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MODELLING THE LOCAL VARIATION OF AQUIFER PARAMETERS WITH THE HELP OF THE ANALYTIC ELEMENT METHOD

Rózsa CSOMA

Department of Hydraulic and Water Resources Engineering
Budapest University of Technology and Economics
H–1521 Budapest, Hungary
Phone: (+36-1) 463-2249, Fax: (+36-1) 463-1879

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Abstract

The Analytic Element Method (AEM) for groundwater flow modelling and also its application possibilities were presented in an earlier volume, [2, 3] . The present paper focuses on one certain element of it, the so-called inhomogeneity. The inhomogeneity describes the local variation of aquifer parameters, like its thickness, base elevation or permeability. After a short introduction of the method, the mathematical description and application conditions for the different approaches of inhomogeneities are given in details. Finally, some recommendations are given to help in the decision which approach to use.

Keywords: groundwater modelling, AEM, inhomogeneity, doublet.

1. Introduction

Analytic element method for groundwater flow modelling was developed about 30 years ago. It is based on the well known potential theory, but instead of the usual velocity potential, it uses its integral over the saturated aquifer, the so-called discharge potential. With the help of it the simplified basic equation of groundwater flow turns to be the *Laplace*-equation whose solutions are well known.

The aquifer itself is subdivided into hydraulic units called the 'elements'. Each element represents one certain individual feature of the aquifer, like a given surface water course, the variation of an aquifer parameter, infiltration, etc. The effects of each element can be described by certain functions and based on the linearity of the governing equation, these functions can be superimposed to each other. This is how the full description of the aquifer is given.

This paper aims to introduce a family of elements that describe the sudden variation of aquifer parameters along a closed border. This family is called the inhomogeneity. To introduce them, first a short description of the method is given. Then detailed derivations are presented to show the different possibilities to build different functions describing inhomogeneities of different shape, character and importance. Detailed evaluation and recommendations are also given to summarize the application possibilities.

2. Theoretical Backgrounds

2.1. The Basic Equation

The basic equation of the model is the steady, shallow groundwater flow equation integrated along the full saturated aquifer:

$$\frac{\partial}{\partial x}\left(-T_x \frac{\partial \varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left(-T_y \frac{\partial \varphi}{\partial y}\right) - N = 0 \tag{1}$$

Before solving the equation, the discharge potential (Φ) has to be introduced. It is actually the integral of the well-known velocity potential integrated along the full thickness of the aquifer. Taking into consideration that the transmissibility (T) is different for shallow confined and unconfined aquifers, the discharge potential is as follows:

$$\text{confined}:\Phi_c = k\varphi H - \tfrac{1}{2}kH^2; \quad \text{unconfined}:\Phi_u = \tfrac{1}{2}k\varphi^2 \tag{2}$$

Due to the application of this potential the differences vanish and the equation turns to be either the *Poisson*-equation, or without infiltration, the *Laplace*-equation:

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \tag{3}$$

Eq. (3) is a simplified equation, that assumes horizontal layers with constant thickness, homogeneous and isotropic soil and a reference level along the aquifer base. The solutions of such a case are the harmonic functions, while the *Poisson*-equation may be solved as the sum of a harmonic function and a particular solution. That is how the infiltration can also be taken into consideration. Other characteristics of the aquifer may be taken into consideration with the help of appropriate elements.

An *element* is a hydraulic unit describing a certain natural feature or artificial intervention of the aquifer. On the other hand, as each element is such a hydraulic unit that can be described by a potential function. After the individual description of each element, the functions must be superimposed to gain the overall description of the full aquifer. For the description of the individual elements conformal mapping offers a handy tool. In this case the real part of the Ω complex potential is the potential function described earlier, $\Phi = \Re(\Omega)$, while the imaginary part is the stream function, $\Psi = \Im(\Omega)$.

A given problem is always connected to a well defined region, though there are also several effects coming from outside. Most of the numerical models include these effects as boundary conditions. But the application of the potential approach considers an infinite plane without any boundaries. To limit this infinite plain and to take the outer effects also into consideration, around the area of interest a certain outer area has to be defined. This outer area provides a transition between the area of interest and the area left out of consideration. Its size, the elements to include or cancel and also the formulation of the included elements may be determined by calibration.

2.2. Some Basic Elements

Any element of the aquifer influencing the flow conditions may be described by harmonic functions. In the following part a short summary of the most important elements are given. More details may be found in an earlier volume [2, 3] CSOMA (2001), in several textbooks and other publications, e.g. [6, 5, 1, 4].

Cross flow helps to describe far field conditions. *Infiltration* from the ground surface or the evaporation of phreatic groundwater is a particular solution of the *Poisson*-equation, that can be approximated with an ellipsis-shaped equipotential. Together with cross flow it is also a useful tool to describe far field conditions.

With the well-known potential of the *source/sink* wells (pumped and recharge) can be modelled. *Line sink* can be defined as the integral of the source/sink along a line. With constant intensity it is a first order line sink, while linearly varying intensity gives the second order one. The first order line sink is used to describe infiltration from smaller water courses with constant water level, while the second order one describes rivers or streams with linearly varying water levels. Usually, a string of them is used. An integration of the source/sink with constant intensity over an area gives the *area sink*. A simple way is to do it over a circle, the integral over a polygon requires much more complicated mathematical description. This second one is a useful tool to model the infiltration from larger lakes, also of irregular shape, reservoirs, or wide rivers which cannot be considered as one-dimensional.

The *doublet* is the resultant flow pattern of a source and a sink of the same intensity located at the same point. A doublet in homogeneous flow can model the sudden changing of aquifer permeability. Further, if a doublet is integrated along a line perpendicular to its moment, the potential of the *line doublet* is obtained. This line doublet has such characteristics, that a string of it forming a closed polygon is a useful tool to model local inhomogeneities of aquifers, like the changing of the permeability mentioned earlier, and also a local step of base elevation or thickness. This paper focuses on the family of elements describing this local variation of aquifer characteristics. This family is called the *inhomogeneity.*

2.3. The Inhomogeneity

While defining the discharge potential Φ the soil was assumed to be homogeneous, with horizontal layers. Therefore some elements have to be introduced that describe the variation of these aquifer parameters along a closed curve. This family of elements is called the inhomogeneity.

Fig. 1 shows an area mentioned above. Within the closed curve of B, inside the domain D all parameters of the aquifer have the upper index "+" , while outside the index is "–".

Along boundary B of the domain both the groundwater levels (φ) and the normal components of the specific discharge (q_n) perpendicular to B have to be

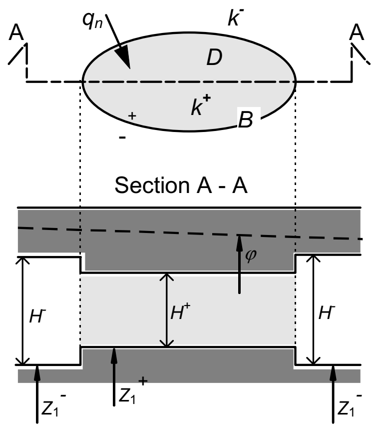


Fig. 1. Inhomogeneity

continuous:

$$\varphi^+ = \varphi^- = \varphi \quad \text{and} \quad q_n^+ = q_n^- = q_n \tag{4}$$

In this case base elevation Z_1 is no more constant (see *Fig. 1*), so it cannot be used as reference level, as it was in case of *Eq.* (3). Using a general reference level, and taking into consideration that the discharge potential requires the distance between the actual base and the groundwater level above it, potential Φ turns to be different inside domain D and outside. In case of confined aquifers the potentials at the two sides of boundary B are as follows:

$$\begin{aligned} \Phi^+ &= k^+ H^+(\varphi - Z_1{}^+) - \tfrac{1}{2} k^+(H^+)^2 \quad \text{and} \\ \varphi^- &= k^- H^-(\varphi - Z_1{}^-) - \tfrac{1}{2} k^-(H^-)^2 \end{aligned} \tag{5}$$

while in case of phreatic aquifer it is:

$$\Phi^+ = \tfrac{1}{2} k^+(\varphi - Z_1{}^+)^2 \quad \text{and} \quad \Phi^- = \tfrac{1}{2} k^-(\varphi - Z_1{}^-)^2 \tag{6}$$

Based on *Eq.*(4) the stream function Ψ must be continuous along B,while *Eqs.* (5) and (6) show that the potentials Φ must be different at the different sides of B. This type of flow pattern may be obtained with the help of doublets.

2.4. The Doublet

Doublet is the resultant of a source and a sink of the same intensity, in case the distance between them vanishes. Let us have a line that makes the angle β with the positive x axis. Along this line a source is located at $z_0 + \Delta$ and a sink at $z_0 - \Delta$ as given in *Fig. 2*. Both have the same intensity of Q. The resultant potential is as follows:

$$\Omega_{ss} = \frac{Q}{2\pi} \ln \frac{z - (z_0 - \Delta)}{z - (z_0 + \Delta)} \tag{7}$$

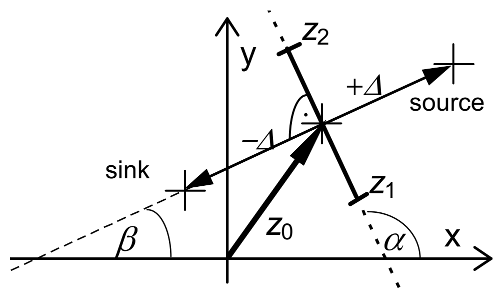


Fig. 2. Sink and source

First using the conjugate of the discharge function of potential *Eq.* (7), then taking the limit $\Delta \to 0$, the conjugated discharge function of the doublet is obtained as

$$\lim_{\Delta\to 0} \frac{d\Omega_{ss}}{dz} = \lim_{\Delta\to 0} q^* = \lim_{\Delta\to 0} \left[-\frac{Q\Delta}{\pi} \frac{1}{(z-z_0)^2 - \Delta^2} \right] = -\frac{M}{(z-z_0)^2} = \frac{d\Omega_d}{dz} \tag{8}$$

where

$$\lim_{\Delta\to 0} (\frac{Q\Delta}{\pi}) = M = M_0 e^{i\beta} = M_{0x} + i M_{0y} = \frac{-\lambda}{2\pi i} \tag{9}$$

is the moment of the doublet. It is a complex expression, and as it is in connection with the vector Δ, it is parallel with it. λ is the strength of the doublet, perpendicular to the moment. Completing the integration the complex potential of the doublet in different forms is as follows:

$$\Omega_d = \frac{M}{z - z_0} = \frac{-1}{2\pi i} \frac{\lambda}{z - z_0}. \tag{10}$$

The different formulations of the doublet's potential is needed for the different applications to be shown later. Some further explanation to *Eq.* (10) can be seen in *Fig. 3*.

The stream and potential functions of the doublet introduced in this way are as follows:

$$\Phi_d = \Re(\Omega_d) = \frac{M_{0x}(x - x_0) + M_{0y} y - y_0}{(x - x_0)^2 + (y - y_0)^2} \tag{11}$$

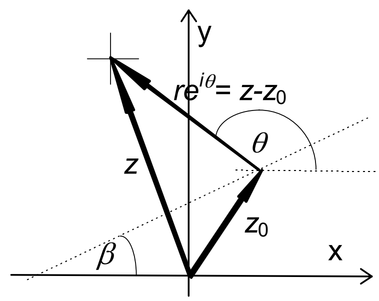


Fig. 3. Doublet at z_0

$$\Phi_d = \Im(\Omega_d) = \frac{M_{0x}(y - y_0) + M_{0y}x - x_0}{(x - x_0)^2 + (y - y_0)^2} \tag{12}$$

Based on the *Eqs.* (11) and (12) it can be understood, that both the potential and streamlines form a series of circles. The only difference between the two sets of curves is the location of the centre. In case of the equipotential lines this centre is located along the line between the sink and source (see *Fig. 2*), while the streamlines have their centre perpendicular to it. This line makes the angle α with the positive x axis. *Fig. 4* shows the flow pattern of such a doublet that $\beta = \pi$. That is why the equipotential lines (continuous line in *Fig. 4*) have the centre along the x axis, while the streamlines (dotted in *Fig. 4*) are centred along the y axis. There is one common point of both families of curves, the centre of the doublet, $z_0 = x_0 + iy_0$. In *Fig. 4* this point is the origin. Due to symmetry, the figure shows only the upper half of the flow pattern.

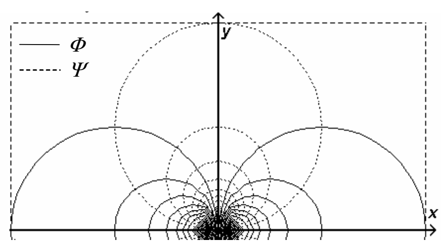


Fig. 4. Doublet at the origin, $\beta = \pi$

The doublet has such characteristics, that it provides the basis for the description of inhomogeneities. All the potentials to be introduced later – general but compound or specific but simple – are based on the potential of the doublet.

3. Line Doublets to Model Inhomogeneities

3.1. Line Doublets

3.1.1. The Integral of the Doublet

If several doublets are placed next to each other, a line doublet is obtained. Line doublets may be defined along curves or straight lines, as well. The potential of a line doublet may be obtained by the integration of the individual doublets.

There are two special straight lines that may be important from the point of view of inhomogeneities. The one is parallel with the moment of the doublet and the other is perpendicular to it. This latter gives such a flow pattern, that the stream function is continuous across it, but the potential has a step. Therefore a line doublet perpendicular to its moment fulfils the requirements given in *point 2.3.* The name 'line doublet' is usually used for this type. The other one with parallel moment makes a continuous potential but a step in the stream function. To make difference between the two elements of similar origin, this second one is called the 'line dipole'. Line dipoles are useful tools to model thin, but highly permeable elements, like a drain.

To determine the complex potential of the line doublet the original doublet has to be integrated along the section of (z_1, z_2) given in *Fig. 2.* This integral can only be accomplished with some approximations, like the application of *Taylor*-series. Using also the geometric transformation given in *Appendix 1.*, the potential in general form is as follows:

$$\Omega_{di} = -\frac{1}{2\pi i}\int_{z_1}^{z_2} \frac{\lambda}{z-\delta} d\zeta = \frac{\lambda(Z)}{2\pi i} \ln \frac{Z-1}{Z+1} + ip(Z) \tag{13}$$

where Z is the transformed location (see *Appendix 1*), *λ(Z)* and $p(Z)$ may be approximated with complex polynomes of real coefficients.

As it was mentioned, this potential provides a flow pattern with a stream function Ψ continuous across the element and a potential function Φ with a jump between the to sides of the element. This potential difference is exactly the strength λ.

3.1.2. The String of Line Doublets

Line doublets may form strings, if the endpoint of a certain element is the beginning of an other one. Of course, string elements cannot cross each other, but they may form closed polygons, as indicated in *Fig. 5.*

Let section *j* be the one between the nodes $(j, j+1)$ of the polygon. The jump of the potential appears along each member of the polygon, so strength *λ(Z)* also has to be defined along the polygon as a function of location. There are several possibilities for it. The simplest way is to define the jump condition at nodal points, at certain points between the two nodes, etc. Therefore the complex polynom *λ(Z)* may be approximated in different ways.

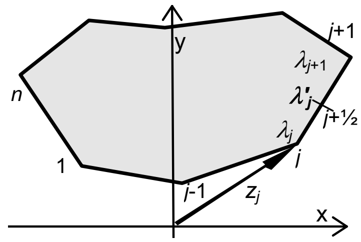


Fig. 5. A string of line doublets

3.1.3. First Order Line Doublets

A possible approximation of the strength *λ(Z)* is that it is prescribed at the nodal points as constant real values. Between them it varies linearly along the given section of the polygon. As it is the function of the complex (transformed) location Z, it remains complex. Along section j it is

$$\lambda(Z_j) = -\frac{1}{2}\left(Z_j - 1\right)\lambda_{f,j} + \frac{1}{2}\left(Z_j + 1\right)\lambda_{f,j+1} \tag{14}$$

Substituting it into *Eq.* (13), the description of the string of line doublets may be obtained as follows:

$$\Omega_f = \frac{1}{2\pi i}\sum_{j=1}^{n}\lambda_{f,j}\left[F(Z_j) + G(Z_{j-1})\right] \tag{15}$$

where the summation covers the full closed polygon (see *Fig. 5*, i.e. if $j = 1$ then $j - 1 = n$. Parameters F and G are also the linear function of the location Z:

$$F(Z_j) = -\frac{1}{2}\left(Z_j - 1\right)\ln\frac{Z_j - 1}{Z_j + 1} - 1 \quad \text{and}$$
$$G(Z_j) = \frac{1}{2}\left(Z_j + 1\right)\ln\frac{Z_j - 1}{Z_j + 1} + 1 \tag{16}$$

Their logarithmic singularities at the endpoints $Z = 1$ and $Z = -1$ may be eliminated with the help of the following limit:

$$\lim_{z\to 0}(z \ln z) = 0 \tag{17}$$

Based on *Eq.* (15) the general form of the potential and stream functions at node j are the following:

$$\Phi_{f,j} = \Re(\Omega_{e,j}) = \lambda_{f,j}\, A_{f,j}(x, y)\; \Psi_{f,j} = \Im(\Omega_{f,j}) \tag{18}$$

where $A_{f,j}(x, y)$ is the function of the location only. More detailed formulation of it may be found in some textbooks and other publications e.g. STRACK [6], HAITJEMA [5], CSOMA [1], etc.

As each term and also the full potential are linear functions of the transformed location Z, it is called the *first order line doublet.*

3.1.4. Second Order Line Doublets

An other possible approximation of the strength is that the jump condition is also prescribed half way between the nodes j and $j+1$, at node "$j+1/2$" as a real constant (see *Fig. 6*). As it is still the function of the complex location Z, it remains complex. But in this case the polynome $\lambda(Z)$ turns to be quadratic, so along section j it is as follows:

$$\lambda(Z_j) = \tfrac{1}{2}\, Z_j \left(Z_j - 1\right) \lambda_{s,j} - \left(Z_j - 1\right)\left(Z_j + 1\right) \lambda'_{s,j} + \tfrac{1}{2}\, Z_j \left(Z_j + 1\right) \lambda_{s,j+1} \tag{19}$$

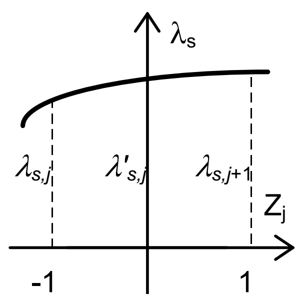


Fig. 6. Second order approximation

Beside $\lambda_{s,j}$, and $\lambda_{s,j+1}$ at the corner points, $\lambda'_{s,j}$ at the centre is also real. With the help of it, the complex potential for the full polygon is

$$\Omega_{s,j} = \frac{1}{2\pi i} \sum_{j=1}^{n} \left\{\lambda_{s,j}\left[F(Z_j) + \tfrac{1}{2}H(Z_j) + G(Z_{j-1}) + \tfrac{1}{2}H(Z_{j-1})\right] - \lambda'_{s,j}\, H(Z_j)\right\} \tag{20}$$

where the summation covers the full closed polygon (see *Fig. 5*), i.e. if j = 1 then j- 1 =n. Parameters F and G are defined by *Eq.* (16). They are linear function of the location Z, but parameter H is quadratic:

$$H(Z_j) = \left(Z_j^2 - 1\right) \ln \frac{Z_j - 1}{Z_j + 1} + 2\, Z_j \qquad (21)$$

This is why the element is called the *second order line doublet.* The singularities can be eliminated like in the former point.

Based on *Eq.* (19) the potential and stream functions at node j are the following:

$$\Phi_{s,j} = \Re(\Omega_{m,j}) = \lambda_{s,j}\, A_{m,j}(x, y) + \lambda'_{s,j}\, A'_{s,j}(x, y)\; \Psi_{s,j} = \Im(\Omega_{s,j}) \qquad (22)$$

where $A_{s,j}(x, y)$ and $A'_{s,j}(x, y)$ are the function of the location, like in case of the first order line doublet.

3.2. Application Possibilities

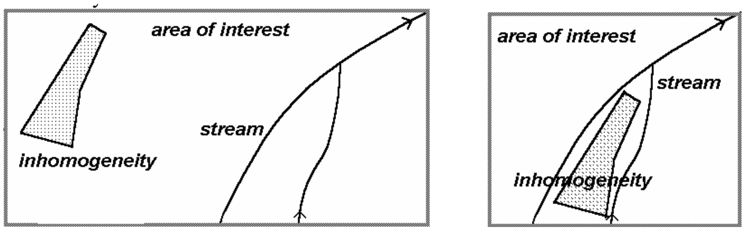
The former point showed two possible ways for the description of inhomogeneities with the help of line doublets. It is obvious, that the second order one gives the better approximation, but it requires more computational resources. Though nowadays, with the help of supercomputers its importance is decreasing, it is worth to make clear the applicability of the two approximations. Sometimes it is rather difficult to decide which one may be sufficient in case of a given problem. This point tries to give some recommendations for it.

As a help, some simple examples are given in *Figs. 7* and *8*. This is a series of problems with an inhomogeneity and some streams. The difference between them is the location of the elements compared to each other within the area of interest *(Fig. 7)* and the location of the area of interest compared to the individual elements (*Fig. 8*).

First order line doublets give a sufficient global description but locally a rough estimation, so they should rather be applied if

- the base elevation Z_1 varies,
- the aquifer thickness Hchanges,
- permeability k varies slightly, rather within one magnitude,
- the dominating element or elements within the area of interest are far away of the inhomogeneity concerned (*Fig. 7/a*),
- the inhomogeneity is far away from the area of interest, that is why its local behaviour is out of importance (*Fig. 8/a*).

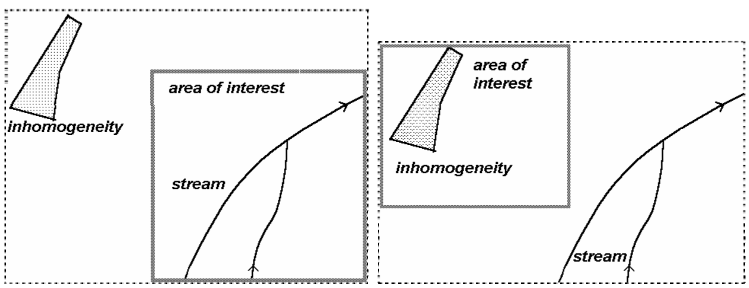
The locally also more accurate second order approximation is required if



a./ elements far *b./ elements together*

Fig. 7. Stream and inhomogeneity in the area of interest

- the variation of seepage coefficient k is sharp, even several magnitudes,
- more parameters vary together within the same border,
- the inhomogeneity and some nearby lying other elements together have dominating effects over the area of interest (*Fig. 7/b*),
- the inhomogeneity itself is of basic importance (*Fig. 8/a*).



a./ outside the area of interest *b./ inside the area of interest*

Fig. 8. Stream and inhomogeneity over the area examined

3.3. Embedded Line Doublets

Embedded inhomogeneities may be applied in case if an inhomogeneity is fully surrounded by an other one. Embedded inhomogeneities require special considerations and care. To demonstrate this, the resultant piezometric heads of a series of tests is given in *Fig. 9*. This series consists of a larger rectangular inhomogeneity of permeability containing a smaller one with an area of one fourth, at different locations, with different permeability, base elevation and aquifer thickness. Due to the symmetry, only half of it is viewed. Based on them and also of those not presented here, the following experiences may be summarized:

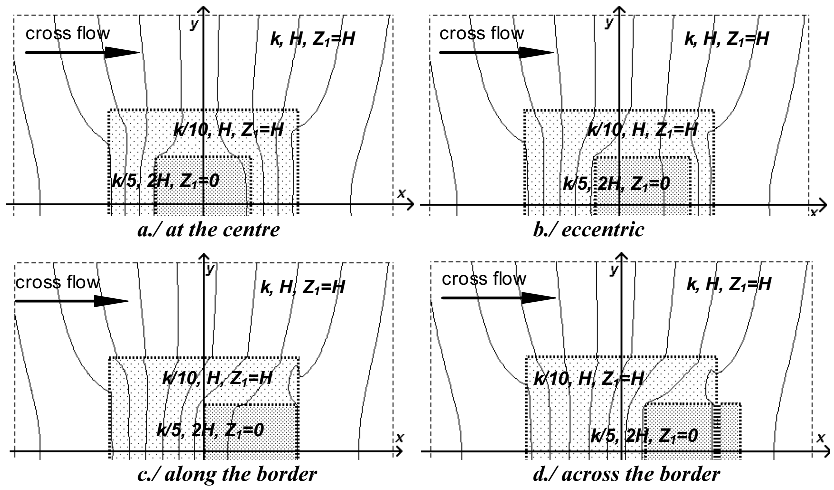


Fig. 9. Embedded inhomogeneity

– any combination of permeability, base elevation and aquifer thickness may result embedded inhomogeneity;

– if the borders of the two or more inhomogeneities are far enough, the application of first or second order line doublets may follow the guidelines given in the former point (*Figs. 9/a* and *9/b*;

– if the borders are rather near, like in *Fig. 9/c*, it is better to apply the second order line doublets, even in case of slight variations;

– the borders of the two inhomogeneities cannot cross each other;

– if the geological conditions require inhomogeneities with crossing borders, like in *Fig. 9/d*, then one of them - usually one with smaller (hydraulic) importance - must be split into two parts in such a way, that within each single inhomogeneity the three parameters (permeability, aquifer thickness, base elevation) are constant;

– along those parts, where the borders of two or more inhomogeneities run near to each other (*Figs. 9/c, 9/d*), a denser string of line doublets is required, while further a sparser one may also be sufficient;

– though it sounds obvious, it has to be emphasized, that one string element may belong only to one inhomogeneity, between the parallel running borders there must be a slight distance;

– the size of this "slight distance" is difficult to determine, but as a first assumption, the half of the aquifer thickness may me assumed, though a sensitivity test is always advisable.

4. Doublet in Homogeneous Flow to Model Inhomogeneities

4.1. Doublet in Homogeneous Flow

Line doublets give a rather good approximation while using them for the modelling of inhomogeneities, even in case of compound shape ones. But *Eqs.* (15) or (20) - even though they look simple - require large computational efforts. Therefore sometimes a less general but simpler solution may also be sufficient, especially in case of large number of similar shape of inhomogeneities. This simpler solution is based on the flow pattern of doublet in homogeneous flow, the well known problem of the flow around a cylinder.

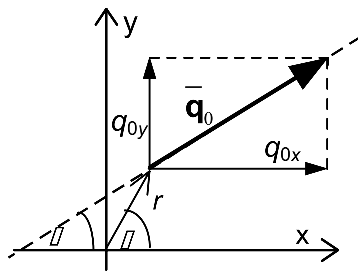


Fig. 10. Homogeneous flow

The potential itself is the superposition of the potentials of the homogeneous (also called cross) flow and the doublet. Cross flow may be characterized by the specific discharge vector $\overline{q}_h = q_{hx} + i q_{hy}$ given in *Fig. 10.* In this case the potential is

$$\Omega_h = -\, q_h\, z\, e^{-i\alpha} \tag{23}$$

If a doublet is placed in this flow, the potential turns to be the sum of *Eqs.* (10) and (23):

$$\Omega_{hd} = \Omega_h + \Omega_d = -\, q_0\, z e^{-i\alpha} + \frac{M}{z - z_0} \tag{24}$$

This is the general form of the well-known potential the *flow around a cylinder*. If the following simplifications are introduced:

1. the flow direction should be in the positive x direction, i.e. $\alpha = 0$,
2. the doublet should have a real moment, against flow direction, i.e. $\beta = \pi$
3. the doublet is placed at the origin, i.e. $z_0 = x_0 + iy_0 = 0$

then the potential turns to be

$$\Omega_c = -\, q_0\, z - \frac{M_0}{z} \tag{25}$$

This is the form that usually appears in textbooks. The flow pattern can be seen in *Fig. 11*, where the equipotential lines are continuous, the streamlines are dotted. Due to symmetry once again the upper half is viewed.

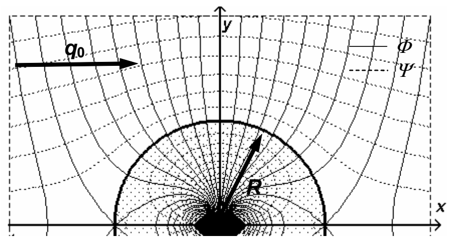


Fig. 11. Flow around a cylinder

There is a special streamline, $\Psi = 0$, that forms a circle of $r^2 = x^2 + y^2 = M_0 / q_0$. This means that there is no flow across the circle of the radius $R = \sqrt{M_0 / q_0}$. It can be considered as an impermeable border, the flow is only around it. This is a special inhomogeneity with zero permeability. With the help of the radius *Eq.* (25) turns to be

$$\Omega_c = -q_0 \left(z + \frac{R^2}{z} \right) \tag{26}$$

4.2. *Single Inhomogeneity of Permeability*

Let us consider a circle of the radius $R_1 \neq R$ at the origin with the permeability k_1 inside, and $k_0$outside *(Fig. 12)*. This is an inhomogeneity of permeability only, like in *Point 2.3.* Let us assume, that out of this circle the flow is similar to the one in the former point, while inside it is a homogeneous flow.

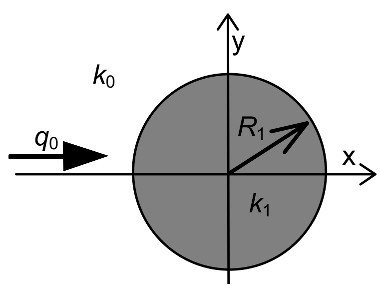


Fig. 12. Circular inhomogeneity

Based on *Eq.* (26) outside the cylinder ($r = \sqrt{x^2 + y^2} \geq R_1$) the potential is

$$\Omega_0 = -q_0(z + \frac{AR_1^2}{z}) + C\Phi_0 \tag{27}$$

where Φ_0 is the potential at the centre, A and C are real constants. Comparing *Eqs.* (26) and (27) it can be understood, that *Eq.* (27) is really a *flow around a cylinder*, with the radius of $R = R_1\sqrt{A}$. But this potential has to be considered not outside the radius R but outside R_1.

Inside the cylinder ($r = \sqrt{x^2 + y^2} \leq R_1$) it is a cross flow of *Eq.* (23). As the flow direction is parallel with the positive x axis and the permeability is not the same as outside, the complex potential is

$$\Omega_i = -Bq_0 z + \Phi_0 \tag{28}$$

where B – like A and C – is a real constant. These three constants can be determined with the help of the conditions in *Point 2.3.* Based on *Eq.* (3) along the circle
$\longrightarrow$ the stream function has to be continuous, so

$$\Psi_0(r = R_1) = \Psi(r = R_1) \tag{29}$$

$\longrightarrow$ the groundwater surface has to be continuous, so

$$\varphi_0(r = R_1) = \varphi_i(r = R_1) = \varphi_R. \tag{30}$$

Based on the definition of the discharge potential (*Eq.* (2)) and its variation along the border of the inhomogeneity (*Eqs.* (4) and (5)), the potential difference along the border is

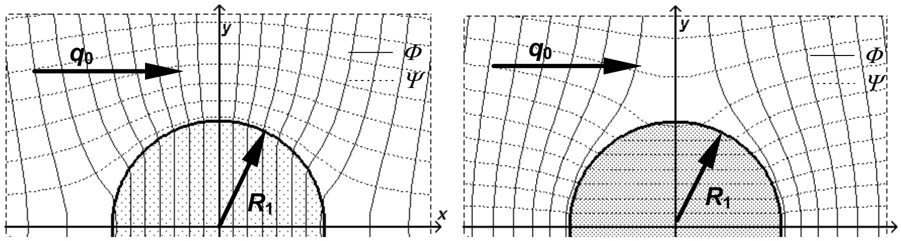
$$\Phi_0(r = R_1) - \Phi_i(r = R_1) = \Delta\Phi = \Phi_0(r = R_1)(1 - \frac{k_1}{k_0}) \tag{31}$$

so the constants are as follows:

$$A = \frac{k_0 - k_1}{k_0 + k_1} \qquad B = \frac{2k_1}{k_0 + k_1} \qquad C = \frac{k_0}{k_1} \tag{32}$$

The constant A is always smaller than 1. It can be positive or negative, while the others must be positive. If the inhomogeneity is due to a less permeable soil ($k_1 < k_0$) then A is positive, B is smaller, C is bigger than 1. Such a flow pattern is given in *Fig. 12/a.* Otherwise, if the permeability of the inhomogeneity is bigger than its surroundings ($k_1 > k_0$), then A is negative, B is bigger, C is smaller than 1. This is demonstrated in *Fig. 12/b.* Both show the equipotential lines as continuous, and the streamlines as dotted. Due to symmetry only the upper half is viewed.

Such a flow pattern is presented in most of the textbooks of the analytic element method (e.g. STRACK [6], HAITJEMA [5]).



a./ $k_0 > k_1, \Rightarrow A > 0$ *b./* $k_0 < k_1 \Rightarrow A < 0$

Fig. 13. Circular inhomogeneities

4.3. The Ring-Shaped Inhomogeneity

Let us consider an embedded inhomogeneity of two concentric circles of the radius R_1 and R_2. Neither of them equals the cylinder radius of *Point 4.1.* The permeability outside is k_0, between the two circles, over the ring k_1 and inside k_2 *(Fig. 14)*. This is an inhomogeneity of only permeability, but now an embedded one. Let us assume, that out of this circle the flow is like in the former point, i.e. doublet in homogeneous flow. Between the two circles, a similar flow pattern may be assumed, just the homogeneous flow is a modified one. And finally, inside there is the third type of cross flow.

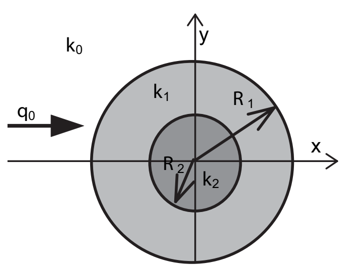


Fig. 14. Ring-shaped inhomogeneity

Similar to *Eq.* (27) outside the bigger circle ($r = \sqrt{x^2 + y^2} \geq R_1$) the potential is

$$\Omega_0 = -q_0\left(z + \frac{AR_1^2}{z}\right) + F\Phi_0 \tag{33}$$

where Φ_0 is the potential at the centre, A and F are real constants. Comparing *Eqs.* (27) and (33) once again it is clear, that *Eq.* (33) is really a *flow around a cylinder* of $R = R_1\sqrt{A}$, just the potential has to be considered outside of R_1.

Between the circles, over the ring ($R_1 \geq r \geq R_2$) the flow is once again *the flow around the cylinder* (see *Eqs.* (27) or (33)), but now the homogeneous flow is different. Actually this cross flow is similar to the inside flow of the former point (see *Eq.* (28)). The potential over this area is

$$\Omega_r = -q_0 B\big(z + \frac{C R_2^2}{z}\big) + E\Phi_0 \tag{34}$$

where B, C and E are real constants. Comparing *Eqs.* (26) and (34) it is once again clear, that *Eq.* (34) is also a *flow around a cylinder*, but now with the radius of $R = R_2\sqrt{BC}$ and the cross flow has the specific discharge of $q = q_0 B$.

Finally, inside the inner cylinder, ($r = \sqrt{x^2 + y^2} \leq R_1$) it is a homogeneous flow (see *Eq.* (23)). Similar to *Eq.* (28) the complex potential is

$$\Omega_i = -Dq_0 z + \Phi_0 \tag{35}$$

where D – like the other constants – is real. They can be determined with the help of *Point 2.3.* Now at both borders (R_1 and R_2) the conditions of *Eq.* (3) has to be fulfilled. They are as follows:

⟶ the stream function has to be continuous, so

$$\Phi_i(r = R_2) = \Phi_r(r = R_2) \text{ and } \Phi_r(r = R_1) = \Phi_0(r = R_1) \tag{36}$$

⟶ the groundwater surface has to be continuous, so

$$\varphi_i(r = R_2) = \varphi_r(r = R_2) \text{ and } \varphi_i(r = R_1) = \varphi_r(r = R_1) \tag{37}$$

Based on the considerations detailed earlier, the potential difference along the circle of R_1 is

$$\Phi_0(r = R_1) = \Delta\Phi_0 = \Phi_0(r = R_1)\big(1 - \frac{k_1}{k_0}\big) \tag{38}$$

while along the inner circle of R_2

$$\Phi_r(r = R_2) = \Delta\Phi_r = \Phi_r(r = R_2)\big(1 - \frac{k_2}{k_1}\big) \tag{39}$$

With the help of the above conditions, the constants are

$$\begin{aligned} &A = \frac{k_0\left(R_1^2+CR_2^2\right)-k_1\left(R_1^2-CR_2^2\right)}{k_0\left(R_1^2+CR_2^2\right)+k_1\left(R_1^2-CR_2^2\right)} \qquad B = \frac{2k_1 R_1^2}{k_0\left(R_1^2+CR_2^2\right)+k_1\left(R_1^2-CR_2^2\right)} \\ &C = \frac{k_1-k_2}{k_1+k_2}, \qquad D = B\,(1-C) \qquad E = \frac{k_1}{k_2} \qquad F = \frac{k_0}{k_1} \end{aligned} \tag{40}$$

In this case it is more difficult to analyse the values of the constants. A and C play a similar role as A of *Point 4.1*, so they may be negative as well, the others are positive. *Fig. 15* shows some examples for it. The figure follows the usual notation as the former ones.

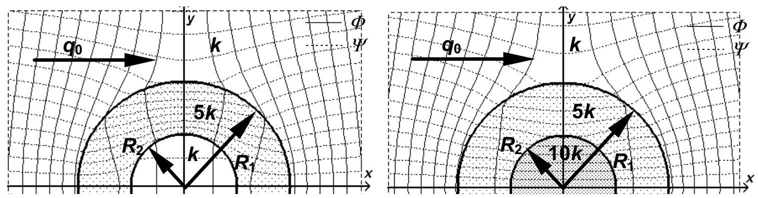


Fig. 15. Examples for ring-shaped inhomogeneities in cross flow

4.4. Application Possibilities

Line doublet inhomogeneities require huge computational efforts to produce a general but approximate result, as the condition for the potential step is fulfilled only at certain points (nodes, centre, etc.) of polygons. Nevertheless, this element can handle all types of inhomogeneities, not only of permeability, but of base elevation, aquifer thickness and any combination.

On the other hand, inhomogeneity as "doublet in cross flow" may describe only the inhomogeneity of permeability, it cannot handle base elevation or aquifer thickness. It works only together with homogeneous flow. It can handle only circular inhomogeneities, and an embedded one has to be concentric. Though in *Points 4.2.* and *4.3.* it is introduced as the centre in the origin and the flow is parallel with the x axis, with a simple transformation it can be located at any point of the plane. But even if its use is limited, it may be a handy tool.

Below some recommendations are given when to apply doublet in cross flow to describe inhomogeneities:

– if the only aquifer parameter that varies is the permeability,
– if the inhomogeneity can be considered as such a plain figure, like a square, a hexagon, etc. , that it can be approximated as a circle,
– if the local behaviour of the inhomogeneity of almost any shape is of minor importance, but its regional effects have to be taken into consideration,
– if embedded inhomogeneities may be considered as concentric,
– if there are a large number of inhomogeneities with smaller or bigger hydraulic importance, like an area with compound hydrogeologic conditions.

And some ideas when line doublet may be more sufficient:

– changing of the base elevation or the aquifer thickness,
– long, narrow or rather compound shape of inhomogeneity,
– the local behaviour of a compound shape of inhomogeneity is of basic importance,
– the embedded inhomogeneities cannot be considered as concentric.

5. Summary and Conclusion

The present paper focused on one certain element of the analytical elements method for the modelling of groundwater flow. The method itself is based on the potential theory using the discharge potential. Each individual feature of the aquifer, the so-called elements, are described by well chosen potentials, and the description of the full aquifer is obtained by the superposition of them.

The element examined is the inhomogeneity that describes local variation of aquifer parameters, as permeability, base elevation and thickness. Such an element may be applied to describe not only different geological formations of the aquifer but structures, as well, like a river barrage with its impermeable fundament plate and sheet piles, as the variation of its thickness and permeability.

After a short introduction of the method a more detailed description of inhomogeneities were given. Then the most important features of the general line doublet inhomogeneities were summarized. The next point described a simpler approach how to use the flow pattern of doublet in cross flow to model inhomogeneities. Though this is a rather handy tool, it has a limited use, so some directions of the further development possibilities can be:

– the description of the variation of the base elevation and the thickness of the aquifer,
– embedded inhomogeneities with gradually varying aquifer parameters, like a sloping base,
– inhomogeneities of other regular plain figures, like an ellipse.

List of Symbols

i:	imaginary unit, $i = \sqrt{-1}$;
$k\ (k_x, k_y)$, m/s :	coefficient of seepage (with its x *and* y components);
$q(q_x, q_y)$, m^2/s :	specific discharge (with its x and y components)
t, s :	time;
x, y, z, m :	the horizontal and vertical co-ordinates;
$z = x + iy$:	complex number;
C:	integration or other constant;
H, m :	thickness of aquifer;
$T\ (T_x, T_y)$, m^2/s :	transmissibility (with its x and y components);
Z_1, Z_2, m :	the lower and upper boundary level of the aquifer;
φ, m :	piezometric head (groundwater level) above a certain reference level;
Φ, m^3/s :	discharge potential;

Λ, (m^3/s, m^2/s or m/s) :	intensity or strength parameter of potentials;
Ψ, m^3/s :	stream function as the conjugated function of discharge potential;
Ω, (m^3/s) :	complex potential, $\Omega = \Phi + i\,\Psi$;
$\Im(\Omega)$, m^3/s :	imaginary part of complex potential;
$\Re(\Omega)$, m^3/s :	real part of complex potential.

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Appendix

Appendix 1.: Lineart Transformation

Line elements are usually obtained by the integration of point elements. While integrating a function along the line with the endpoints (z_j, z_{j+1}), it may be though useful to introduce the transformation given in *Fig. A1*. This transforms the centre point to the origin of the co-ordinate system, and the endpoints (z_j, z_{j+1}) go to the points +1 and -1 along the real axis. The transformation is linear, its function is: $Z = X + iY = \frac{z - \frac{1}{2}(z_1 + z_2)}{\frac{1}{2}(z_2 - z_1)}$. For the notation see the figure.

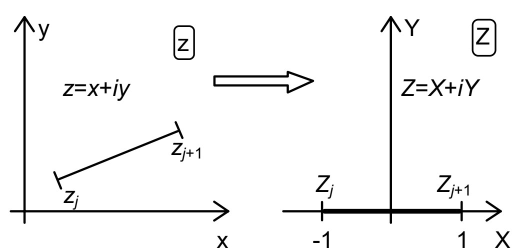

Fig. A1 Linear transformation