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APPROXIMATION OF THE BOUNDARY CONDITIONS AT THE COMPUTATION OF FLOOD WAVES

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Abstract

The computation of wave phenomena belongs to the subject of the unsteady flow problems. To the computations boundary conditions are required. The initial conditions, usually, can be very easily produced. Generally, the upstream boundary condition is given, and this presents the actual wave for the effected and calculated lower river stretch.

It is a general and difficult problem to produce the downstream boundary condition – especially for shorter river stretches – which is often substituted by the normal (steady flow) discharge curve: $Q_0 = Q_0(h)$. This approach – depending on the type of the flood wave – is very often not precise enough. The goal of this study is to justify that the downstream boundary condition, instead of the $Q_0 = Q_0(h)$ discharge curve, can be substituted by the $Q = Q(h)$ loop, which is also a rating curve, but for unsteady flow.

The computational results justified the statements of the authors. It was also numerically justified, that during a total wave flow the maximum time order of the main hydraulic parameters is: $S_{\max}$, $v_{\max}$, $h_{\max}$ (or $Z_{\max}$) and $Q_{\max}$.

Keywords: hydraulics, flood waves, unsteady flow.

1. Basic Equations

The flood wave is a gradually varying unsteady flow, which is determined by a *continuity equation* (KOZÁK 1977, ABBOTT 1979, RÁTKY 1989):

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0, \tag{1}$$

and a *dynamic equation*:

$$\frac{v^2}{C^2 R} - \left(S_0 - \frac{\partial h}{\partial x}\right) + \frac{1}{g}\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) = 0\,, \tag{2}$$

where, Q – the discharge, A – the wetted cross-sectional area, v – the mean velocity, S_0 – the bottom slope, h – the depth, R – the hydraulic radius, C – the velocity coefficient, g – the acceleration due to gravity, x – the longitudinal co-ordinate of cross-section and t – the time.

Of course, the dynamic equation should have many different forms (KOZÁK 1977, CUNGE–HOLLY–VERWEY 1980) depending on the conditions, but these conditions do not change the substance of our idea. For sake of simplicity in this case, we supposed a prismatic channel.

The general solution of the basic equations (1 and 2) are (KOZÁK 1977, RÁTKY 1989):

$$h = h(x, t) \text{ and } v = v(x, t) \text{ or } Q = Q(x, t), \tag{3}$$

which – in our present case – are not important.

2. The Approximation of the $Q(h)$ Loop

From the basic hydraulic conditions of steady flow it is well known that the discharge formula is:

$$Q_0 = AC\sqrt{RS_0} \tag{4}$$

while, *for an unsteady flow*, with a very good approximation

$$Q = AC\sqrt{RS} = AC\sqrt{R\left(S_0 + \frac{\partial h}{\partial x}\right)}, \tag{5}$$

where (for prismatic channel):

$$S = S_0 + \frac{\partial h}{\partial x} \tag{6}$$

is the *actual slope of the water surface* (approximately) and

$$\frac{\partial h}{\partial x} \tag{7}$$

is the *additional surface slope* (KOZÁK 1958), which shows the longitudinal variability of the depth. (For non-prismatic channel the $Z_0 = Z_0(x)$ and $Z = Z(x)$ surface curves have to be applied.)

From our former research work (KOZÁK 1958) it was justified, that

$$\frac{\partial h}{\partial x} = -\frac{1}{W}\frac{\partial h}{\partial t} - \frac{Q_0}{2BS_kW}\frac{\partial^2 h}{\partial x^2}; \tag{8}$$

$$S_k = \sqrt{S_0 S}; \qquad W = W_0\sqrt{\frac{S}{S_0}}; \qquad W_0 = \frac{1}{B}\frac{\mathrm{d}Q_0}{\mathrm{d}h}, \tag{9}$$

where, $\partial h/\partial t$ – the tangent of the $h = h(t)$ flood hydrograph, B – the width of the water surface, W – the wave velocity (KOZÁK 1958) and W_0 – the tangent of the normal $Q_0 = Q_0(h)$ discharge curve. From the equations of (4) and (5) the actual discharge can be derived

$$Q = Q_0\sqrt{\frac{S}{S_0}} = Q_0\sqrt{1 + \frac{1}{S_0}\frac{\partial h}{\partial x}} \tag{10}$$

from the steady flow discharge (Q_0) and the additional slope ($\partial h/\partial x$).

During the unsteady flow computational processes we have to solve the two (1 and 2) basic equations in every discretised time (Δt) cycle. As a result we will have – in every time cycle – calculated values at every model point $[n, j]$ for $[h_n^j, Q_n^j, v_n^j, \text{ etc.}]_{n=1+N}^{j=1+N}$

From these, using a *retrograde scheme*, the additional slope can be expressed by the following form:

$$\left(\frac{\partial h}{\partial x}\right) \approx \frac{h_n^j - h_{n-1}^j}{x_n - x_{n-1}}. \tag{11}$$

Substituting this value $(\partial h/\partial x)$ to the *Eq.* (10) the linear equation of unsteady flow (KOZÁK 1977, RÁTKY 1989) can be solved. After the solution we get new values for the water depth $[h]^{j+1}$. With these new depth values we can calculate a more exact value for the additional slope $(\partial h/\partial x)$. Finally for the time t^{j+1} we can get a more exactly approximated value for $[h - Q]^{j+1}$ in every cross section. This iteration can be repeated as many times, as required.

3. Example and Approximation

The following example will justify the applicability of the proposed method. The main characters and hydraulic parameters of the analysed flow are as follows. The 300 km long *channel* has a *prismatic* cross section where:

$$S_0 = 0.00005, \quad k = 40\ \mathrm{m}^{1/3}/\mathrm{s}, \quad B = 100\ \mathrm{m}, \quad h_0 = 4\ \mathrm{m}, \quad Q_0 = 270.83\ \mathrm{m}^3/\mathrm{s}.$$

For the computation of the unsteady flow, the well known implicit method was applied, using

$$\Delta x = 1000\ \mathrm{m}, \qquad \Delta t = 15\ \mathrm{min}$$

discretisation network (KOZÁK 1977).

The *upstream boundary condition* is given in *Fig. 1* in $Q = Q(t)$ form, at $x = 300$ km section.

The *downstream boundary condition* (in section $x = 0$ or $x = 10$ km) was given under the following conditions, in cases ***A***, ***B*** and ***C***:

A,	at	$x = 0$ km,	$Q_0 = Q_0(h)$	with *Eq.* (4).
B,	at	$x = 0$ km,	$Q = Q(h)$	with *Eq.* (10).
C,	at	$x = 10$ km,	$Q = Q(h)$	with *Eq.* (10).

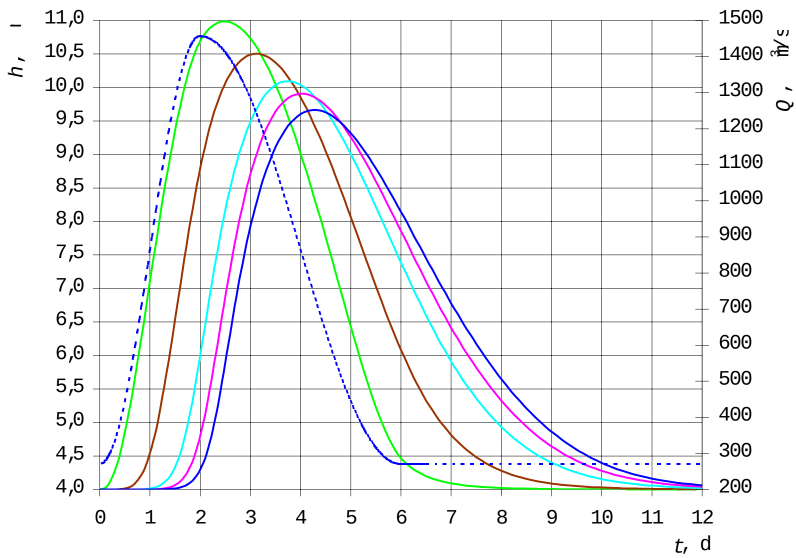

Fig. 1. The calculated h = h(t) water depths and Q = Q(t) at upper boundary section

The results of the computations are presented in *Figs.* 1 - 6. and the conclusions are the following.

Fig. 1 presents not only the upstream boundary condition – $Q = Q(t)$ with dashed line – but the calculated $h = h(t)$ water depths in sequence at $x =$ 300, 200, 100, 50 and 0 km sections (continuous lines).

Fig. 2 presents the steady and the calculated unsteady discharge curve, namely loop – $Q_0 = Q_0(h)$ and $Q = Q(h)$ – at $x = 0$ km (downstream boundary condition). *Fig.* 3 presents the calculated $h = h(t)$ hydrograph also in the lower cross section ($x = 0$ km). The calculated curves show *significant difference between the two methods* if

- the lower boundary is $Q_0 = Q_0(h)$ or
- the lower boundary is $Q = Q(h)$ (i.e., it is a loop curve).

The difference between the two curves is more than 50 cm! The reason of the lower water level is the greater unsteady discharge ($Q > Q_0$), which causes significant suction effect for the upstream channel stretch.

Fig. 4 presents the same $Q - h$ relations, but in $x = 10$ km section. The continuous line $\langle 3 \rangle$ presents the unsteady (Q loop) and the dashed line $\langle 2 \rangle$ the steady (Q_0) downstream boundary condition ($x = 0$ km). $Q_0 = Q_0(h)$ was supposed for the downstream boundary and the loop number $\langle 2 \rangle$ was calculated. The difference between curves $\langle 1 \rangle$ and $\langle 2 \rangle$ is quite considerable. *Fig.* 4 also presents the permanent $Q - h$ curve $\langle 1 \rangle$ in the $x = 10$ km section, with dotted line.

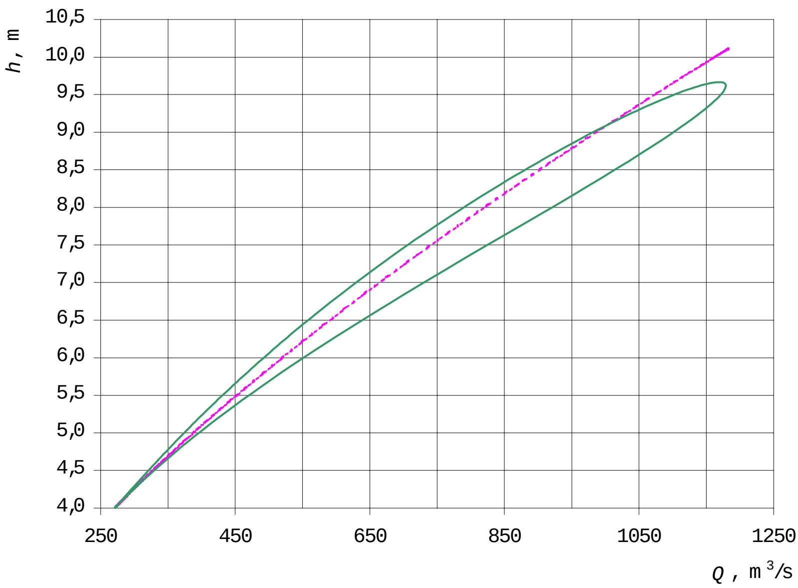

Fig. 2. The steady rating curve and unsteady loop curve at the x = 0 km section

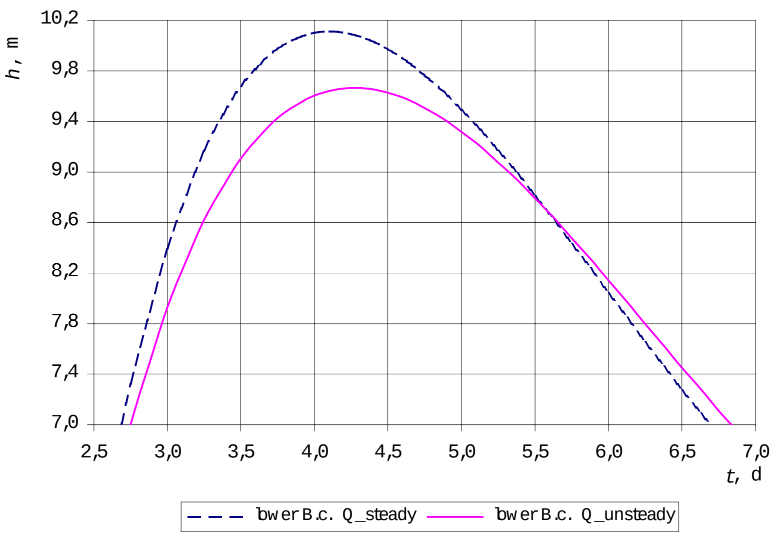

Fig. 3. The $h = h(t)$ hydrographs at the $x = 0$ km lower boundary section

Fig. 5 presents the calculated $h = h(t)$ hydrograph also in section $x = 10$ km, under the former (*Fig. 4*) conditions. The difference between the depth values (Δh)

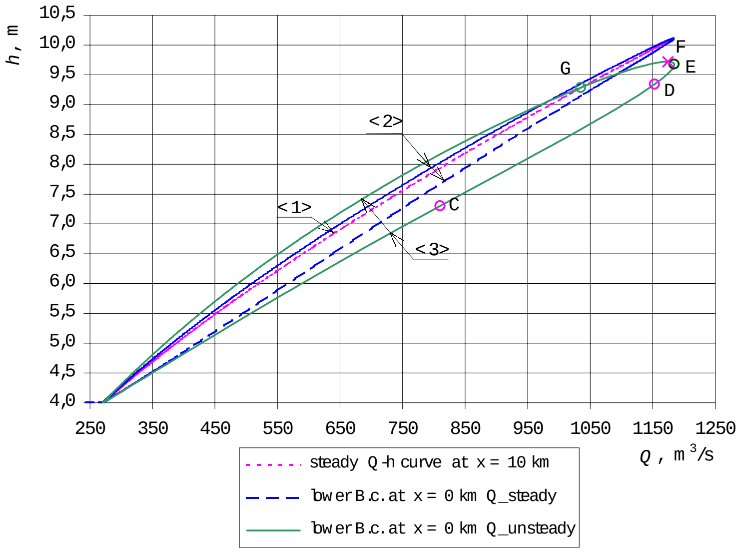

Fig. 4.

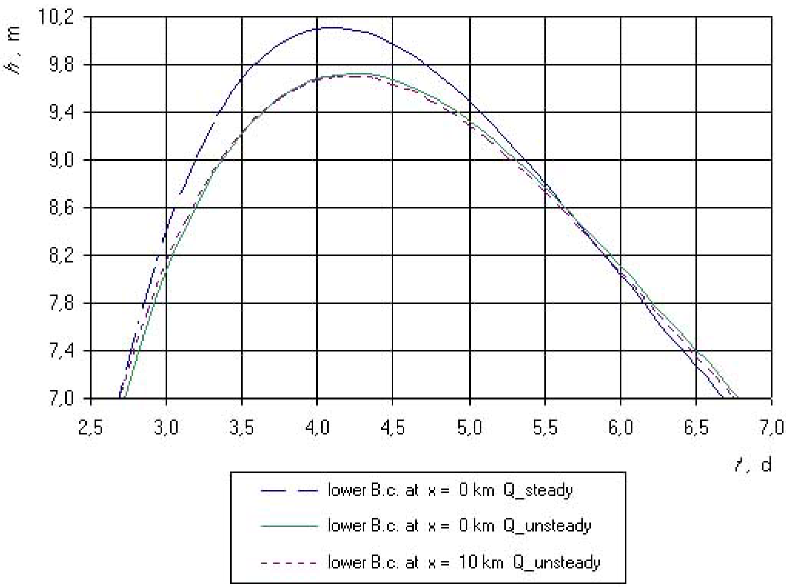

Fig. 5. The h = h(t) hydrographs at the x = 10 km lower boundary section

are still significant, but they decrease upwards. The careful analysis of the same $(h - t)$ curves in sections $x = 50$, 100 and 150 km proves that the *difference* (Δh)

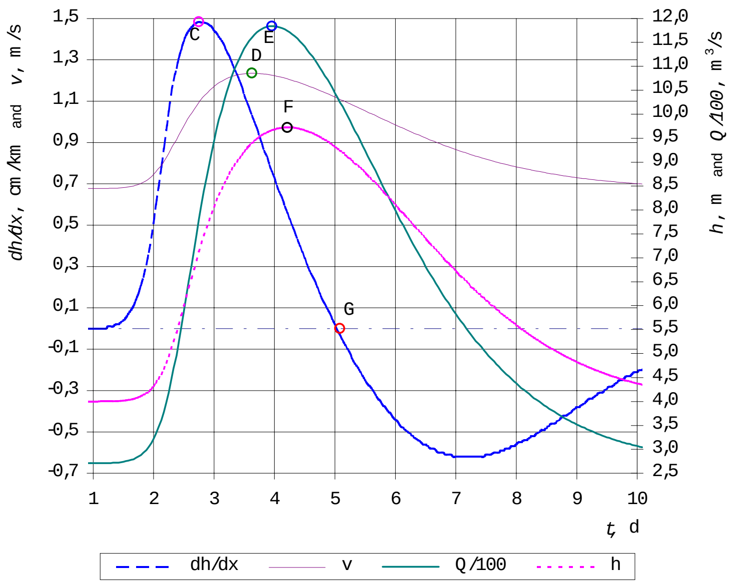

Fig. 6.

gradually decrease upwards from the downstream boundary. This means that the difference between the two (steady or loop downstream boundaries) methods has the maximum values at $x = 0$ km and it is decreasing upwards.

From hydraulic point of view these mean the following: the different lower boundaries are able to influence only the shorter neighbouring stretches of the flow.

The calculated $h = h(t)$, $S = S(t)$, $v = v(t)$ and $Q = Q(t)$ curves in *Fig.* 6 numerically prove the hydraulic features of the unsteady phenomenon during a flood wave according to many authors' (KOZÁK 1958, VÁGÁS 1984, SZIGYÁRTÓ 1985) opinion.

First the slope (S, or $\partial h/\partial x$), then the velocity (v) and the discharge (Q) and finally the depth (h) reach their maxim values ($S_{\max}$, $v_{\max}$, $Q_{\max}$, $h_{\max}$: C, D, E and F points), which is corresponding to the theory (KOZÁK 1958).

The special $F - G$ stretch of the loop in *Fig.* 4 can be analysed in the following way by the application of *Eq.* (8). At the point $F\,\partial h/\partial t = 0$ therefore the value of the additional slope ($\partial h/\partial x$) is just the function of the curvature of the wave surface expressed by $\partial^2 h/\partial x^2$ (here $h = h_{\max}$). At the point G of *Fig.* 6 $\partial h/\partial x = 0$ (and $Q = Q_0$) therefore from *Eq.* (8):

$$\frac{\partial h}{ht} = -\frac{Q_0}{2BS_k}\frac{\partial^2 h}{\partial x62}. \qquad (12)$$

4. Conclusion

The study presented an approximate method to calculate the loop (discharge curve for unsteady flow). The calculations *proved* that in case of unsteady flow the *downstream boundary conditions* – instead of the steady discharge curve – can be substituted *by a loop* and this *gives much better results*. The difference (Δh) between the steady (Q_0) and unsteady (Q) downstream boundary conditions attain their maximum in the neighbouring ($\Delta x = 0 - 40$ km) area of the downstream boundary, but these differences (Δh) gradually decrease as the function of the distance. The calculation also *justified* that during a flood wave the *logical sequence* of the main hydraulic parameters is: $S_{\max}$, $v_{\max}$, $Q_{\max}$, $h_{\max}$ (*Fig.* 6).

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