

# THE ANALYTIC ELEMENT METHOD FOR GROUNDWATER FLOW MODELLING

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## Abstract

The analytic element method is developed to model shallow groundwater flow. It is based on the well known potential theory, though it uses the discharge potential. Its basic feature is the description of each aquifer element influencing the flow with a harmonic function. These individual functions are superimposed to obtain an overall formulation. This presentation of the method is connected to a comparison with the well-known finite difference solution.

*Keywords:* groundwater modelling, analytic elements, discharge potential

## 1. Introduction

Groundwater is one of the most important resources of drinking water, therefore its protection is of high importance. In the last decades there are several groundwater models developed to examine the effects of intervention into the underground environment. This environment is a porous two phase medium, if the aquifer is saturated, but it may be also three phases, if the unsaturated zone of the soil is examined. The modelling of such a medium may involve flow, transport or deformation processes with a stochastic or deterministic approach.

This paper focuses on deterministic flow models of the two phase, saturated medium. The most important fields of its application are as follows:

- the protection and utilisation of underground water bases, as,
  - the operation of production wells or well fields to ensure drinking or other water,
  - the effects of their drawdown on the groundwater regime of the region, on the neighbouring aquifers, and on the water household of the unsaturated zone which is usually the root zone of the vegetation, as well,
  - the examination of bank infiltration, etc.
- the examination of the effects of several interventions in surface waters on the groundwater both on local and regional scales, as
  - the examination of scouring or deposition in the bed due to regulation, which may influence the surface water – groundwater connection;

- the operation of hydraulic engineering structures, that may influence the groundwater head in the surrounding region;
- the setting of new artificial surface waters, as canals, reservoirs, gravel pits, etc.;
- as the water is the transporting medium of any dissolved foreign material in the soil, flow models provide basic data for the transport models, as well.

The aim of this paper is to introduce a relatively new way of groundwater modelling, the *analytic elements method (AEM)*. This introduction is based on the comparison with one of the oldest and most reliable modelling approaches, *the finite differences (FDM)*. After a short introduction of the groundwater flow with all the basic assumption and a review of modelling possibilities, a more detailed description is given about AEM. Then a wide range comparison of the two methods is presented, and as a conclusion their applicability will be discussed.

## 2. Theoretical Backgrounds

### 2.1. Basic Assumptions

The movement of water in the soil is considered as gravity dominated laminar flow. The velocity conditions of such a flow may be described by the well-known *Darcy* law:

$$\bar{v} = -k \cdot \text{grad } \varphi. \quad (1)$$

This virtual velocity, which is rather a specific discharge over a unit area of full soil, may be transferred to the effective velocity with the help of the porosity. Replacing *Darcy* velocity into the continuity equation, and taking also elastic storage in soil into consideration, the basic equation of seepage may be obtained:

$$\frac{\partial}{\partial x} \left( -k_x \frac{\partial \varphi}{\partial x} \right) + \frac{\partial}{\partial y} \left( -k_y \frac{\partial \varphi}{\partial y} \right) + \frac{\partial}{\partial z} \left( -k_z \frac{\partial \varphi}{\partial z} \right) = -S_{0,\varphi} \frac{\partial \varphi}{\partial t}. \quad (2)$$

In case of homogeneous, isotropic soil without storage this turns to be the *Laplace* equation.

This description takes seepage into consideration, as a three dimensional, time dependent process. Based on this equation the piezometric head or velocity can be determined in any point of the aquifer. If the parameters of flow are constant in one direction, the flow is a two-dimensional plain flow.

On larger, regional scale such detailed approach is not always necessary. In most cases it is a good approximation to consider the aquifers and also the regional flow as horizontal, though vertical effects may also have an important role. Such vertical effects may be the infiltration from the ground surface, leakage through not fully impermeable layers, the drawdown close around wells or surface water courses, a sudden change in the aquifer parameters, etc. That is why the groundwater

flow cannot be considered as a two-dimensional plain flow. But the scale of the examinations makes it possible to apply average and integrated aquifer parameters. They are defined as an average or an integral along the vertical saturated thickness of the aquifer. Taking a general parameter  $f$  into consideration, the connection between this average and the integral quantities is as follows:

$$f_{\text{ave}} = \frac{1}{H} \int_{z_1}^{z_2} f \cdot dz. \quad (3)$$

So instead of soil parameters like the permeability ( $k$ ) the aquifer parameters, like the transmissibility ( $T$ ) may be applied, and also instead of the *Darcy* velocity ( $v$ ) the specific discharge ( $q$ ) and the average head is taken into consideration.

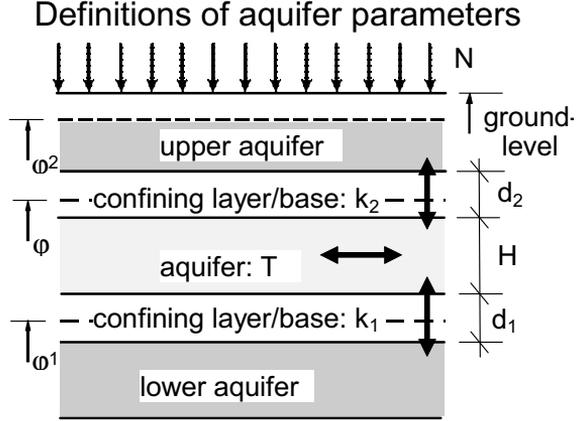
## 2.2. Aquifer Approach

The integral of Eq. (2). is to be applied for the following types of aquifers:

1. *Confined aquifers*: they are bordered by impermeable layers below (base) and above (confining layer). Across these layers there is no or negligible flow. The saturated part is the full thickness of the aquifer, and the piezometric head in it is higher than the lower surface of confining layer. If the confining layers in Fig. 1 are impermeable ( $k_1 = k_2 \cong 0$ ), then the middle and lower aquifers can be considered as confined.
2. *Unconfined (phreatic) aquifers*: they are bordered by an impermeable base below, and by the unsaturated zone of soil above. Across the base there is no flow, but there may be infiltration ( $N$ ) from the unsaturated zone. The upper boundary of the saturated part is the groundwater surface with atmospheric pressure. Though the groundwater surface is usually not horizontal, it is so slightly curved, that based on the *Dupuit–Forchheimer* assumption, the flow may be approximated as horizontal. Such an aquifer is the upper aquifer in Fig. 1, if its base is unconfined.
3. *Leaky aquifers*: they are bordered by such base and/or confining layer, that their permeability cannot be neglected any more, though it is much less, than that of the aquifer itself. These bordering layers are considered as leaky layers. If there is a further aquifer on the other side of the leaky layer, across it there is flow. This flow may be assumed as vertical. Leaky aquifers may be phreatic, if their base is leaky, or confined, if any of their bordering layers is leaky. The upper aquifer in Fig. 1. is a leaky phreatic aquifer, while the middle and lower aquifers are leaky confined ones.

## 2.3. Basic Equation of Groundwater Flow

Integrating Eq. (2) along any of the above aquifers, and also taking the vertical flow components, infiltration and leakage into consideration, the basic equation of



*Fig. 1.* Definitions of aquifer parameters

groundwater flow may be obtained:

$$\frac{\partial}{\partial x} \left( -T_x \frac{\partial \varphi}{\partial x} \right) + \frac{\partial}{\partial y} \left( -T_y \frac{\partial \varphi}{\partial y} \right) - N + \frac{\varphi - \varphi_1}{c_1} + \frac{\varphi - \varphi_2}{c_2} = -S \frac{\partial \varphi}{\partial t}. \quad (4)$$

The indexes 1 and 2 refer to the parameters of the layers below and above respectively, while the ones without such an index refer to the aquifer examined. Several of these parameters of Eq. (4). can be seen in Fig. 1.

The first two terms of the left side describe the horizontal flow in the aquifer, the third one stands for the infiltration in case of unconfined aquifer, while the fourth and fifth are the leakage through the base and confining layer, if there is any. The simultaneous consideration of all the last three terms is physically impossible, as a given layer is either confined or unconfined. The term on the right side describes storage in the layer.

All the parameters in the equation describe certain characteristics of no more the soil, but the different layers. The aquifer parameters are  $T$  and  $S$ , they are different in confined and unconfined case, while  $c$  describes the base and/or confining layer. The transmissibility  $T$  can be determined based on Eq. (3) for both aquifer types. It is as follows:

$$\text{confined: } T = k \cdot H, \quad \text{unconfined: } T = k(\varphi - Z_1). \quad (5)$$

This helps to define the specific discharge  $q$ . Based on the integral of Darcy law along the vertical, in scalar form it is as follows:

$$q_x = \int_{Z_1}^{Z_2} v_x \, dz = -T_x \frac{\partial \varphi}{\partial x}, \quad q_y = \int_{Z_1}^{Z_2} v_y \, dz = -T_y \frac{\partial \varphi}{\partial y}. \quad (6)$$

In case of confined aquifer the storativity  $S$  represents the elastic storage with rather small value, while in phreatic aquifers it comes from the saturation of the unsaturated zone, so its magnitude is like that of the porosity, though it is smaller.

The resistance of the leaky layers  $c$  is the ratio of their thickness and permeability,  $c = d/k$ . It has a dimension of time, and describes the duration needed for the water to cross the leaky layer at a unit difference of head.

The boundary conditions for *Eq. (4)* may be as follows:

- given piezometric head, e.g. at surface water courses with known water level (*Dirichlet-type*),
- given flow, e.g. in case of known extraction of a pumped well (*Neumann-type*),
- connection between head and flow, e.g. colmated river bed (mixed type).

#### 2.4. Solution Possibilities

There are several ways to develop numerical groundwater models to solve *Eq. (4)* with its boundary conditions.

In case of simple or simplified aquifers with certain boundary conditions there are analytic solutions of closed form or finite/infinite series. They provide continuous solutions for some special forms of *Eq. (4)*. Though there is a wide range of such solutions, due to the assumptions and simplifications, their use in larger scale compound problems is limited.

Another group of solutions is the numerical ones. They divide the full region into smaller subregions (elements) or apply grids of usually geometric form. They solve the general basic equation with different mathematical assumptions and tools in discrete points. The most well known ones are the finite difference, the finite elements, the boundary integral element, etc. For the later comparison an alternating direction implicit iterative finite difference model with rectangular grid will be used. The details of such a model may be found in several textbooks, e.g. BEAR (1979), BEAR – VERRUIJT (1987), KINZELBACH (1986), KINZELBACH – RAUSCH (1995), WANG – ANDERSON (1982), etc. The application of such models is wide range in solving groundwater flow problems.

Finally there are several semi-analytic models, which use on local scale the above mentioned analytic solutions to characterise one single feature of the aquifer, and for the description of the interaction between them some numerical tool is also applied. An example with nowadays growing range of application is the analytic elements method.

### 3. The Analytic Elements Method

#### 3.1. Basic Features

The analytic elements method has a fast development in the last 25 years. Its first description was introduced by O. D. L. STRACK. Since then several applications have appeared, especially in the United States, HAITJEMA (1995), STRACK (1987, 1989), STRACK et al (1987), etc.

The basis of the method is to describe each feature of the aquifer individually so, that the description satisfies a simplified form of the basic equation. These individual features influencing the flow are called the elements. They are, e.g.:

- natural or artificial surface water courses like rivers, lakes, canals, etc.
- pumped wells and well fields,
- recharge from infiltration or leakage from neighbouring aquifers,
- natural formations of the aquifer, like the changing of the permeability or aquifer thickness, etc.

The simplifications of the basic equation involve

- steady flow,
- horizontal layers with constant thickness,
- homogeneous, isotropic soil,
- no vertical recharge from infiltration or leakage.

Several of these simplifications, like constant thickness, homogeneous soil, no vertical recharge may be released by sufficient elements. Some others, like unsteady flow or anisotropy may be taken into consideration indirectly.

Beside these simplifications the so-called *discharge potential* is also introduced. In accordance with the considerations of point 2.1 it is actually the vertical integral of the well-known velocity potential. The application of the discharge potential turns the simplified Eq. (4). into the *Laplace* equation, whose solutions are the harmonic functions. The potential function and the stream function perpendicular to each other form the conjugated harmonic functions. An efficient tool to determine such functions is the conformal mapping.

Each of the above features of the aquifer can be formulated individually as a potential function. Based on the linearity of Laplace equation these individual features may be superimposed to obtain a full description of the aquifer.

#### 3.2. The Discharge Potential

The discharge potential  $\Phi$ , as an integral of the well-known velocity potential gives the  $x$  and  $y$  components of the specific discharge:

$$q_x = -\frac{\partial \Phi}{\partial x}, \quad q_y = -\frac{\partial \Phi}{\partial y}. \quad (7)$$

It may be defined as a special case of the *Girinski* potential, BUSCH – LUCKNER (1972), VARGA – CSOMA (1995):

$$\Phi_G = \int_{Z_1}^{Z_2} k(z)(\phi - z) dz. \quad (8)$$

Taking the horizontal aquifer base as the reference ( $Z_1 = 0$ ) and an average permeability along the vertical ( $k(z) \cong k_{ave}$ ), based on the assumptions of the former point, with  $Z_2 = H$  for confined and with  $Z_2 = \phi$  for unconfined aquifers, the discharge potential is as follows:

confined:

$$\Phi_c = \int_0^H k(\phi - z) dz = k\phi H - \frac{1}{2}kH^2, \quad (9)$$

unconfined:

$$\Phi_u = \int_0^\phi k(\phi - z) dz = \frac{1}{2}k\phi^2.$$

Applying either the potential for confined aquifer or the one for unconfined flow, with all the simplifications introduced earlier, Eq. (4) turns to be the same for both cases:

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0. \quad (10)$$

This is the *Laplace* equation for discharge potential, whose solutions are the harmonic functions. If there is a constant on the right-hand side e.g. for the infiltration, one obtains the *Poisson* equation. Its solution is always the sum of a harmonic function and a particular solution. That is how the infiltration can also be taken into consideration.

Confined and unconfined groundwater flow

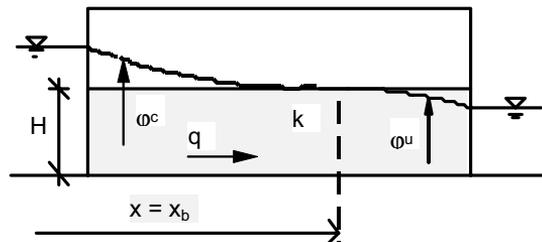


Fig. 2. Confined and unconfined groundwater flow

The application of discharge potential makes it possible to make no difference between the further description of confined and unconfined aquifers. It is rather

advantageous, especially in case if a certain aquifer may be of both types, like the one in *Fig. 2*. In such cases the location of the dividing line ( $x_b$ ) between the two parts is usually unknown. To solve this problem the potentials of *Eq. 9* are formulated so, that

- they are continuous across  $x_b$ ,
- the flow  $q$  is continuous across  $x_b$ ,
- at  $x = x_b$  the piezometric head of the confined part and the groundwater level of the unconfined part equals the thickness of the layer,  $\varphi_c = \varphi_u = H$ .

### 3.3. Some Analytical Elements

Each feature of the aquifer represents an element, and their description is an analytic function. The element with its description together is called the analytic element. In this point some analytic elements will be introduced focusing rather on their use than on the full mathematical description. Such a description can be found in several text books, e.g. BUSCH – LUCKNER (1972), HAITJEMA (1995), HÁLEK – ŠVEC (1979), KOVÁCS (1972), NÉMETH (1963), STRACK (1989), etc.

Most elements are going to be given in the form of complex potential,  $\Omega = \Phi + i\Psi$ , whose real part is the potential function  $\Phi = \Re(\Omega)$ , and the imaginary part is the stream function,  $\Psi = \Im(\Omega)$ .

*Cross flow*  $\Phi_c$  is one of the basic flow patterns. It helps to describe far field conditions.

To describe *infiltration from the ground surface* or the evaporation of phreatic groundwater,  $\Phi_e$ , the particular solution of the *Poisson*-equation as an ellipsis shaped equipotential is applied. This assumption makes possible to define not only ellipses, but circles or parallel lines as equipotentials, just the proper ratio of the two axes has to be given. The application of more such ellipses makes it possible to assume variable infiltration, and together with cross flow it is also a useful tool to describe far field conditions.

With the well known potential of the *source/sink* wells can be modelled. As an example, the complex potential of a well located at  $z_0 = x_0 + iy_0$  with the pumping rate of  $Q_w$  is as follows:

$$\Omega_w = \frac{Q_w}{2\pi} \ln(z - z_0) + C \quad (11)$$

and the potential function is:

$$\Phi_w = \Re(\Omega_w) = \frac{Q_w}{4\pi} \ln[(x - x_0)^2 + (y - y_0)^2] + C = Q_w A_w(x, y) + C, \quad (12)$$

where the term  $A_w(x, y)$  depends purely on the geometric conditions of the well.

*Line sink* can be defined as the integral of Eq. (11) along a line. If its intensity is constant along the line, it is the first order line sink, while linearly varying intensity gives the second order line sink.

First or second order means, that the complex co-ordinate  $z$  appears in the potential  $\Omega$  of the first or second order.

The potential of the first order line sink  $\Phi_d$  describes infiltration from smaller water courses with constant water level, like ditches, creeks, etc. The second order one,  $\Phi_r$  describes rivers or streams with linearly varying water levels. A string of such line sinks is a useful tool to describe longer water courses with variable water level. With well chosen members of such a string both the horizontal layout and the level of a river can be followed rather well. The form of both potentials  $\Phi_d$  and  $\Phi_r$  can be transformed to the same form, as Eq. (12), i.e. the product of an intensity and a term containing geometrical layout plus a constant.

An integration of the sink with constant intensity over an area gives the *area sink*. A simple way is to do it over a circle to get the potential for the infiltration from smaller, circle shaped ponds,  $\Phi_p$ . The same integral over a polygon requires much more complicated mathematical tools, and requires more time consuming computations. This potential  $\Phi_l$  is a useful tool to model the infiltration from larger, irregular shape of lakes, reservoirs, or wide rivers which cannot be considered as one-dimensional. Both the potentials  $\Phi_p$  and  $\Phi_l$  have two parts, one valid below the lake, where there is infiltration, and the other outside of it, where there is no infiltration. But both parts can be transformed to the same form, as Eq. (12).

If a doublet is integrated along a line perpendicular to its momentum, the potential of the *line doublet* is obtained. The stream function of the line doublet is continuous across the line, but there is a jump in the potential function. Such a jump may be observed in the discharge potential of Eq. (9) due to the sudden changing of permeability, thickness or base elevation of aquifer. This jump in the potential though causes continuous piezometric head and flow. So a string of line doublets of closed polygon is a useful tool to model local inhomogeneities of aquifers, like the one in Fig. 3.

The jump in the potential is actually the strength of the doublet, the absolute value of its momentum. If the strength varies linearly along the side of the polygon, it is a first order line doublet  $\Phi_{i1}$  and if the variation is quadratic, then the line doublet is of second order,  $\Phi_{i2}$ . The form of both potentials may also be transformed to the usual form, just in this case instead of the intensity the strength appears, multiplied with the usual term containing geometrical layout plus a constant.

Line doublets are applied to model local aquifer inhomogeneities. The first order one is suitable to describe inhomogeneities of the base elevation and aquifer thickness and also a smaller variation of the permeability. Sharper inhomogeneities of the permeability are better to describe with the more accurate second order line doublets. If the inhomogeneity of Fig. 3 appears in a cross flow from left to right, the piezometric contours of Fig. 4 may be obtained. On the figure the circular inhomogeneity is modelled by a polygon of 24 sides. Though the two sets of

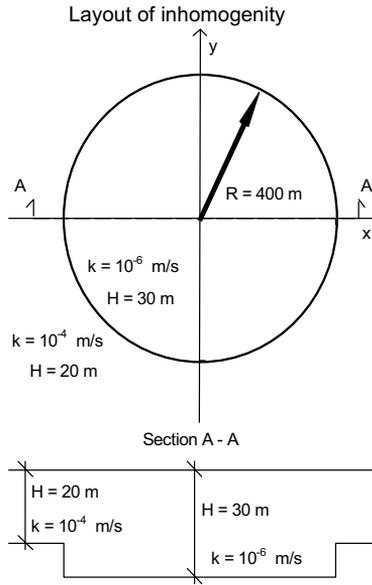


Fig. 3. Layout of inhomogeneity

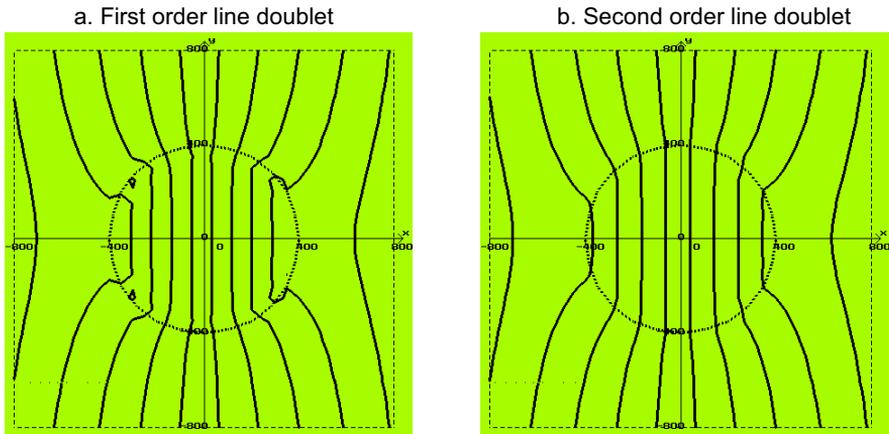


Fig. 4. a. First order line doublet, b. Second order line doublet

contours are rather similar, the left one modelled with first order line doublet shows some instability around its border. This inaccuracy vanishes on the right one due to the second order line sink. Further away from the inhomogeneity itself the two sets of contours are almost identical.

The short list introduced above shows only some examples to demonstrate the

major potential functions. It cannot be considered as a closed set, there are several other potentials derived or to be derived to solve a wide range of groundwater flow problems.

### 3.4. The Comprehensive Potential, the Boundary Conditions and the Solution

All the potentials of point 3.3 represent one particular feature of the aquifer. To obtain a full description these particular features must be superimposed. Due to the linearity of the *Laplace*-equation it means a simple sum of the individual potentials. So the comprehensive potential describing the aquifer is:

$$\Phi = \Phi_c + \sum_{ne} \Phi_e + \sum_{nw} \Phi_w + \sum_{nd} \Phi_d + \sum_{nr} \Phi_r + \sum_{np} \Phi_p + \sum_{nl} \Phi_l + \sum_{ni1} \Phi_{i1} + \sum_{ni2} \Phi_{i1} + C. \quad (13)$$

The summation in several terms stands for all the elements of the same type, while the last term  $C$  represents the sum of the constants appearing in each potential.

Some of these potentials may be determined at any point of the aquifer, as all their parameters are known, such as a well with a given pumping rate  $Q_w$ . They are the known terms of the comprehensive potential, signed with  $U$ .

Other parts of  $\Phi$  contain such parameters that are not known. There are three groups of such parameters:

- point, line and area sinks with given head, but unknown intensity, like wells with given drawdown, surface water courses with given water levels, etc.;
- inhomogeneities, whose strength is unknown, and neither the head nor a flow is given, only the jump condition of the potential;
- the constant  $C$ .

The unknown term of the comprehensive potential is signed with  $V$ . The particular potentials are defined so, that the unknown parameters of  $V$  appear always on the first order. To determine these parameters control points with boundary conditions are needed. They may be:

- a point on the well screen with given head,
- corner points of polygonal strings of line sinks with given water level,
- centroid of area sinks with given water level,
- corner points of inhomogeneities with the jump condition,
- an extra point for the constant  $C$ .

So many control points are needed as many unknown parameters in  $V$  appear. In case of a control point  $(x_c, y_c)$  with a given head  $\varphi(x_c, y_c)$  Eq. (13) turns to be in terms of  $U$  and  $V$ :

$$\Phi[\varphi(x_c, y_c)] = U(x_c, y_c) + V(x_c, y_c),$$

where

$$V(x_c, y_c) = \sum_i \Lambda_i A_i(x_c, y_c) + C. \quad (14)$$

$i$  is the number of elements with unknown parameters,  $\Lambda_i$  is the unknown strength or intensity like the unknown infiltration from a lake, and  $A_i$  is the term depending only on geometric conditions, like the one in Eq. (12). The left hand side of the potential can be determined by Eq. (9).

Eq. (14) is a linear equation with  $i + 1$  unknowns. Such an equation can be applied at each control point with given head and also similar ones may be used at inhomogeneities with the jump conditions. So finally a set of linear equations may be obtained with so many equations as many  $\Lambda_i$  unknowns. For the solution there are several methods, though only such ones may be taken into account which handle a full matrix efficiently.

After solving this set of equations the comprehensive potential may be calculated at any point of the aquifer and with the help of Eq. (9) it can be transformed to piezometric head or groundwater level. Also the stream function  $\Psi$  containing the same  $\Lambda_i$  parameters may be determined, so a full flow pattern is also provided. Based on Eq. (7) the  $q_x$  and  $q_y$  components of the specific discharge are available, as well.

### 3.5. The Model Area

Though it is not emphasised earlier, each particular potential describes some hydraulic features over an infinite plain. On the other hand, groundwater flow problems are never of infinite extent. There is always a certain area of interest where that problem has to be solved, and outer areas are out of interest. In the numerical models of point 2.4 this area of interest is bordered by given boundary conditions. These boundary conditions also involve the effects of outer areas.

In case of the AEM considering infinite plain there are no such boundary conditions, though the infinite plain should be limited without neglecting the effects coming from the outer area. Therefore the model area is always larger, than the real area of interest. It involves all those elements that may have any influence on the flow over the area of interest. The extent of the model area has to be determined by trial and error.

It seems that the identification of the outer area requires numerous extra computations. But it requires less accuracy on the local scale, only its global effects are of decisive importance. Therefore the application of well chosen cross flow and infiltration may be effective to describe far-field conditions. Also the application of simpler, lower order of elements and other simplifications is advisable, e.g.

- a far away well field may be replaced by an equivalent well of the same pumping rate,
- compound shape of lakes may be modelled by a circular one,

- first order inhomogeneity may be applied even in case of sharp changes in the aquifer parameters, etc.

to save up numerous calculations. Though these elements will not provide high accuracy locally over the outer area, they provide sufficient information over the further lying area of interest. The contours around an inhomogeneity in *Fig. 4* demonstrate this as well.

Such simplifications as introduced earlier are useful over the outer area, but they cannot be applied within the area of interest. This region needs the most effective description.

This trial and error does not require much more effort than the ordinary calibration, that each model needs. It can be made parallel with the usual calibration.

## 4. Comparative Evaluation

### 4.1. *The Way of Comparison*

The comparison of the two modelling methods, the analytic elements and the finite differences may be performed in several ways, based on several aspects. They are the following:

- theoretical considerations to compare all the hydraulic and mathematical assumptions of the two models,
- the comparison of data requirements,
- the performance of numerous, simple tests and their evaluation with both models, and the comparison of the results to each other, and to other methods, especially to analytic solutions, if there is any;
- experiences in solving fictitious and real problems using both methods.

This point aims to summarise all the experiences obtained by the ways mentioned above. Neither the numerous tests, nor the compared problems will be discussed in detail, only a short example will be given. For more detail see CSOMA (1995).

At several points the experiences to be presented are rather similar to those obtained by ZAADNOORDIJK (1990) while comparing AEM with a finite element model. He did not give a full, overall comparison, he just summarised his experiences gained during the parallel solution of a case study. The similar experiences are coming from the similarity of the applied models. The finite element and the finite difference methods are based on the same basic equation, only the applied mathematical tool to solve it is different.

#### 4.2. The Basic Equation and the Applied Assumptions

Both methods describe the same phenomenon, the shallow groundwater flow. But their different considerations and assumptions result in numerous differences. These differences are discussed in this point.

While FDM applies the general basic equation (Eq. (4)) and for the solution applies mathematical approximations, AEM applies a hydraulically simplified equation (Eq. (10)). But their solutions, the harmonic functions are mathematically accurate.

FDM defines its solutions at discrete points of a grid, while AEM provides a continuous groundwater surface. This surface is determined by several natural or artificial influences within the area examined. FDM takes these influences as boundary conditions (point 2.3) indirectly into consideration. For example a river may be a given head boundary at certain grid points. AEM on the other hand describes them directly with such a potential that is the most suitable. So the same river turns to be a string of second order line sinks. To choose the most suited potential gives a modeller a high level of freedom. And if neither seems to be suitable, even a new one may also be defined to solve a particular problem. That gives the model a high level of flexibility in the development and use. On the other hand, FDM applies the general basic equation, so it should be suitable for almost any kind of problems.

The discharge potential applied by AEM makes it possible to handle confined and unconfined aquifers the same way, only the transformation to piezometric head is different. This is a rather useful feature, especially if the extent of confined and unconfined parts of a given aquifer is not known in advance, or subject to changes in case of different situations. On the other hand, the basic equation of FDM contains such a transmissibility (see Eq. (5)) that is different in the two cases, and in case of unconfined transmissibility Eq. (4) turns to be quadratic. Of course there are several tools in FDM for the solution of this problem, but it needs a careful model development.

Due to the mathematically accurate harmonic solutions AEM is suitable to follow sharp changes of the groundwater table. As FDM considers the groundwater surface only at discrete points, this may be done with a rather dense grid, at least over the region of sharp variation.

To illustrate this feature, a simple example is given in Fig. 5. A well field of four wells with the same drawdown in a phreatic aquifer with infiltration is modelled. The model area is a circle with the diameter of 2500 m with a given head around. It was rather simple to model the full area with AEM, while with FDM due to the symmetrical location of the wells along the axes only one quarter was taken into consideration. The groundwater divides along the axes were taken as impermeable boundary. This helped to save up computational time and computer storage. Two different rectangular grids were applied,  $50 \times 50$  and  $25 \times 25$  m.

The results in the form of piezometric head along the  $x$  axis are given in Fig. 6. The symbol FDM-25 stands for the denser and FDM-50 for the sparser grid.

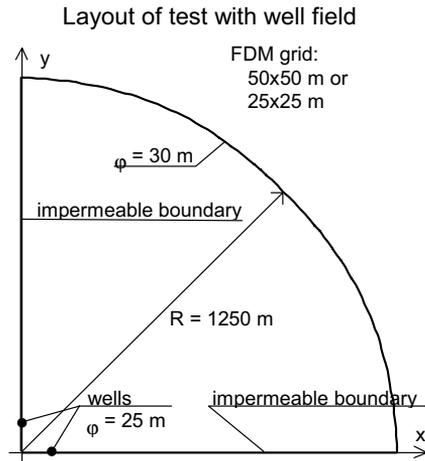


Fig. 5. Layout of test with well field

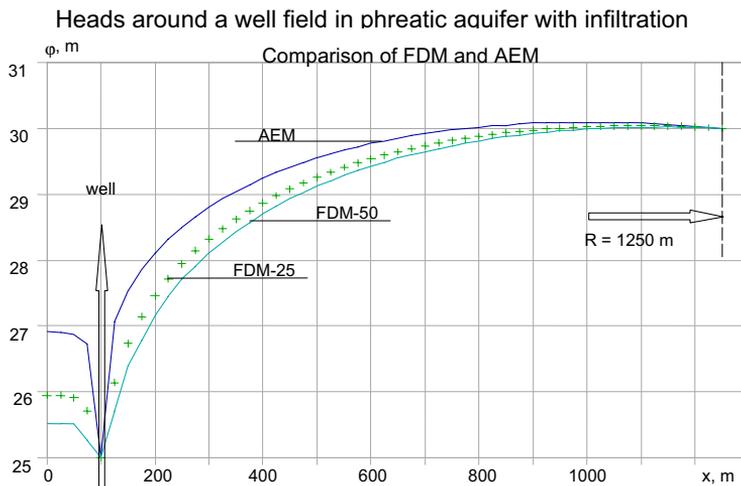


Fig. 6. Heads around a well field in phreatic aquifer with infiltration

Though the AEM model provides continuous groundwater surface, the figure was constructed with values only at the same points as FDM-25. It can be seen that the differences are reasonable near the wells, especially between them. Though FDM-25 provides the better results, in this region its grid is not dense enough.

Finally, two features are mentioned, that are rather effective in FDM. Both the spatial variation of aquifer parameters, as permeability, thickness, base elevation. etc. and the unsteady processes are rather simple to model. In AEM variable

aquifer parameters may be taken into consideration as inhomogeneities. They are suitable to describe local changes, but it is complicated to model larger scale ones, e.g. a sloped aquifer base. Unsteady processes cannot be described by the *Laplace* equation. Though there are references e.g. to include the *Theiss* solution for unsteady well, it is not yet fully developed.

The summary of all these considerations is given in *Table 1*.

*Table 1.* The comparison of model features

feature	FDM	AEM
basic equation	general	with hydraulic approximations
way of solution	with mathematical approximations	mathematically accurate
groundwater surface	given in discrete points (grid points)	continuous
the consideration of elements influencing the flow	indirect, with boundary conditions	direct
structure of model	general	flexible
confined/unconfined aquifers	different due to transmissibility	same potential
modelling sharp changes in groundwater surface	only with a rather dense grid	reliable
description of the variation of aquifer parameters	simple	complicated
description of unsteady flow	simple	not yet fully developed

#### 4.3. Data Requirements and Boundary Conditions

Data and boundary condition requirements are basic elements of the applicability of a model. These features are given in this point.

FDM requires given, fix boundaries around the area of interest. On each boundary a properly defined boundary condition (point 2.3) has to be given. In case of rivers, rock outcrops or fix groundwater divides, like in the former point, the boundaries are obvious, but there may be such situations that finding a proper boundary is almost impossible. AEM requires no fix boundaries, but a stripe, the outer area. But this outer area can be identified only with trial and error, which may sometimes be laborious.

Both of the methods apply a subdivision of the area of interest. But FDM needs a grid – usually rectangular –, which is a geometrical subdivision. In place of plain figures, AEM considers all the individual features influencing the flow as elements. Such a subdivision has a hydraulic-hydrogeologic basis. The geometrical subdivision defines grid points everywhere over the area of interest, though denser, if the hydraulic conditions require it, or sparser, if not. The hydraulic subdivision defines only those elements, which have real influence on the flow. Therefore the number of elements in this case is usually less than the number of grid points.

As FDM applies the basic equation at each grid point, all the aquifer parameters and other data have to be given at every point, or an algorithm is needed to calculate them. AEM also requires data connected to the elements and requires a general description of the aquifer. This actually means the same information in both cases, just presented in a different way.

As mentioned earlier, FDM requires well defined boundary conditions all around, while the boundary conditions of AEM are connected to the elements, as control points. This difference in boundary conditions can be considered as interpolation from outer boundaries in case of FDM, and extrapolation from inner boundaries in case of AEM.

Finally, in case of the rectangular grid a compound boundary line may be difficult to follow. Sometimes a well chosen local co-ordinate system makes the grid lines parallel with – at least some of – the boundaries, which may increase the efficiency of a model. Such a local co-ordinate system is not needed in case of AEM, though it is also not needed in case of triangular grid, e.g. of finite element method.

*Table 2* summarises the above considerations.

*Table 2.* The comparison of data requirements

feature	FDM	AEM
boundaries of the area of interest	given, fix outer boundaries	no fix outer boundaries
subdivision of the area of interest	geometrical (grid)	based on hydraulic considerations
layout of grid points or elements	everywhere over the full area	only where it is necessary
number of grid points or elements	relatively high	usually smaller
data requirements	at grid points	at elements
boundary condition requirements	at each outer boundary	at elements
local co-ordinate system	may be useful	unconcerned

#### 4.4. Program Development

Both models require computer programs. In this point some experiences of program development and application are given.

Though FDM usually requires a larger amount of data, this connected to grid points. So data processing means usually the transformation of geological and hydrogeological maps to a uniform data set. On the other hand, the data requirements of AEM are connected to the elements, and each type of elements may require a different way of data processing.

The set of linear equations in case of FDM has a sparse matrix, with non-zero elements usually around the main diagonal, while that of AEM has a full matrix. Their solution may be the same elimination, though in case of FDM relaxation may also be effective.

The full matrix of AEM seems to be unfavourable from the point of view of computational time and storage requirements. But in this case the number of unknown values is much smaller, than in case of FDM with sometimes several thousands of grid points. This reduced number of unknown values balances the disadvantages of the full matrix in such an extent, that both computational time and storage capacity demand of AEM seem to be smaller.

The presentation of the computation results is usually the same in both cases, i.e. piezometric contours, flow patterns, specific discharges, etc. To obtain them from the grid values of FDM usually interpolation is needed. As AEM determines a continuous potential and stream function, so post-processing usually requires less efforts.

In *Table 3* a summary of the above considerations is given.

*Table 3.* Program development

feature	FDM	AEM
data processing	simple, general	may be different at each element
set of linear equations	bend matrix	full matrix
computer storage capacity requirement	usually bigger	depends on the number of element, usually smaller
computation time requirement	usually bigger	depends on the number of elements, usually smaller
presentation of results	interpolation	continuous functions

## 5. Summary and Conclusions

The paper introduces the analytic elements method to model groundwater flow. After a short summary of the phenomenon concerned, the method itself is described. The most important features and the applicability of the method is presented as a comparison with the finite difference method. Based on the detailed comparison of point 4 it can be seen that both methods are rather useful tools for the groundwater modelling. Both of them have several advantages and disadvantages, and always the requirements of a given problem will determine which one to use. Though, the following recommendations may give some guidelines.

The finite difference method should be applied, if:

- the layers (aquifers, aquitards, etc.) cannot be considered or approximated as horizontal, they are sloped, or they are inhomogeneous, their permeability cannot be described or approximated with a constant value, as in this case the application of analytical elements (inhomogeneities) is complicated;
- at all the borders around the area of interest there is a well defined boundary to be applied, as in this case the laborious trial and error to identify the outer area of AEM can be avoided;

- there is no sharp variation of groundwater surface over the area of interest, which requires a locally dense grid in case of FDM;
- time dependent process is modelled, as it is not yet fully developed in case of AEM.

Analytic elements method is preferable, if:

- there is no sufficient information about the borders of the area of interest, as the method does not require given boundary conditions around the area;
- several surface water courses or other structures (e.g. wells) influence or determine the groundwater flow over the area of interest, as the description of their joint effects with the corresponding elements is simple and sufficient;
- in case of sharp variation of the groundwater level, e.g. around wells or well fields, as the method can model both their local and regional effects rather accurately;
- the process to be modelled can be considered as steady flow, as there is hardly any experience in case of unsteady groundwater flow.

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### List of Symbols

$c, s$	: resistance of leaky layer;
$d, m$	: thickness of leaky layer;
$i$	: imaginary unit, $i = \sqrt{-1}$
$k(k_x, k_y, k_z), m/s$	: coefficient of seepage (with its $x, y$ and $z$ components);
$q(q_x, q_y), m^2/s$	: specific discharge (with its $x$ and $y$ components)
$t, s$	: time;
$v(v_x, v_y, v_z), m/s$	: a <i>Darcy</i> velocity of seepage (with its $x, y$ and $z$ compo- nents);
$x, y, z, m$	: the horizontal and vertical co-ordinates;
$z = x + iy$	: complex number;
$C$	: integration or other constant;
$H, m$	: thickness of aquifer;
$N, m^3/s, m^2$	: infiltration rate;
$Q, m^3/s$	: discharge;
$S, -$	: storativity;
$S_{0\varphi}, m^{-1}$	: specific storage;
$T(T_x, T_y), m^2/s$	: transmissibility (with its $x$ and $y$ components);
$U, m^3/s$	: known term of potential;
$V, m^3/s$	: term of potential containing unknown;
$Z_1, Z_2, m$	: the lower and upper boundary level of the aquifer;
$\varphi, m$	: piezometric head or groundwater level above a certain reference level;
$\Phi, m^3/s$	: discharge potential;
$\Lambda, (m^3/s, m^2/s \text{ or } m/s)$	: unknown intensity or strength parameter potential;
$\Psi, m^3/s$	: stream function as the conjugated function of discharge potential;
$\Omega, m^3/s$	: complex potential, $\Omega = \Phi + i\Psi$ ;
$\Im(\Omega), (m^3/s)$	: imaginary part of complex potential;
$\Re(\Omega), m^3/s$	: real part of complex potential.