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TOPOLOGY OPTIMIZATION IN CASE OF VARIABLE SUPPORT COST

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Abstract

Optimal design with thousands of variables is a great challenge in engineering calculations. In this paper a solution technique is introduced for the topology optimization of elastic disks under single parametric static loading and variable support conditions. Different boundary conditions (elastic and fix supports) with their cost and thousands of design variables are applied. Due to a simple mesh construction technique the checker-board pattern is avoided. The Michell-type problem is investigated minimizing the modified weight of the structure subjected to a compliance condition. The numerical procedure is based on iterative formula which is formed by the use of the first order optimality condition of the Lagrangian function.

Keywords: topology optimization, optimal design, SIMP method, support cost.

1. Introduction

Recently the topology optimization is the most "popular" topic in the field of optimal design and a great number of papers indicate the importance of the topic [1]-[7]. Due to the complex nature of the problems, it is necessary to apply difficult mathematical and mechanical tools for the solution even in case of simple structures. The limitation of the available mathematical programming tools (the programs work with limited number of design variables) requires an iterative solution technique.

This paper discusses the problem of optimizing structural topologies when some of the external forces are variable and they have a nonzero cost and the 'fictitious weight' of the structure what contains the cost modified weight of the elements is the overall measure of the problem. Such forces may represent a reaction at a support, a force generated by passive control or a ballast (weight) used for increasing cantilever action or modifying natural frequencies.

Classical theories of variable force (mostly support) optimization, based on optimality criteria and adjoint displacement fields, were developed in the mid-seventies (e.g. ROZVANY and MROZ [8]). Topology optimization for variable external forces will be first discussed in terms of the exact optimal truss topologies, taking the cost of external forces (e.g. at supports) into consideration. In the present study, it is assumed that the cost of external forces depends on their magnitude, and this theory is demonstrated in the context of a linear force (or support) cost function [9]. BUHL [10] assumed that the support costs are independent of the reactions. POMEZANSKI [11] introduced a new aspect of the support optimization in case of truss structures.

In the following an iterative technique (which is named SIMP method) and the connected numerical examples will be discussed in detail. The object of the design (so-called ground structure) is a rectangular disk with given loading (one parametric static) and support conditions (fix or/and elastic bars). The material is linearly elastic. The design variables are the thickness or/and cross-sectional area of the finite elements. To obtain the correct optimal topology some filtering method has to be applied to avoid the so-called 'checker-board pattern'.

2. Iterative Formulation

2.1. Problem Definition

In the classical MICHELL [12] truss theory, the total truss weight is minimized for a single load condition, subject to constant tensile and compressive permissible stresses, but without allowance for the cost of supports. The basic topology optimization problem is to minimize the penalized weight of the structure which is subjected to a given compliance and side constraints. This work is a continuation of the basic and extended topology optimization procedures given in [13, 14], respectively. If there are extra stiffening bars as supports (elastic bars) then the original problem has to be modified. The new objective function contains the weight of the bars, as well:

$$\tilde{W} = \sum_{g=1}^{G} \gamma_g A_g t_g^{\frac{1}{p}} + \sum_{s=1}^{Gb} \gamma_s A_s l_s. \tag{1}$$

Here:

- G is number of the ground elements of the discretized panel structure,
- γ_g is the specific weight of the ground element,
- A_g is the area of ground element,
- t_g is the thickness of ground element,
- p is the penalty parameter,
- Gb is the number of bars,
- γ_s is the specific weight of the bar element,
- A_s is the cross sectional area of the bar element,
- l_s is the length of the bar element.

The supports (bars) could be added to the plate's internal elements or to the external elements. If a bar is added internally then its both ends are connected to the plate elements, but if it is added externally, only one end is connected to the plate's element. Since the support elements are classified according to their connection types, the formulation of weight of the supports can be modified as below:

$$\sum_{s=1}^{Gb} \gamma_s A_s l_s = \sum_{si=1}^{Gbi} \gamma_{si} A_{si} l_{si} + \sum_{se=1}^{Gbe} \gamma_{se} A_{se} l_{se}, \tag{2}$$

where:

- Gbi is the number of the internal supports,
- γ_{si} is the specific weight of the internal bar element
- A_{si} is the cross sectional area of the internal bar element,
- l_{si} is the length of the internal bar element,
- Gbe is the number of the external supports
- γ_{se} is the specific weight of the external bar element
- A_{se} is cross sectional area of the external bar element,
- l_{se} is the length of the external bar element,

and $Gb = Gbi + Gbe$ gives the total number of supports (internal plus external).

Introducing the following notations, $l_{si} = l_{osi}\, t_{si}^{\frac{1}{p}}$ and $l_{se} = l_{ose}\, t_{se}^{\frac{1}{p}}$, the *Eq.* (2) can be modified as follows:

$$\sum_{s=1}^{Gb} \gamma_s A_s l_s = \sum_{si=1}^{Gbi} \gamma_{si} A_{si} l_{osi}\, t_{si}^{\frac{1}{p}} + \sum_{se=1}^{Gbe} \gamma_{se} A_{se} l_{ose} t_{se}^{\frac{1}{p}}. \tag{3}$$

Let γ_{si} and γ_{se} express the cost of the corresponding support's cost and if we introduce $A_{gsi} = A_{si} l_{osi}$ and $A_{gse} = A_{se} l_{ose}$ then, the original weight function can be formally simplified.

In this way the following compact weight function can be formed:

$$\tilde{W} = \sum_{g=1}^{G} \gamma_g A_g t_g^{\frac{1}{p}} + \sum_{si=1}^{Gbi} \gamma_{si} A_{si} l_{osi}\, t_{si}^{\frac{1}{p}} + \sum_{se=1}^{Gbe} \gamma_{se} A_{se} l_{ose} t_{se}^{\frac{1}{p}} = \sum_{g=1}^{GG} \gamma_g A_g t_g^{\frac{1}{p}}, \tag{4}$$

where *GG=G+Gbi+Gbe.* The unknowns are the 'thickness' of the elements. The topology optimization problem with compliance and side constraints can be written as follows:

$$\left.\begin{array}{ll} & \min\ \tilde{W} = \min \sum\limits_{g=1}^{GG} \gamma_g A_g t_g^{\frac{1}{p}} \\ \text{subjected to :} & \\ & \mathbf{u}^{\mathbf{T}}\mathbf{K}\mathbf{u} - C \le 0; \\ & -t_g + t_{\min} \le 0; \quad (\text{for } g = 1, ..., GG)\ , \\ & t_g - t_{\max} \le 0; \quad (\text{for } g = 1, ..., GG)\ . \end{array}\right\} \tag{5}$$

where:

- $t_{\min}$ is the minimum allowable thickness for ground element,
- $t_{\max}$ is the maximum allowable thickness for ground element,
- $\mathbf{u}$ is the nodal displacement vector associated with the load,
- C is the compliance.

In *Eq.* (5) the nodal displacement vector **u** associated with the load **P** is calculated from $\mathbf{Ku} = \mathbf{P}$ linear system.

In topology optimization the checker board pattern, a numerical artifact with artificially high stiffness is a big problem. To avoid this problem, as an optimal solution here a simple procedure was used which was published by GÁSPÁR, LÓGÓ, ROZVANY [2]. The key point is that all the ground elements (a primary meshing provides the so-called ground elements) should be sub-divided into further finite elements (secondary elements) with the same thickness. For the subdivision it is enough to use 2 by 2 elements. Further number of sub-elements cannot improve significantly the final result.

2.2. Lagrange Function

Using the Lagrange multipliers υ,α_g,β_g and slack variables h_1, h_{2g}, h_{3g} for the constraints in problem (5), the following Lagrange function can be formed:

$$\begin{gathered}£\left(t_g, \upsilon, \alpha_g, \beta_g, h_1, h_{2g}, h_{3g}\right) = \sum_{g=1}^{GG} \gamma_g A_g t_g^{\frac{1}{p}} + \upsilon\left(\mathbf{u}^{\mathbf{T}}\mathrm{K}\mathbf{u} - C + h_1^2\right) + \\ \sum_{g=1}^{GG} \alpha_g \left(-t_g + t_{\min} + h_{2g}^2\right) + \sum_{g=1}^{GG} \beta_g \left(t_g - t_{\max} + h_{3g}^2\right).\end{gathered} \tag{6}$$

2.3. Design Formulation

By using the first order optimality conditions (Kunh-Tucker conditions) one can obtain a closed form for the optimal design.

$$\frac{\partial £}{\partial t_g} = \frac{1}{p}\gamma_g A_g t_g^{\frac{1-p}{p}} + \nu\left(\frac{\partial \mathbf{u}^T}{\partial t_g}\mathrm{K}\mathbf{u} + \mathbf{u}^T\frac{\partial \mathbf{K}}{\partial t_g}\mathbf{u} + \mathbf{u}^T\mathbf{K}\frac{\partial \mathbf{u}}{\partial t_g}\right) - \alpha_g + \beta_g = 0,$$

$$(g = 1, ..., GG). \tag{7a}$$

Due to symmetry of the stiffness matrix **K** the *Eq.* (6) can be replaced by the following relation:

$$\frac{\partial £}{\partial t_g} = \frac{1}{p}\gamma_g A_g t_g^{\frac{1-p}{p}} - \nu \sum_{e=1}^{E_s} u_{ge}^{T} \frac{\partial \mathrm{Kge}}{\partial t_g} u_{ge} - \alpha_g + \beta_g = 0, \quad (g = 1, ..., GG)\ , \tag{7b}$$

where the subscript *ge* refers to the *e*-th finite element of the g-th ground element, E_s is the number of the sub-elements belonging to the g-th ground element.

If the 'normalized' element stiffness matrix is $\tilde{\mathbf{K}}_{ge}$ (e.g. calculated for a unit thickness ($t_g = 1$)), then the element stiffness matrix $\mathbf{K}_{ge}$ for actual thickness t_g is expressed by $\mathbf{K}_{ge} = t_g \tilde{\mathbf{K}}_{ge}$ due to the linear relation and $\frac{\partial K_{ge}}{\partial t_g} = \tilde{\mathbf{K}}_{ge}$. Introducing the following notation $R_g = t_g^2 \sum_{e=1}^{E_g} \mathbf{u}_{ge}^T \tilde{\mathbf{K}}_{ge} \mathbf{u}_{ge}$ the *Eq.* (7b) becomes very simple

$$\frac{1}{p}\gamma_g A_g t_g^{\frac{1-p}{p}} - \upsilon \frac{R_g}{t_g^2} - \alpha_g + \beta_g = 0. \tag{7c}$$

Continuing the derivations:

$$\frac{\partial \pounds}{\partial \nu} = \mathbf{u}^{\mathbf{T}}\mathrm{K}\mathbf{u} - C + h_1^2 = 0; \quad \frac{\partial \pounds}{\partial h_1} = 2\nu h_1 \overset{=}{} 0; \tag{8}$$

$$\frac{\partial \pounds}{\partial \alpha_g} = -t_g + t_{\min} + h_{2g}^2 = 0; \quad \frac{\partial \pounds}{\partial h_{2g}} = 2\alpha_g h_{2g} \overset{=}{} 0; \tag{9}$$

$$\frac{\partial \pounds}{\partial \beta_g} = t_g - t_{\max} + h_{3g}^2 = 0; \quad \frac{\partial \pounds}{\partial h_{3g}} = 2\alpha_g h_{3g} \overset{=}{} 0; \tag{10}$$

Omitting the details from *Eqs.* (7c), (8), (9) and (10.a-b) the values of the Lagrange multipliers, slack variables and the thickness values t_g can be calculated iteratively.

As it is in COC type methods [13], before the calculation of the Lagrange multiplier ν, one needs to define a range for the thickness: a set of active ($\mathcal{A}$)and passive ($\mathcal{P}$) thicknesses.

There exist three possibilities:

If $t_{\min} < t_g < t_{\max}$ (the ground element is 'active', $g \in A$) then $\alpha_g = \beta_g = 0$ and by (7c) the following formula can be obtained:

$$t_g = \left(\frac{\upsilon p R_g}{A_g \gamma_g}\right)^{\frac{p}{p+1}}. \tag{11}$$

In case of $t_g = t_{\min}$ the corresponding Lagrange multipliers are $\alpha_g \geq 0$, $h_{2g} = 0$ and (7c) implies

$$t_g \geq \left(\frac{\upsilon p R_g}{A_g \gamma_g}\right)^{\frac{p}{p+1}}. \tag{12}$$

This means that if (11) gives a t_g - value which is smaller than $t_{\min}$ then (7c) is satisfied by $t_g = t_{\min}$. Similarly, in case of $t_g = t_{\max}$ the corresponding Lagrange multipliers are $\beta_g \geq 0$, $h_{3g} = 0$ and then (7c) implies

$$t_g \leq \left(\frac{\upsilon p R_g}{A_g \gamma_g}\right)^{\frac{p}{p+1}}, \tag{13}$$

which allows $t_g = t_{\max}$when (11) gives a t_g - value which is greater than $\mathrm{t_{max}}$. If $t_g = t_{\min}$ or $t_g = t_{\max}$ we call the ground element 'passive' ($g \in \mathcal{P}$).

2.4. *Calculation of the Final Iterative Formulas*

In order to keep the number and layout of ground elements constant and avoid the ill-conditioned stiffness matrix, one can replace the zero element thickness (t_{min}) with a small, but finite value (e.g. $t_{\min} = 10^{-6}$). If the compliance constraint is active in problem (5) (e.g. satisfies the equality sign) the following form holds

$$C - \sum_{g=1}^{GG} \frac{R_g}{t_g} = 0. \tag{14}$$

Since the thickness value for passive elements ($g \in \mathcal{P}$) is given and for active elements ($g \in \mathcal{A}$), it can be calculated from (11), then

$$C - \sum_{g\in\mathcal{P}} \frac{R_g}{t_g} = \sum_{g\in\mathcal{A}} \frac{R_g}{t_g} = \sum_{g\in\mathcal{A}} \frac{R_g}{\left(\frac{\upsilon p R_g}{\mathcal{A}_g \gamma_g}\right)^{\frac{p}{p+1}}} \tag{15}$$

implying

$$\upsilon^{\frac{p}{p+1}} = \frac{\sum\limits_{g\in\mathcal{A}} \left(\frac{\mathcal{A}_g \gamma_g}{p}\right)^{\frac{p}{p+1}} R_g^{\frac{1}{p+1}}}{C - \sum\limits_{g\in\mathcal{P}}^{\frac{R_g}{t_g}}} \text{ (for } \mathcal{A} \neq 0). \tag{16}$$

The optimal solution can be obtained by calculating iteratively the thickness values t_g and the Lagrange-multiplier from (11) and (16).

2.4.1. *The Applied SIMP Algorithm can be defined as follows:*

1. Specify the Max and Min value of t_g, ($t_{\max} = 1$, $t_{\min} = 10^{-6}$).
2. Specify a maximum of C (compliance), of say 150% of the C value corresponding to $t_g = t_{\max}$ for all elements.
3. Set the penalty value p=1, later this value will be incremented to p = 1.5, 2, etc. and specify the maximum value of the penalty parameter.
4. Specify design domain, including supports and loading.
5. Specify the cost of the internal and external supports.
6. Carry out FEM.
7. Extract displacement field **u** for entire structure .
8. Calculate elemental compliance $\bar{C}_e$ and R_g with displacement vector based on current element solution set t_g, but using the stiffness matrix for the elements as if it had t_g=1.

$$\bar{C}_e = \{\mathbf{u}_e\}^T \left[\tilde{\mathbf{K}}_e\right] \{\mathbf{u}_e\}. \tag{17}$$

9. Calculate Lagrange multiplier υ:

$$\upsilon = \frac{\sum\limits_{g\in\mathcal{A}} \left(\frac{\mathcal{A}_g \gamma_g}{p}\right) R_g^{\frac{1}{p}}}{\left(C - \sum\limits_{g\in\mathcal{P}} \frac{R_g}{t_g}\right)^{\frac{p+1}{p}}} \text{ (for } \mathcal{A} \neq 0). \tag{18}$$

10. Calculate new element solution set: $t_{g,new} = \left(\frac{\upsilon p R_g}{\mathcal{A}_g \gamma_g}\right)^{\frac{p}{p+1}}$;

    where $\upsilon^{\frac{p}{p+1}}$ is the Lagrange multiplier calculated in step 8 with the correct power.
11. Determine the set of active and passive elements by the following element limit set:

    $t_{g,new} = t_{\min}$ if $t_{g,new} \leq t_{\min} = 10^{-6}$; $e \in \mathcal{P}$,
    $t_{g,new} = t_{\max}$ if $t_{g,new} \geq t_{\max} = 1$; $e \in \mathcal{P}$,
    $t_{g,new} = t_{g,new}$ if $t_{\min} \leq t_g \leq t_{\max} = 1$; $e \in \mathcal{A}$.
12. If active set has changed in the previous iteration, go to step 5, else if active set has not changed from pervious iteration go to step 12.
13. Increase p until all the elements become passive or reach the limit for p, using the following formula: $p = p + increment$ (step size is controlled).

3. Numerical Examples

A simple rectangular homogenous plate structure which is supported at the left side of the design domain is examined with different boundary conditions (*Fig. 3*) in order to find the optimum topology for the ground structure. To demonstrate the method and the algorithm, several cases were examined. The design domain (or the ground structure) is 10×40 units. The applied concentrated load is 100 units and it acts at the middle of the right edge of the ground structure. The Poisson ratio is set to 0. The penalty parameter 'p' is varied from 1.0 to 3.0 (from p=1.0 to p=1.5 with smooth increasing (increment is set to 0.1) and later penalty parameter is increased up to p=3.0 with a larger increment (0.5)). For the continuum design domain 4-nodes quadrilateral FE's are used. Total number of finite elements are 6400 by the use of 20x80 ground elements and 2x2 sub-elements.

Two groups of examples are investigated. In the first group of examples *Fig. 3* where the basic topology problem definition is applied, a homogenous plate is supported by fix supports on one side (*Fig. 3*). Here the effect of variation of the supports rigidities are investigated for the optimal topology. At the beginning, all the side supports are set to be rigid with same rigidity (equal costs), but later on, two supports -at the top and bottom- are kept rigid and the middle support's rigidity decreased gradually from infinite to zero (equivalent with increasing the support costs). Using this trick, made a possibility to find the optimal topology shape of the ground structure in function of the support rigidities. The results show that most expensive case is when all the side supports are taken as rigid (*Fig. 2d*), and cheapest case is when the middle supports are taken as rigid (*Fig. 2a*). As one can see on the results, the first result gives the cheapest possibility and the last one shows the most expensive solution (if the disk, as structure is considered only).

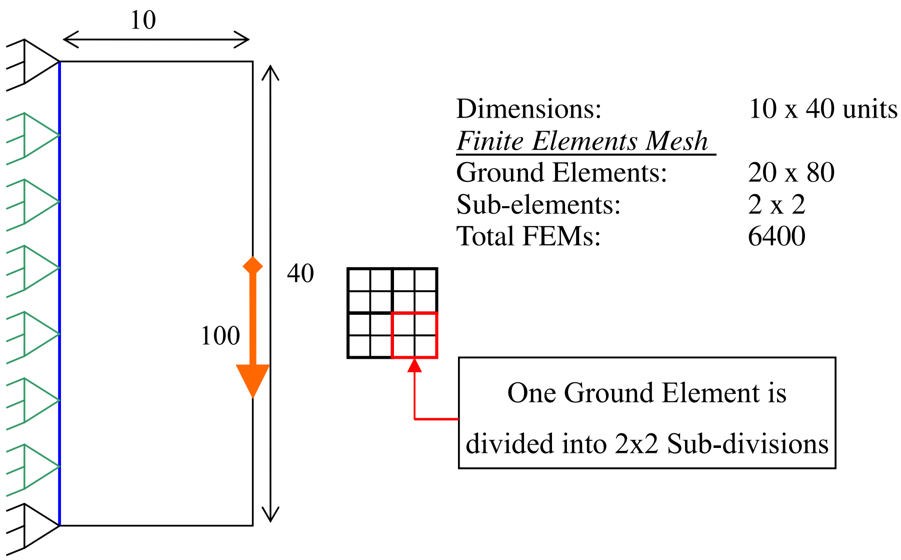
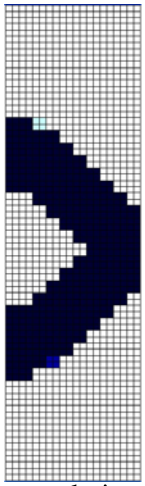


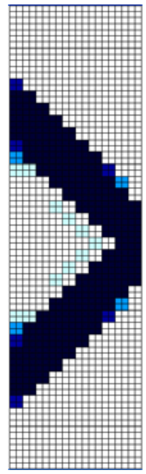
Fig. 1. Design domain and boundary conditions

The optimal topologies due to the intermediate costs can be seen in *Figs. 2b-c*. In *Figs. 2b-c* the optimal topology of the disk is presented where the costs of the supports gradually decrease from the middle to the bottom and to the top, respectively.

In the second group of examples (*Fig. 3*.) the extended topology problem definition is applied (*Eq.*5), the middle supports are substituted with bars which are connected externally to the disk within each 5 degrees in order to have bars almost in all directions. The costs of the supports are varied from 1 (zero rigidity) to 10000 (infinite rigid). Using modified cost function, made a possibility to monitor topology optimization change over the plate and find the working bar against external force. The cost function includes cost of the supports as well as cost of the ground structure (volume). Almost identical results are obtained as the preceding examples. As it is shown below, the first result would show the cheapest solution, where the external bars is rigid. Later the rigidity of the bars is symmetrically and gradually decreased from the middle of side support outward till one gets the most expensive solution where all the bars are set to have small rigidity. This is shown in *Fig. 4d*. The results of the second example are very similar to the optimal topologies coming from the

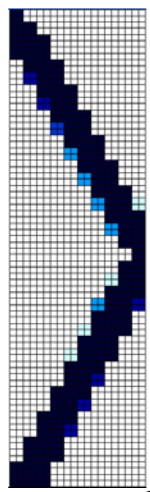


a) The cheapest solution, where all the supports have same rigidity.



b) Mid supports are set to be softer and outer supports are rigid.

c) Slowly mid supports are set to softer.



d) All mid supports are set to be significantly softer than the top and bottom supports. This is most expensive!

Fig. 2. a–d

previous example. All the results are in good agreement with the expected ones.

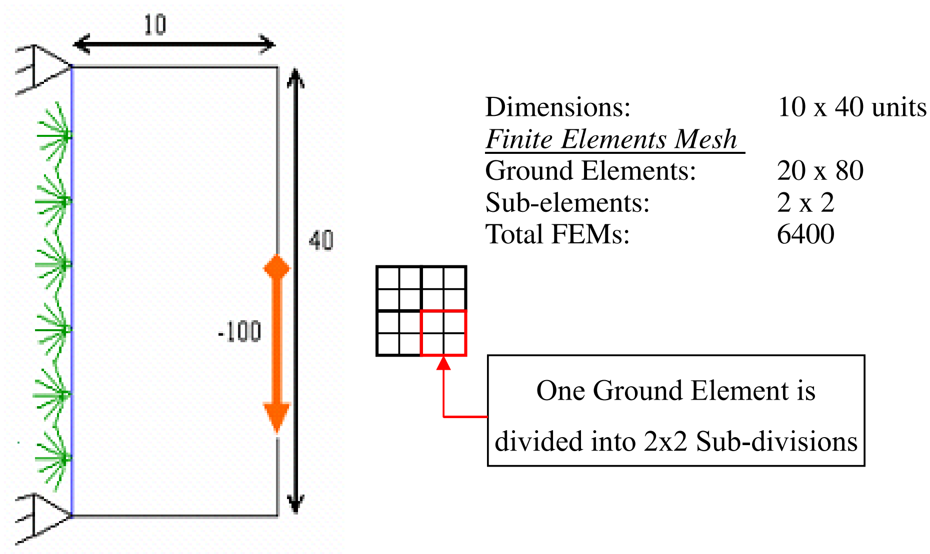
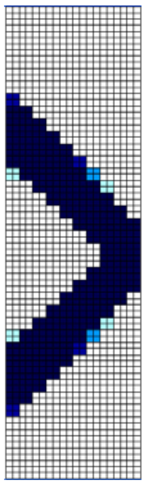


Fig. 3. Design domain and bar supports

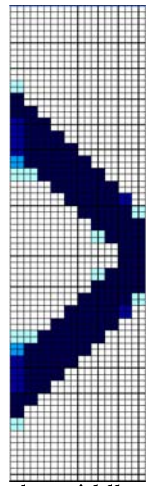
The most interesting point in the second example is the following: in all calculations, as the rigidity of the bars were diminished symmetrically and gradually from the middle of side support outward, only the bars which were in direction of the resultant force were working. The rest of the bars were not taking role on resisting the external force.

4. Conclusions

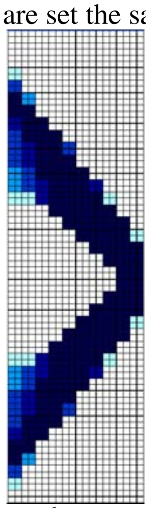
A very efficient iterative algorithm was presented for topology design of continuum type structures with variable support cost and having a compliance constraint. The applied meshing provides a good technique to avoid the checkerboard pattern. By the use of the smooth penalization increment the obtained numerical solutions are in good agreement with the expectations. Conceptually this topology design is simple, since the algorithm does not require intensive computer storage. The number of the design variables (thousands) significantly exceeds the maximum number of variables what can be used in any kind of mathematical programming algorithm. The main disadvantage is that the buckling and other constraints are not taken into consideration during the optimization procedure but the obtained numerical topologies are good starting points for further optimal design. The support optimization technique is suitable to demonstrate the effects of strengthening of structures.



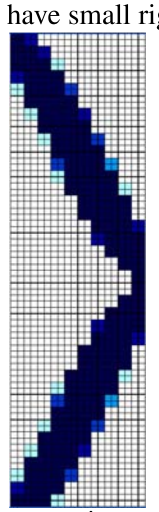
a) The cheapest solution, where the bar elements are set the same cost.



b)The bars from the middle outward have, gradually set to have small rigidity.



c) More and more bar set to have small rigidity.



d) All bars are expensive, only outer fix supports are cheap.

Fig. 4. a–d

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