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THE CONVOLUTION OF SERIES AND ITS APPLICATION ON BAR STRUCTURE

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Abstract

The cascade model is widely applied in water-currents flow calculations. The mathematical background of the cascade models is convolution. The convolution, especially in the case of continuous functions, is usually solved by Laplace transformation, which is handled with considerable difficulty.

This study principally deals with convolution of series, rather than continuous functions. The convolution is traced back to the multiplication of the series' characteristic polynomials. To find a suitable solution for this task, instead of the traditional definition of the linear space a more general definition was adopted. According to this adopted definition, the linear combination created by elements of module from vectors under addition in a way that the external relationship -instead of multiplying the vectors with real numbers- is solved by square matrixes. To cast the external relationship into a matrix multiplication form is possible because the algebraic structure found in the external relationship sufficient to be 'ring' according to the general definition of the linear space, and it is not required to be 'field' (body) as what used to be the common concept in traditional engineering practice.

The method can be applied on any model that can be described by the linear differential equations with constant coefficients. To make it easy to follow each step of implementing the procedure it was demonstrated by application on one of the most simple bar structures, that is a Kelvin-Voigt type material model of cantilever.

Keywords: cascade model, series, convolution, bar structure.

1. Introduction

The cascade model is an easy-to-handle solution suitable for linear differential equations with constant coefficients in their description to different processes. The essence of this method is that to find the general solution (2) for the differential equation (1), [2, 3, 7], the function $f(t)$ being considered to be periodically constant can be taken out of the integration process. By doing so, the remaining integrant can be integrated simply and independently from the process function:

$$\dot{x} = ax + f(t), \tag{1}$$

$$x(t) = e^{at}x_{\circ} + e^{at}\int_{0}^{t} e^{-a\tau} f(\tau)\cdot d\tau. \tag{2}$$

In this paper, the transform from continuous functions to numeric series is presented first, then the convolution of series is defined [1, 3, 5, 6] , and finally the background model is generalized to include differential equation system. Instead of considering the usual representation of the linear space, attention was shifted to a vector module over ring of square matrixes. This arrangement leads easily to the ability of modeling the task at hand by the linear combination of the vectors and matrixes.

In this developed procedure, due to the fact that the general solution (4) of the differential equations system (3) is similar to the solution discussed in (2). and because the emphasis is on 'system' so the whole discussion and its logic can be generalized to be applied in multidimensional concept.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{f}(t), \tag{3}$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}x_{\circ} + e^{\mathbf{A}t}\int_{0}^{t} e^{-\mathbf{A}\tau}\mathbf{f}(\tau)\cdot d\tau. \tag{4}$$

2. Numerical Series and their Convolution

Let us have the series (5), which is finite, but can be continued at will:

$$a = [a_0, a_1, \ldots, a_n]\,. \tag{5}$$

Similarly, the series (6) can be defined, and can be considered as a series of answer of unit causes:

$$b = [b_0, b_1, \ldots, b_m]\,. \tag{6}$$

These assumptions are illustrated in *Fig. 2*.

If equation (5) is considered to be the series of the real causes, then the series of the effects can be constructed according to the following arrangement, the effects

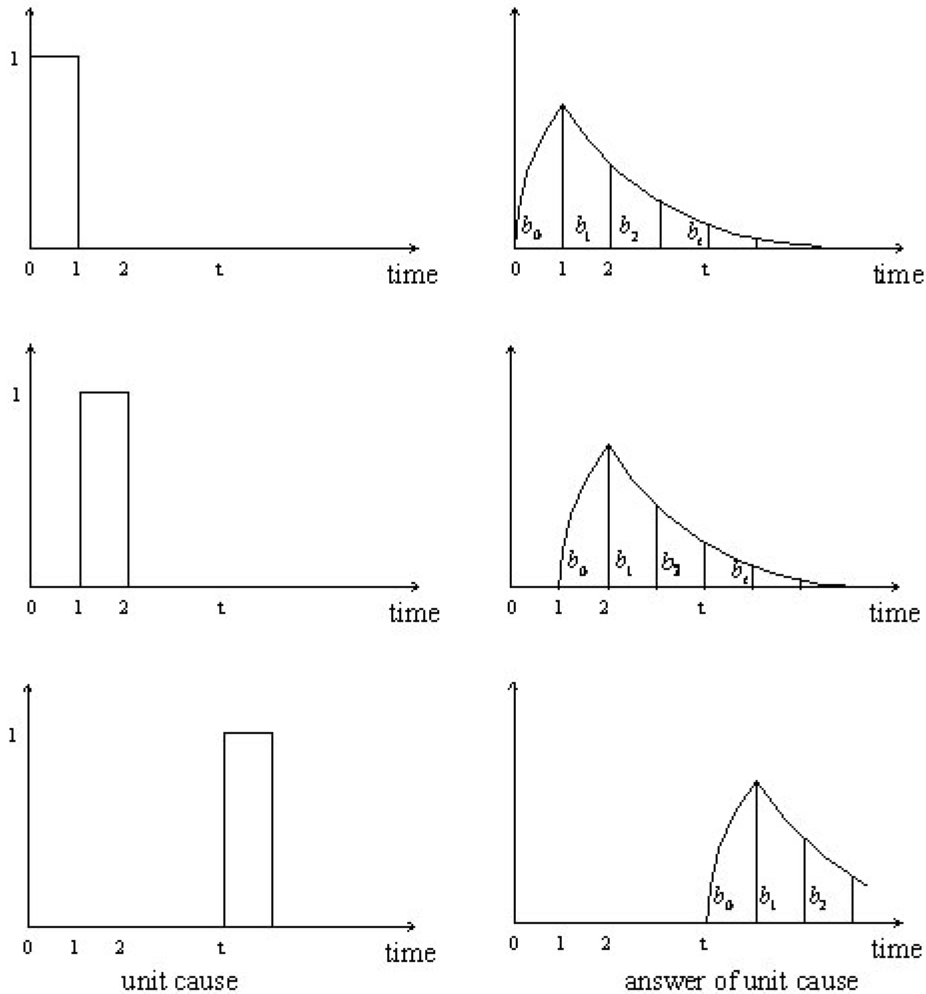


Fig. 1. Succession of sequential unit cause and the series of their reactions

are illustrated in *Fig. 2*.

$$
\begin{array}{llll}
n=0 & 1 & 2 & 3 \\
a_0b_0 & a_0b_1 & a_0b_2 & a_0b_3 \\
 & a_1b_0 & a_1b_1 & a_1b_2 \\
 & & a_2b_0 & a_2b_1 \\
 & & & a_3b_0 \\
\hline
\sum_{i=0}^{n=0} a_i b_{n-i}, & \sum_{i=0}^{n=1} a_i b_{n-i}, & \sum_{i=0}^{n=2} a_i b_{n-i}, & \sum_{i=0}^{n=3} a_i b_{n-i}, \dots
\end{array}
\tag{7}
$$

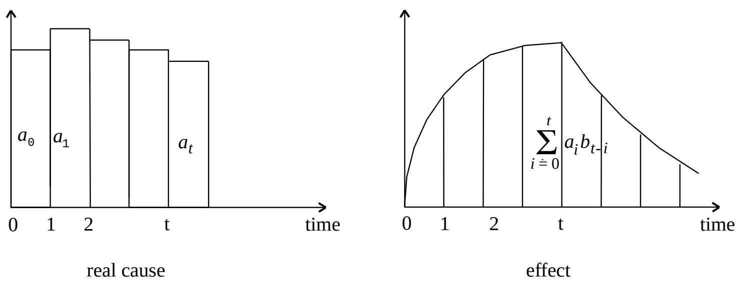


Fig. 2. The real 'causes' and the series of their effects

If we have the continuous functions $f(x)$ and $g(x)$, their convolution can be determined as follows [2, 3, 7]:

$$k(t) = f(x) * g(x) = \int_0^t f(x) \cdot g(t-x) \cdot dx = \int_0^t f(t-x) \cdot g(x) \cdot dx. \quad (8)$$

Therefore literature [5]associates the series with characteristic functions (9):

$$\begin{aligned} A(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n \\ B(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots + b_m x^m \end{aligned} \quad (9)$$

Equations (9) have a multiplication polynomial of the form:

$$A(x) \cdot B(x) = \sum_{i=0}^{n} \sum_{j=0}^{m} a_i b_j x^{i+j} = \sum_{k=0}^{t} a_k b_{t-k} \cdot x^t, \; t = 0..n+m. \quad (10)$$

The series of coefficients of *Eq.* (10) is equal to the series of sums in *Eq.* (7), and is called the convolution of a and b series. This was what led to the decision to consider the numerical series rather than the continuous functions [6].

3. The Linear Space and the Vector Module Over Ring

In mathematics, it is possible according to *Fig.*3, to define a special structure [9]. In this structure internal relationship is understood to be a kind of relationship in which two elements (operands) produce third element (value) in a way that the first two elements and the third one are all members of the same set (strictly speaking, the relationship is understood as operation). In the case presented in this study, in a set of vectors of given dimension, the addition is understood as internal relationship or operation. External relationship on the other hand, is taken as a kind of relationship in which the connection is constructed between elements from different sets (loosely this connection is taken as operation too). Here, it is worthy to mention that determining the linear combination of vectors is as if the sum of vectors multiplied by scalar is being constructed.

The vectors under addition (internal relationship) constitute module (one-operation algebraic structure). If another algebraic structure is taken into consideration, let it be the 'field' of a real number (two-operation algebraic structure), then with the use of these algebraic structure's elements, it is possible to construct the vectors linear combination as an external relationship (multiplication). However, there is more general algebraic structure than the field which was previously used in constructing the external relationship, that is called a 'ring', which is a set S together with two binary operators '+' and '$\cdot$' (commonly interpreted as addition and multiplication, respectively) satisfying the following condition:

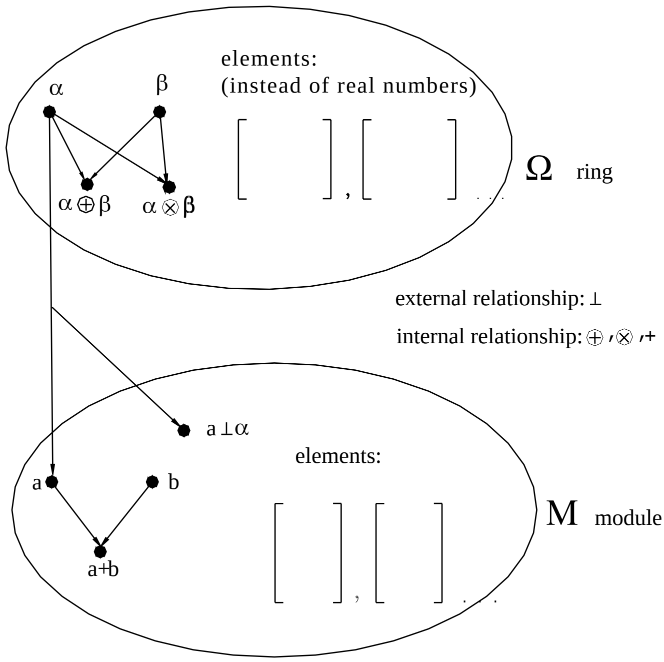
1. Additive associativity: For all $a, b, c \in S$, $(a+b)+c = a + (b+c)$,
2. Additive commutativity: For all $a, b \in S$, $a+b = b+a$,
3. Additive identity: There exists an element $0 \in S$ such that for all $a \in S$, $0 + a = a + 0 = a$,
4. Additive inverse: For every $a \in S$ there exists $-a \in S$ such that $a + (-a) = (-a) + a = 0$,
5. Multiplicative associativity: For all $a, b, c \in S$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,
6. Left and right distributivity: For all $a, b, c \in S$, $ad \cdot (b+c) = (a \cdot b) + (a \cdot c)$ and $(b+c) \cdot a = (b \cdot a) + (c \cdot a)$.

A module taking its coefficients in a ring R is called a module over ring, or an R-module.

This exactly satisfies the needs of the task at hand if the vectors combination is being constructed by the use of square matrixes of suitable order. Here in this study, instead of taking into consideration the traditional conception of the linear space, a more 'general' conception is being utilized in which the linear space is constructed by determining the sum of the vectors multiplied by the square matrixes. This 'generalized' linear space makes it possible to expand number series combination to convolution of matrix series and vector series with complete accuracy.

4. Application of the Procedure on Bar Structure

Let one consider the cantilever shown in *Fig. 4*. The external force applied on the structure produces internal forces and moments resulting in the displacements of the bar's endpoint. In the case of ideal elastic material, where the Hooke's law is valid, the relationship between the displacements and the internal forces can be written [4, 11] as:



$\alpha, \beta \in \Omega, a, b \in M$
1)$\alpha \perp (a + b) = \alpha \perp a + \alpha \perp b$
2) $(\alpha \oplus \beta) \perp a = \alpha \perp a + \beta \perp a$
3)$(\alpha \otimes \beta) \perp a = \alpha \perp (\beta \perp a)$
4) $e \perp a = a$

Fig. 3. The ‘general’ linear space

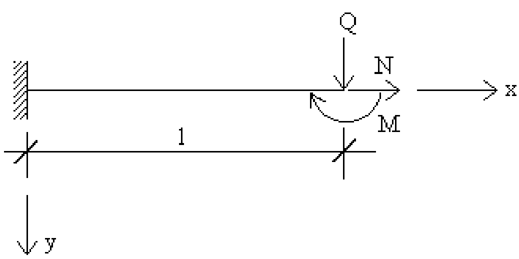


Fig. 4. Cantilever construction and force applied

$$\mathbf{u} = \mathbf{F} \cdot \mathbf{s}. \tag{11}$$

F is the flexibility matrix of an l-long constant stiffness cantilever, which in the case of plane cantilever is given as:

$$\mathbf{F} = \begin{bmatrix} \frac{l}{EA} & & \\ & \frac{l^3}{3EJ} & \frac{l^2}{2EJ} \\ & \frac{l^2}{2EJ} & \frac{l}{EJ} \end{bmatrix}, \tag{12}$$

where A is the area of the bar's cross section, E is the modulus of elasticity, and J is the moment of inertia.

If the bar's material is of visco-elastic type and the Kelvin–Voigt material law can be applied on it, then *Eq.* (11) can take the form:

$$\mathbf{u} + tk \cdot \dot{\mathbf{u}} = \mathbf{F} \cdot \mathbf{s}, \tag{13}$$

where $\dot{\mathbf{u}}$ is the time derivative of displacement vector, and tk is retardation time.

The answer function of unit load can be determined by finding the solution of differential equation system (13) for the initial condition $\mathbf{u}(0)=0$, in a way that unit loads are applied during unit time at the endpoint of the cantilever.

The loading case and unloading case can be handled at the same time by the help of Heaviside function (if $t > 0$ then $\Delta(t)=1$, otherwise $\Delta(t)=0$).

First, where N=1 kN, ($k = 10^3$), and Q and M are zero, the *Eq.* (13) takes the form:

$$\begin{bmatrix} u_{Nx} \\ u_{Ny} \\ \varphi_N \end{bmatrix} + tk \begin{bmatrix} \dot{u}_{Nx} \\ \dot{u}_{Ny} \\ \dot{\varphi}_N \end{bmatrix} = \begin{bmatrix} l/EA \\ 0 \\ 0 \end{bmatrix} \cdot [\Delta(t) - \Delta(t-1)]. \tag{14}$$

The solution of *Eq.* (14) can be given in the following form:

$$\begin{bmatrix} u_{Nx}(t) \\ u_{Ny}(t) \\ \varphi_N(t) \end{bmatrix} = \begin{bmatrix} l/EA \\ 0 \\ 0 \end{bmatrix} \cdot v(t), \tag{15}$$

where

$$v(t) = \left(1 - e^{\frac{-t}{tk}}\right) \cdot \Delta(t) - \left(1 - e^{-\frac{t-1}{tk}}\right) \cdot \Delta(t-1).$$

Second, where Q=1 kN, and N and M are zero, the *Eq.* (13) takes the form:

$$\begin{bmatrix} u_{Qx} \\ u_{Qy} \\ \varphi_Q \end{bmatrix} + tk \begin{bmatrix} \dot{u}_{Qx} \\ \dot{u}_{Qy} \\ \dot{\varphi}_Q \end{bmatrix} = \begin{bmatrix} 0 \\ l^3/3EJ \\ l^2/2EJ \end{bmatrix} \cdot [\Delta(t) - \Delta(t-1)]. \tag{16}$$

The solution of *Eq.* (16) can take the form:

$$\begin{bmatrix} u_{Qx}(t) \\ u_{Qy}(t) \\ \varphi_Q(t) \end{bmatrix} = \begin{bmatrix} 0 \\ l^3/3EJ \\ l^2/2EJ \end{bmatrix} \cdot v(t)\,. \tag{17}$$

Finally, where M=1 kNm, and N and Q are zero, the *Eq.* (13) takes the form:

$$\begin{bmatrix} u_{Mx} \\ u_{My} \\ \varphi_M \end{bmatrix} + tk \begin{bmatrix} \dot{u}_{Mx} \\ \dot{u}_{My} \\ \dot{\varphi}_M \end{bmatrix} = \begin{bmatrix} 0 \\ l^2/2EJ \\ l/EJ \end{bmatrix} \cdot [\Delta(t) - \Delta(t-1)]\,. \tag{18}$$

The solution of *Eq.* (18) then can be given as:

$$\begin{bmatrix} u_{Mx}(t) \\ u_{My}(t) \\ \varphi_M(t) \end{bmatrix} = \begin{bmatrix} 0 \\ l^2/2EJ \\ l/EJ \end{bmatrix} \cdot v(t)\,. \tag{19}$$

It is clear from equations (15), (17) and (19) that it is possible to obtain the answer function of unit loads by multiplying the function $v(t)$ by the elements of flexibility matrix, *Fig.*5.

If the previous discussion is incorporated into the equation system of the state change of the bar structures [4, 11], that in this case means that the **F** block has to be multiplied by $v(t)$ function, this is shown in *Eq.* (20):

$$\begin{bmatrix} & \mathbf{G} \\ \mathbf{G}^{\mathbf{T}} & \mathbf{F} \cdot v(t) \end{bmatrix} \cdot \begin{bmatrix} \mathbf{u} \\ \mathbf{s} \end{bmatrix} + \begin{bmatrix} \mathbf{q} \\ \mathbf{t} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}. \tag{20}$$

In a way suitable for the displacement method, if **s** vector from the compatibility equation is determined and then substituted in the equilibrium equation, the following *Eq.* (21) is obtained:

$$\mathbf{u} = \mathbf{K}^{-1} \cdot v(t) \cdot \mathbf{q}, \tag{21}$$

where

$$\mathbf{K} = \mathbf{G} \cdot \mathbf{F}^{-1} \cdot \mathbf{G}^{\mathbf{T}}.$$

where **K** is stiffness matrix, **G** is geometric matrix, **q** is nodal reduced load vector and **t** is kinematical load vector.

The answer series of the unit load is obtained by multiplying the series constructed from the function $v(t)$ by the inverse of **K**. The result is shown in *Eq.* (22).

$$\mathbf{A}_t = [\mathbf{A}_0, \mathbf{A}_1, \cdots, \mathbf{A}_i, \cdots, \mathbf{A}_n]\,, \tag{22}$$

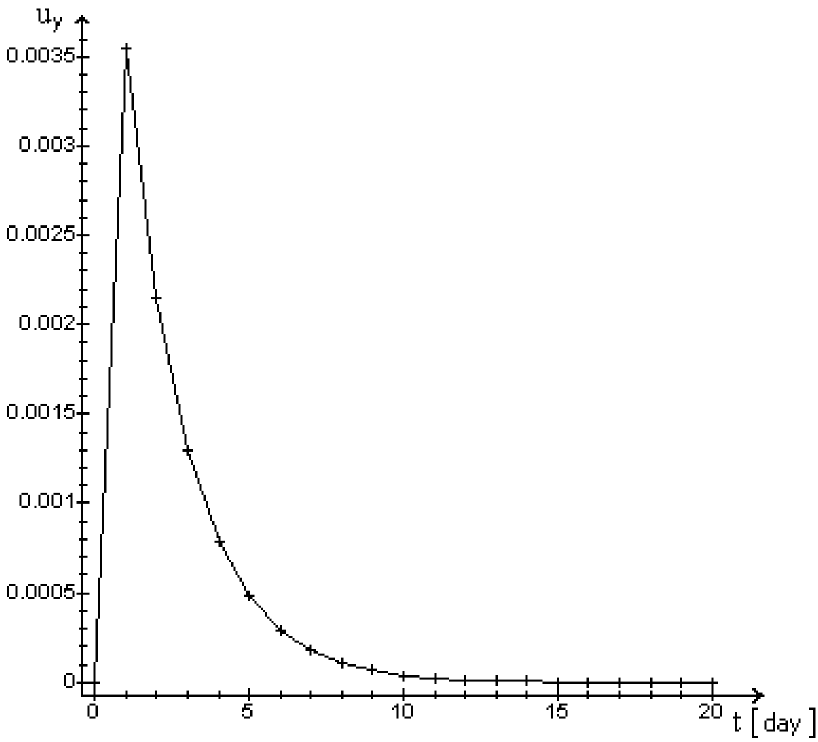


Fig. 5. Answer of unit load of cantilever's endpoint

where

$$\mathbf{A}_i = v\,(i) \cdot \mathbf{K}^{-1}.$$

In the present case the vector series of real cause is the series of the nodal reduced loads.

$$\mathbf{q}_t = \left[\mathbf{q}_0, \mathbf{q}_1, \ldots, \mathbf{q}_i, \ldots, \mathbf{q}_m\right] \tag{23}$$

Finally, the vector series of the displacement is determined by the convolution of the matrix series (22) and the vector series (23). The pattern of this convolution is illustrated in *Fig. 6*, and its calculation can be done using the relationship constructed in *Eq*. (24).

$$\mathbf{u}_t = \sum_{k=0}^{t} \mathbf{A}_k \cdot \mathbf{q}_{t-k} \tag{24}$$

$$
\begin{array}{cccccccccc}
 & & t=0 & & t=1 & & & t=2 & & t=3 \\
 & \boxed{\mathbf{B}_0} & \boxed{\mathbf{a}_0} & & \boxed{\mathbf{B}_1} & \boxed{\mathbf{a}_0} & & \boxed{\mathbf{B}_2} & \boxed{\mathbf{a}_0} & --- \\
 & & & & \boxed{\mathbf{B}_0} & \boxed{\mathbf{a}_1} & & \boxed{\mathbf{B}_1} & \boxed{\mathbf{a}_1} & --- \\
 & & & & & & & \boxed{\mathbf{B}_0} & \boxed{\mathbf{a}_2} & --- \\
\hline
\sum_{k=0}^{t=0} & \boxed{\mathbf{B}_k} & \boxed{\mathbf{a}_{t-k}} & \sum_{k=0}^{t=1} & \boxed{\mathbf{B}_k} & \boxed{\mathbf{a}_{t-k}} & \sum_{k=0}^{t=2} & \boxed{\mathbf{B}_k} & \boxed{\mathbf{a}_{t-k}} & ---
\end{array}
$$

Fig. 6. Linear combination represented by matrixes

Example:

Determining the displacement experienced at the endpoint of the cantilever shown in *Fig. 7* is illustrated in the following solution.

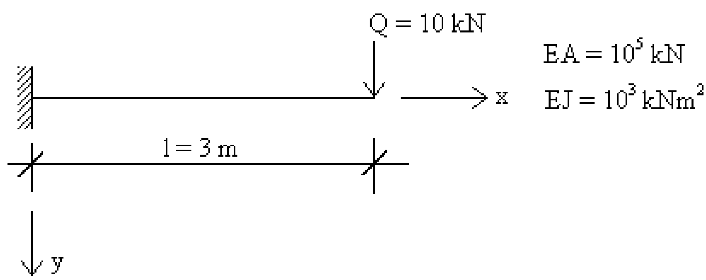


Fig. 7. Force applied at the cantilever's endpoint

Solution:

The function $v(t)$ can be calculated at a chosen time interval Δt.

Let Δt=1 day and tk= 2 days.

Let reading the result be done through a period of 20 days (e.g. 0-20 day), this arrangement is shown in equations (25):

$$
\begin{array}{cccccccc}
 & t=0 & t=1 & t=2 & & & & t=20 \\
v_t = & [\,0 & 0,3934 & 0,23865 & . & . & . & 0,00003]
\end{array}
\tag{25}
$$

Then v_t is multiplied by the flexibility matrix **F**, this will give the answer series of the unit loads $\mathbf{A}_t$. The vector series of the real cause which results from applying the constant loads up to the fourteenth day. The application of loads on the structure is ceased after this day:

$$q_t = \begin{array}{ccccccc} t=0 & & t=14 & t=15 & & t=20 \\ \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix} & \dots & \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & \dots & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{array} \tag{26}$$

Finally the displacements can be calculated by the help of *Eq.* (24). *Fig.* 8 shows that the displacements take maximum value on the fourteenth day, and then after this day they are going to take the value zero because the loads are ceased after this date.

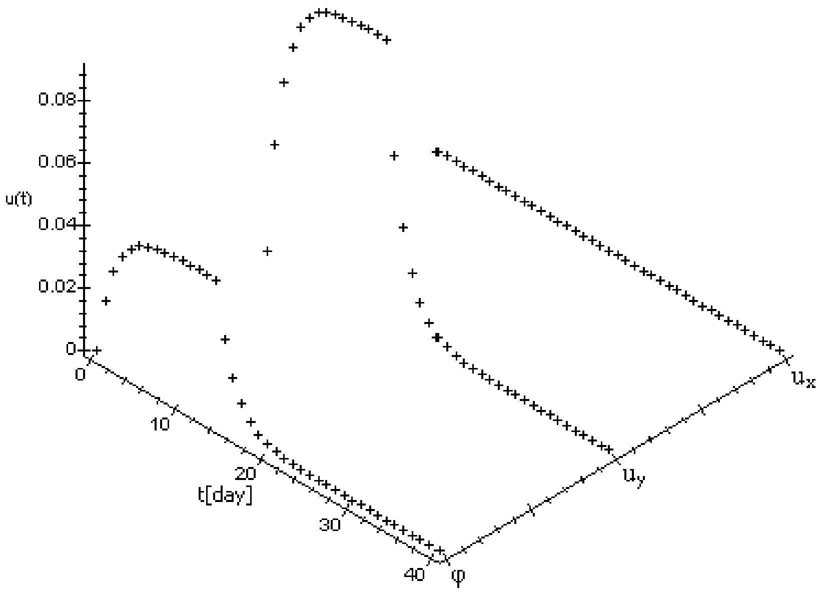


Fig. 8. Displacements taking place at the cantilever's endpoint

5. Conclusions

Computers deal with discrete situations, for this reason in the case of implementing machine-based calculations it is worthy to utilize discrete models (abstract algebra) rather than continuous analysis. Utilization of series (sequence of discrete elements) makes it possible to process the ‘one after another’ arriving measurement results, and in case of necessity to intervene in the process.

In this study, one-dimensional convolution, which is not so common in engineering practice, is traced back to the multiplication of series of characteristic function. Here, one of the characteristic polynomials was matrix-coefficient polynomial; the second was vector-coefficient polynomial. With the definition of an expanded linear space, it was possible to convert the convolution of matrix series and vector series into a multiplication of polynomials.

The application of the procedure introduced in this study was illustrated in detail on cantilever suitable for modeling creep where Kelvin-Voigt material law is valid. Other than this illustrated example, the procedure can be applied on any case of linear differential equations system with constant coefficients.

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