PERIODICA POLYTECHNICA SER. CIV. ENG. VOL. 46, NO. 2, PP. 213–219 (2002)

THE COMPARISON OF FIVE ADJUSTMENTS OF THE HUNGARIAN FIRST ORDER LEVELLING NETWORK USING GEOPOTENTIAL VALUES

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Received: Jan. 25, 2002

Abstract

Although the GPS technology is getting better, it is not precise enough to complete first order levelling. This technology is still an important issue these days. Since 1995 it is possible to compute the geopotential values of levelling points in Hungary. The geopotential numbers can be used for geoid determination and to determine any kind of metric heights. In this article some methods for adjusting levelling networks using geopotential numbers are presented and compared. The 'best' solution can be used later for any purposes.

Keywords: geopotential values, first order levelling networks, singular value decomposition

1. Introduction

The connection of the Hungarian First Order Levelling Network to the UELN (United European Levelling Network) in 1995 (ÁDÁM et al., 1999) made the determination of geopotential numbers and heights possible in a common system covering almost all Europe. This fact has a great importance, because the only theoretically correct way to determine the vertical position of a point is to tell its geopotential number. The geopotential number is not really suitable for practical purposes (disregarding some exceptions), but can be used to determine any kind of metric height (BIRÓ, 1985).

Our German colleagues sent us the geopotential numbers of 39 points, 35 of them are part of the Hungarian first order levelling network (*Fig. 1*), the others are connecting points near the border (SACHER et al., 1998; TOKOS, 1998). The potential numbers are related to the zero point of the tide gauge in Amsterdam. So it became possible to adjust the network using this data and adjust the network independently and comparing the results.

In the next part, the used adjustment methods and their results will be introduced.

2. The Used Methods

The treatment of potentials and potential differences is similar to the managing heights and height differences; therefore every computational method that is used for levelling networks can be used also in the case of adjusting geopotential numbers. The problem of singularity is common. It means that the determinant of the normal matrix of the equation system is zero. This problem can be solved in several ways. The simplest solution is to fix the value (height, geopotential) of at least one parameter. One of the solutions was made with 32 fixed parameters (the values came from the UELN 95/10 adjustment).

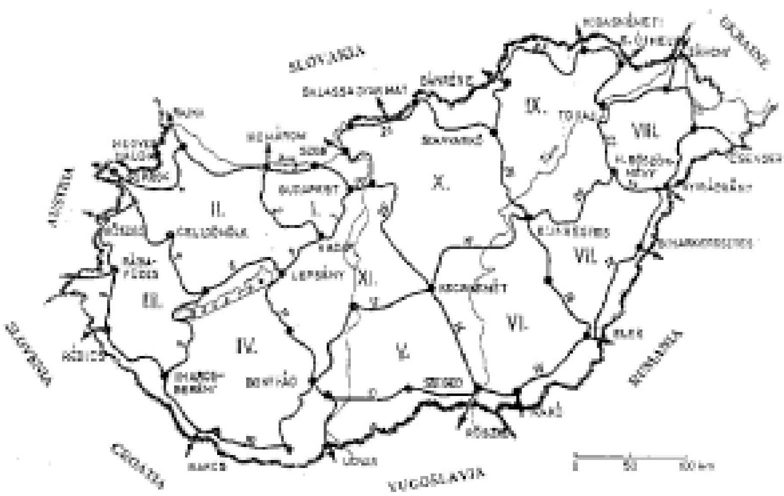

Fig. 1. The Hungarian First Order Levelling Network

Another option is to use generalized inverses (STRANG). The solution with conditions

$$v^T P v = \min \quad \text{and} \quad x^T x = \min \tag{1}$$

is given by

$$\hat{x} = (A^T P A)^+ A^T P l, \tag{2}$$

where $+$ denotes the Moore–Penrose generalized inverse.

The Moore–Penrose generalized inverse can be determined by the following equation.

$$(A^T P A)^+ = Q \Lambda^+ Q^T = \sum_{i=1}^{n} \lambda_i^+ q_i q_i^T, \tag{3}$$

where Q and Λ are from the Singular Value Decomposition (SVD) of the $A^T P A$ matrix.

Substituting *Eq.* (3) into (2) gives the formula

$$\hat{x}(\lambda) = \sum_{i=1}^{n} \lambda_i^{+} \tilde{x}_i, \tag{4}$$

where

$$\tilde{x}_i = q_i q_i^T A^T P l, \qquad i = 1, 2, \ldots, n. \tag{5}$$

Using *Eq.* (4) we can introduce other estimates of vector x:

$$\tilde{x}(\omega) = \sum_{i=1}^{n} \omega_i \tilde{x}_i. \tag{6}$$

As shown in *Eq.* (6), $\tilde{x}(\omega)$ is a linear combination, using the ω_i weights as coefficients. If we choose $\omega_i = \lambda_i^{+}$ we obtain the least squares method. But there are other options (STRANG, TOKOS, 1998).

These are:

a) CPC (Combinative Principal Components) estimation (QINGMING–XURONG, 1993; ZÁVOTI, 1999):

$$\begin{aligned} &\hat{x}(\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_k^{-1}, 1, \ldots, 1) \\ &\quad = Q * \operatorname{diag}(\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_k^{-1}, \lambda_{k+1}, \lambda_{n_0}, 1, \ldots, 1) Q^T \hat{x}(\lambda). \end{aligned} \tag{7}$$

b) SPPC (Single Parametric Principal Components) estimation (QINGMING–XURONG, 1993; ZÁVOTI, 1999):

$$\begin{aligned} &\hat{x}\left(\frac{\lambda_1 - 1 + \vartheta}{\lambda_1^2}, \frac{\lambda_2 - 1 + \vartheta}{\lambda_2^2}, \ldots, \frac{\lambda_k - 1 + \vartheta}{\lambda_k^2}, \vartheta, \ldots, \vartheta\right) = \\ &\quad = Q * \operatorname{diag}\left(\frac{\lambda_1 - 1 + \vartheta}{\lambda_1}, \frac{\lambda_k - 1 + \vartheta}{\lambda_k}, \vartheta\lambda_{k+1}, \ldots,\right. \\ &\qquad \left.\vartheta\lambda_{n_0}, 1, \ldots, 1\right) Q^T \hat{x}(\lambda). \end{aligned} \tag{8}$$

The weighting method was usual, the p_i weights came from

$$p_i = \frac{1}{t_i}, \tag{9}$$

where t_i is the length of the levelling section i (ÁDÁM et al., 1999; TOKOS, 1998).

The computations were carried out using my own programs (written in language 'C') and a well-known mathematical software named Matlab. Here is a simple example of the programming in Matlab.

```
load A.txt;
load P.txt;
```

```
load l.txt;
P=diag(P);
x=inv(A'*P*A)*A'*P*l;
```

In this case matrix ‘A’, and vector ‘l’ must be full. Here matrix ‘P’ is a vector and this program transforms it into a diagonal matrix (line no. 4). The software can handle sparse matrices too. This kind of storing can save lots of space.

3. Results of the Computations

Before presenting the results, it is useful to show the data used for the computations. The results of the **UELN 95/10** adjustment are the following: geopotential numbers of the main points of the Hungarian First Order Levelling Network and standard deviations, potential differences and their standard deviations along levelling lines, a table that shows the different point numbers. The number of the points is 35 and four points are near the border (for connecting other networks).

The given data of the **EOMA** (the Hungarian First Order Levelling Network) include the Φ and Λ co-ordinate of the points, length of the levelling sections, date of measurement, measured height differences, measured g values and the ΔK potential difference of levelling sections in GPU. The number of the points is 742, 32 of them are the main points of the network; the others are so-called KKP (crustal movement monitoring) points.

Using these data altogether 5 adjustments were made. The following part shows the results.

3.1. Using One Known Parameter

The simplest solution contains very few constraints and gives potential numbers and their standard deviation. The known point was NadapII ($K_{\mathrm{NadapII}} = 172.98629$ GPU, 1 GPU potential difference $\approx$ 1 m height difference). Because of this method, the standard deviation of the potential numbers increases with the distance from the known point. The highest value is 0.00769 GPU ($\approx$ 7.8 mm), the mean is 0.00552 GPU ($\approx$ 5.6 mm). The number of points: 741. The problem with this solution is that there is only one point that has the same geopotential number as in UELN.

3.2. Using 32 Points with Given Geopotential Numbers (from UELN 95/10 Adjustment)

This solution contains the strongest constraint, because of the 32 fixed parameters. These points are part of the UELN (naturally they didn't use all of the EOMA points, only these). The EOMA point number of the used points are: 1–2, 4–6, 8–14, 16, 20–21, 23–29, 31–37, 39–40, 5001 and no. 18032 (altogether 32). Like the previous method, also this method gives potential numbers and their standard deviations.

The greatest standard deviation is 0.00223 GPU ($\approx$ 2.3 mm), the mean is 0.00129 GPU ($\approx$ 1.3 mm). The number of points: 710. It is an important advantage that there are 32 points keeping their UELN potential values. The others fit in this 'frame'.

3.3. Using Moore – Penrose Pseudoinverse

This solution does not contain any constraints. It has two conditions: $v^T P v \rightarrow$ min and $x^T x \rightarrow$ min. This solution gives potential numbers and their standard deviations, but these potential values are not real owing to lack of known points. So we have to add a constant value to the potential numbers to shift the network. The mentioned 32 points were used to compute an average value. The greatest standard deviation is 0.00563 GPU ($\approx$ 5.7 mm), the mean is 0.00434 GPU ($\approx$ 4.4 mm). The estimation of standard deviations is much better in this case than in the previous cases. The number of points: 742.

3.4. Using CPC [4, 8], 5. SPPC [4, 8]

With these methods the potential values of altogether 742 points were determined. Unfortunately, these methods do not make possible to determine the standard deviations of potential values, only the accuracy of the method can be estimated.

4. Comparison of the Results

Special attention must be paid to the standard deviations, because sometimes their estimation is not really correct. In this case their order of magnitude is around 10^{-3} GPU (few mm). In the UELN they reach the dm level. It has several reasons, one is the extension of the network, another is the adjustment method. The standard deviations increase with the distance from Amsterdam, because the only one known point was used, Amsterdam. So the UELN standard deviations show the quality of our levelling network related to Amsterdam, not the quality of the network itself.

Fig. 2 contains the standard deviations only of the 32 used known points, because it would be hard to show all the data. Method 1 shows the greatest standard deviations, method 3 has limited values between 0.003 and 0.006 GPU. Only 32 points are represented, because using all would result a confused diagram. And as all these points are known, the thick horizontal line is evident. It is clearly visible that the standard deviations of method 3 (using Moore–Penrose pseudoinverse) are

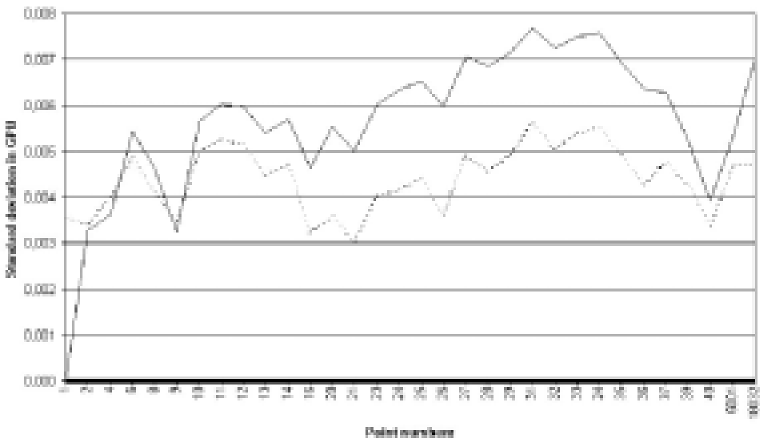

Fig. 2. A posteriori standard deviation of the potential values in the 32 used known points (method 1: continuous thin line, method 2: horizontal thick line, method 3: dotted line)

all in a narrow belt between 0.003 and 0.006 GPU. Method 1 gives bigger standard deviations (max. 0.008).

Table 1. Statistical properties of the standard deviations of sol. 1–3 in GPU (using only nonzero values!)

Solution	Minimum	Maximum	Mean	Std. dev.
1	0.00089	0.00769	0.00552	0.00129
2	0.00029	0.00223	0.00129	0.00041
3	0.00304	0.00563	0.00434	0.00064

Table 1 agrees with *Fig. 1*. The effect of solutions on standard deviations is clearly visible. The most important results of the computations are the potential values, so it is useful to investigate them too. For further use we have to choose one of the solutions, because it is highly recommended using only one dataset. The potential values of the main points of the chosen solution must agree with their UELN values. *Therefore only solution 2 can be the chosen one* (some purposes need more than one dataset, see Summary). So this is recommended for further use, and use it as a reference for the examination of the solutions. If we compare the solution using Moore–Penrose pseudoinverse to the solution computed with 32 known points we find differences in the potential values of around 5 mm (maximal).

Table 2. Differences in potential values referred to solution 2 (in GPU)

Solution	Minimum	Maximum	Mean	Std. dev.
1	−0.00697	0.00295	−0.00227	0.00208
3	−0.00416	0.00577	0.00055	0.00208
4	−0.00697	0.00296	−0.00227	0.00208
5	−0.00113	0.00065	0.00000	0.00013

This result is consistent with the investigations that can be found in (TOKOS, 1998).

The differences are not too high (around 7 mm maximum), so if we need smaller accuracy (1 cm or more), any of the solutions can be used. The fifth column is interesting, because three of four values are the same. The reason is not clear, this may be accidental. If we need higher accuracy, the chosen solution should be used.

5. Summary

At the moment these six solutions are available. For further use solution 2 can be used. The most important areas for applying the potential values include height determination, comparing height systems and geoid determination (FREISTRITZER, 1998). The first and the third problem can be solved employing the chosen solution, for the second also other data are needed. In the near future a geoid determination will be computed at the Department of Geodesy and Surveying at the TUB.

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