9028	266.8466	301.6849	365.7476
l=4	257.6088	260.3075	263.8635	268.3819	273.9570	280.6702	333.0903	417.9087

In the first row of *Table 1*, complex values of α_{kl} can be seen. As mentioned before, cropping up complex roots of Eq(59) does not mean at all that the method fails in these cases. The solutions of the frequency equation stay real; the only difference is that Bessel and modified Bessel functional components of H_{k0} (and also of w_{k0} and F_{k0}) switch to Thomson functions [3]. However, the big jumps in the values of ω_{kl} from $l = 0$ to $l = 1$ shows a qualitatively varying contribution of the rigidities in determining the natural frequencies. This difference may get a plausible explanation by surveying the modes of vibration.

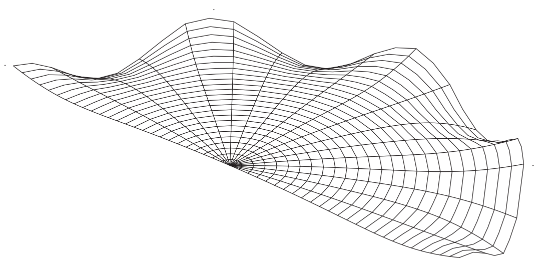

Fig. 2. Relief of w for $l = 0$, and $k = 6$

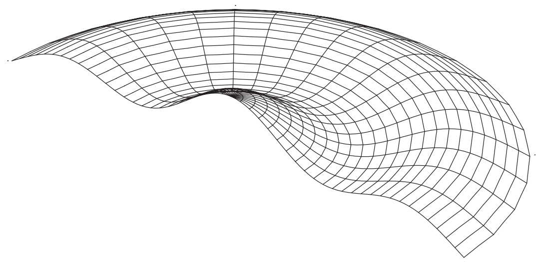

Fig. 3. Relief of w for $l = 3$, and $k = 0$

As *Figs.* 2 and 3 show, parameter l has a geometric sense: it gives the number of antinodes of surface lines in radial direction. For $l = 0$ antinodes are not formed and the deformations resemble to inextensional deformations of the paraboloid.

For the sake of a deeper insight, two comparisons have been made. One with the natural frequencies of a circular plate having the same boundary radius, thickness and material properties as the paraboloid, and another with the natural frequencies of the same plate resting on a fictitious elastic foundation with the Winkler coefficient C defined by *Eq* (20) natural frequencies of the unsupported circular plate are listed in *Table* 3, those of the elastically supported plate can be seen in *Table* 4.

Table 3. Natural frequencies ω_{kl} [1/s] of the circular plate

	k=0	k=1	k=2	k=3	k=4	k=5	k=10	k=15
l=0	–	–	1.6271	3.7870	6.6593	10.2287	38.2033	82.6320
l=1	2.8116	6.3561	10.9176	16.3922	22.7275	29.8903	77.4695	143.7297
l=2	11.9314	18.5458	26.1400	34.6665	44.0903	54.3847	118.3017	201.9630
l=3	27.2063	36.8704	47.5020	59.0704	71.5505	84.9212	164.5988	264.6976
l=4	48.6034	61.3188	74.9932	89.6050	105.1350	121.5665	216.7849	332.8933

Table 4. Natural frequencies ω_{kl} [1/s] of the plate on elastic foundation

	k=0	k=1	k=2	k=3	k=4	k=5	k=10	k=15
l=0	252.9822	252.9822	252.9874	253.0106	253.0698	253.1889	255.8505	266.1354
l=1	252.9978	253.0620	253.2177	253.5127	254.0011	254.7419	264.5780	290.9608
l=2	253.2634	253.6611	254.3291	255.3464	256.7955	258.7619	279.2764	323.7114
l=3	254.4409	255.6549	257.4033	259.7871	262.9058	266.8550	301.8158	366.1486
l=4	257.6088	260.3075	263.8636	268.3823	273.9587	280.6749	333.1602	418.1124

Table 4 shows equal natural frequencies in the first two entries of the row $l = 0$. These values belong to rigid body motions of the elastically supported plate and equal to

$$\omega_{spring} = \sqrt{\frac{C}{\rho t}}. \tag{61}$$

which is the lower bound of natural frequencies of the elastically supported plate.

The comparison of corresponding values in the first rows of *Tables* 2 and 3 proves that natural frequencies $\omega_{k,0}$ of the paraboloid, and the circular plate are fairly close to each other, but for small values of k, and $l \neq 0$, the natural frequencies strongly differ. That means, for $l = 0$, natural frequencies of the paraboloid can be estimated as those of the unsupported plate, but for $l \neq 0$ another estimation is needed.

The comparison of corresponding values of *Tables* 2 and 4 yields a complementary conclusion: in cases $l > 0$, natural frequencies of the paraboloid can be excellently estimated using those of the elastically supported circular plate.

5. Conclusions

The adequate results of the dynamic analysis of the shallow paraboloid of revolution proves that the method of generator functions can be successfully used for solving partial differential equation systems too.

Natural frequencies of the paraboloid shell belonging to modes with and without antinodes in radial direction differ by magnitudes and can be estimated using basically different models.

According to Rayleigh's classification [4], vibrations with inextensional, and extensional deformations fall under different classes. Though vibrations of a paraboloid shell are not perfectly inextensional, modes without antinodes are similar to inextensional deformations and the corresponding natural frequencies are close to those of a replacing unsupported plate. natural frequencies corresponding to modes with antinodes in radial direction can be estimated as those of a circular plate resting on a fictitious elastic foundation. Winkler coefficient of the fictitious foundation is the same for each mode of this kind and can be calculated from the elastic constants of the material and the geometric data of the shell.

On the basis of finite element calculations, similar conclusions have been drawn in [5].

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