PERIODICA POLYTECHNICA SER. CIV. ENG. VOL. 44, NO. 1, PP. 3–12 (2000)

DETERMINATION OF THE COEFFICIENTS OF THE GREEN-TYPE MATERIAL MODEL BY NON-LINEAR PROGRAMMING

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Received: December 20, 1998

Abstract

Material planning is an important and developing part of applied mechanics. The problem is usually solved by FEM, which leads to highly non-linear equations causing several numerical problems. This paper presents a method for the determination of the coefficients of a Green-type material model using mathematical programming based on the theorem of virtual force. The material model is included in the compatibility equations. The material parameters appear as unknowns in the non-linear objective function, while the conditions remain linear. The uniqueness and stability of the material model (e.g. Drucker postulates) are assured by inequality conditions. The boundary conditions are considered. Sample problems were run by a program for non-linear mathematical programming problems written in Fortran 77.

Keywords: mathematical programming, material design, non-linear material models, Green model.

1. Introduction

Material design is now an important and developing part of applied mechanics. There are several ways to achieve our design goals, but the problem is usually solved with FEM. The greatest disadvantages of this method are that they lead to highly non-linear equations causing several numerical problems [7]. On the other hand, the problem can be formulated using the concept of virtual energy. This formulation leads to a mathematical programming problem [6].

The state of equilibrium of a structure is usually determined by solving the system of the equilibrium and compatibility equations. Another possibility is to use the extremum principles of virtual energy with the strain energy and complementary strain energy function [9].

At first we assume that the principle of small displacement is valid and the constitutive law is linear. The primal problem expresses the virtual force theorem by the help of a quadratic programming problem [2], where the equality conditions are the equilibrium equations concerning the structure and the objective function is

the complementary strain energy. In this problem, we are looking for the minimum of the objective function.

Formulating the dual problem, one can find that it contains the compatibility equations as equality condition with the strain energy function as objective function. Again, the problem is to find the minimum of the objective function. The complementary problem gives the usual formulation.

This formulation allows wide opportunities in

- delimiting stresses and displacements,
- satisfying the boundary conditions,
- defining yield domain,
- choosing either elastic or elasto-plastic constitutive laws.

They can be built into the mathematical programming problem as equality or inequality conditions, iff the conditions are consistent.

The main advantage of the mathematical programming formulation can be seen if we introduce different material parameters (e.g. elastic modulus, cross-section parameters, etc.) as unknowns. Using the primal formulation, the objective function becomes of higher order even in the case of linear material law, but the equality conditions remain linear.

In the rest of the paper this formulation is shown, and applied to a simple, statically indeterminate beam structure with a Green-type hyperelastic material model [1]

$$\varepsilon_{ij} = \frac{\partial \Omega}{\partial \sigma_{ij}}.$$

The integration of the stresses (σ) concerning the volume was calculated by the Gauss–Legendre scheme, and thus the coefficients can be represented by the diagonal matrix $\langle \rho \rangle$. These coefficients are independent of the stresses and therefore the order of the derivation and the numerical integration can be changed.

2. The Mathematical Programming Problem

The elastic state of a structure can be determined through the equilibrium and compatibility equations in the following form [5,8]:

$$\mathbf{T}^* \cdot \mathbf{B}^* \cdot \langle \rho \rangle \cdot \sigma + \mathbf{P} = 0, \qquad \text{(equilibrium equations)}$$

$$\mathbf{B} \cdot \mathbf{T} \cdot \mathbf{u} + \langle \rho \rangle \cdot \nabla \Omega(\sigma) = 0, \qquad \text{(compatibility equations)}$$

$$\mathbf{P}\Big|_{S_P} = 0, \qquad \text{(stress boundary conditions)}$$

$$\mathbf{u}\Big|_{S_u} = 0. \qquad \text{(displacement bound. conditions)}$$

Here σ is the stress vector, **P** is the vector of external forces, **u** is the displacement vector, **B** is the geometric matrix, ε is the strain vector, **T** is the transformation matrix between the local and global coordinate systems. The unknowns are vectors **u** and σ.

The structure of this system of equations can be visualized as in *Fig. 1*. Matrix **F** is hyperdiagonal, and if the constitutive law is linear, the elements are constants, otherwise they are functions.

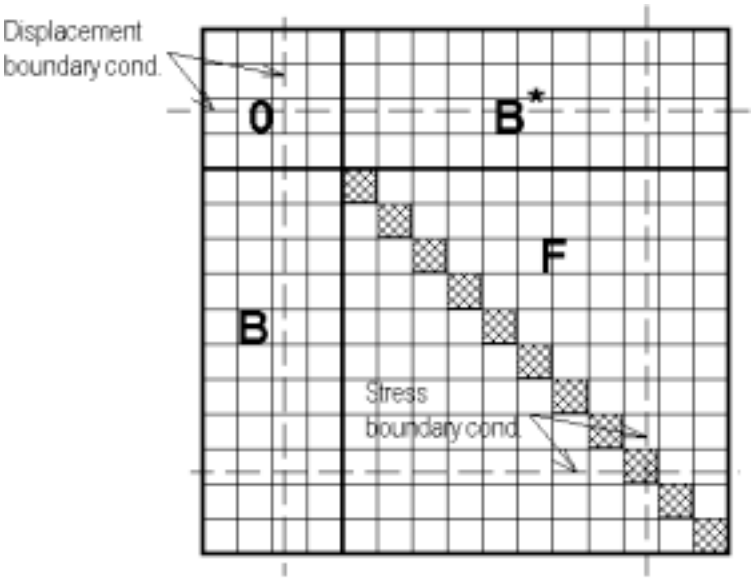


Fig. 1. The structure of the equilibrium and compatibility equations. The boundary conditions can be considered by clearing the appropriate rows and columns.

The static and kinematic boundary conditions can be considered at the adequate equations. The simplest method in the case of prescribed zero displacement or zero stress is to clear the appropriate row and column as shown in *Fig. 1*. This is impossible unless the constitutive law is linear and thus matrix **F** can be explicitly stated. In this case, the appropriate stress in the material law must be substituted with zero, but in the geometric matrix **B** the appropriate row can be deleted. Otherwise, the boundary conditions can be included as additional equality constraints.

Based on the principle of virtual force theorem, it is possible to give a pair of mathematical programming problems, either of them defines the equilibrium state of a structure, and which are in primal–dual relationship. The primal problem expresses the minimum of virtual complementary energy

$$\min\left\{\int_V \Omega(\sigma)\,\mathrm{d}V\right\}$$

$$\mathbf{T}^* \cdot \mathbf{B}^* \cdot \langle\rho\rangle \cdot \sigma + \mathbf{P} = 0, \qquad \text{(equilibrium equations)}$$

$$\mathbf{P}\Big|_{S_P} = 0, \qquad \text{(stress boundary conditions)}$$

$$\mathbf{u}\Big|_{S_u} = 0. \qquad \text{(displacement bound. conditions)}$$

The Wolf-dual of this problem is:

$$\min\left\{\int_V W(\varepsilon)\,\mathrm{d}V - \mathbf{P}^*\mathbf{u}\right\}$$

$$\mathbf{B} \cdot \mathbf{T} \cdot \mathbf{u} + \langle\rho\rangle \cdot \nabla\Omega(\sigma) = 0, \qquad \text{(compatibility equations)}$$

$$\mathbf{P}\Big|_{S_P} = 0, \qquad \text{(stress boundary conditions)}$$

$$\mathbf{u}\Big|_{S_u} = 0. \qquad \text{(displacement bound. conditions)}$$

which expresses the minimum of the virtual energy. The complementary problem, which includes the equality conditions of both the primal and dual formulation, recalls the usual system of equilibrium and compatibility equations.

The theorem of virtual forces and virtual displacements gives a necessary condition for the existence of the optimum for the primal and dual problem, respectively. On the other hand, the convexity of the feasible domain defines the sufficient condition. Moreover, if the energy function is convex in the domain of material stability, the optimum is global optimum and it is the same for the primal and dual problem.

3. The Material Model

The most simple and convenient material model is based on Hook's law, and expresses a linear function between strains and stresses:

$$\varepsilon = \nabla\Omega(\sigma) = \mathbf{F} \cdot \sigma.$$

In this case, the equality constraints are linear either in the primal or in the dual problem, and the objective function is quadratic.

$$\min\left\{\sum_{\text{elements}}\frac{1}{2}\sigma^*\cdot\mathbf{F}\cdot\sigma\right\} \qquad \text{(primal problem)}$$
$$\mathbf{T}^*\cdot\mathbf{B}^*\cdot\langle\rho\rangle\cdot\sigma+\mathbf{P}=0, \qquad \text{(equilibrium equations)}$$
$$\min\left\{\sum_{\text{elements}}\frac{1}{2}\sigma^*\cdot\mathbf{F}\cdot\sigma-\mathbf{P}^*\cdot\mathbf{u}\right\} \qquad \text{(dual problem)}$$
$$\mathbf{B}\cdot\mathbf{T}\cdot\mathbf{u}+\langle\rho\rangle\cdot\nabla\Omega(\sigma)=0, \qquad \text{(compatibility equations)}$$

The boundary conditions for both problems are:

$$\mathbf{P}\Big|_{S_P}=0, \qquad \text{(stress boundary conditions)}$$
$$\mathbf{u}\Big|_{S_u}=0. \qquad \text{(displacement bound. conditions)}$$

In the linear case, the stability of the material is automatically satisfied. From the computational point of view, one of the most simple non-linear material behaviors is a Green-type hyperelastic model. In a three-parameter quadratic model, the relation between stresses and strains can be stated as

$$\varepsilon_{ij}=\nabla\Omega(\sigma_{ij})=\Phi_1\delta_{ij}+\Phi_2\sigma_{ij}+\Phi_3\sigma_{ik}\sigma_{jk},$$

where Φ_1, Φ_2 and Φ_3 are material parameters. The complementary virtual energy is:

$$\Omega(\sigma)=\int_0^{\sigma}\nabla\Omega(\sigma_{ij})\,\mathrm{d}\sigma_{ij}\,.$$

The integral was calculated numerically with a three-point Gauss–Legendre scheme:

$$\Omega(\sigma)=\frac{\sigma}{2}\sum_{i=1}^{3}c_i\nabla\Omega\left[\frac{\sigma}{2}(\xi_i+1)\right].$$

Here, c_i are the Gauss weights at the integration points ξ_i, and σ is the unknown upper limit of the integration.

As it can be seen in *Fig. 2*, where $\Phi_1=0$ and $\Phi_3=$ constant, the effect of normal stress is influenced by the value of its coefficient at a given value of the energy function very strongly.

If the constitutive law is non-linear, the stability of the material is not automatically satisfied. The stability can be ensured by the Drucker postulate (*Fig. 3*) [3], which states that the work of any external effect rate ($\dot{T}_i$, $\dot{F}_i$) on the displacement rate ($\dot{u}_i$) is positive,

$$\int_A\dot{T}_i\dot{u}_i\,\mathrm{d}A+\int_V\dot{F}_i\dot{u}_i\,\mathrm{d}V>0.$$

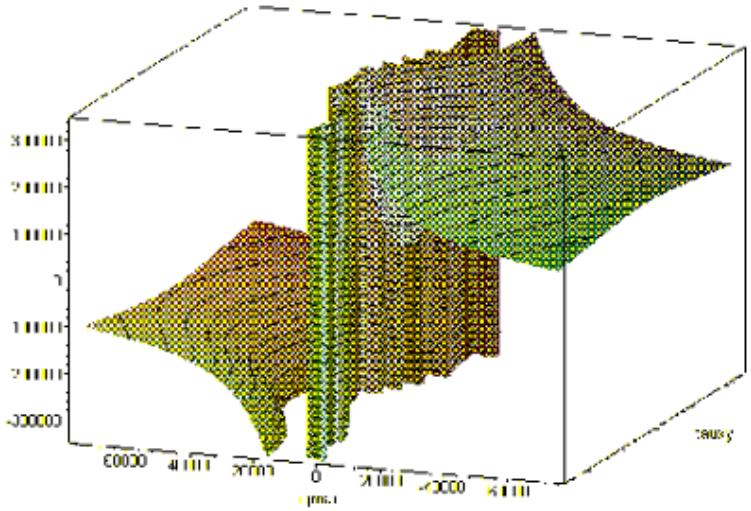


Fig. 2. The linear coefficient (Φ_2) as function of stresses at constant energy level

Using the principle of virtual work, one can rewrite this in the following form:

$$\dot{\sigma}_{ij}\dot{\varepsilon}_{ij} > 0.$$

The rate of stresses and strains can be understood as the change of the effect in each iteration step (r) of the non-linear programming solver and therefore

$$\Delta\sigma_{ij} = \sigma_{ij}^{r+1} - \sigma_{ij}^{r}.$$

Using the definition of the Green model one can write

$$\mathbf{H}'_{ijkl}\,\Delta\sigma_{ij}\,\Delta\sigma_{kl} > 0,$$

where $\mathbf{H}'_{ijkl}$ is the Hessian of the complementary energy function. This non-linear inequality condition ensures the stability and convexity of the material, however, it is stricter than it would follow form the laws of thermodynamics.

4. A Simple Beam Structure

The above described method was applied at the calculation of the a very simple, 1D, statically indeterminate structure shown in *Fig. 4*. The structure consists of two elements, each with three internal Gauss–Legendre integration points for the integration of the energy function. Each node has three degrees of freedom (ε_x, ε_y, γ_{xy})

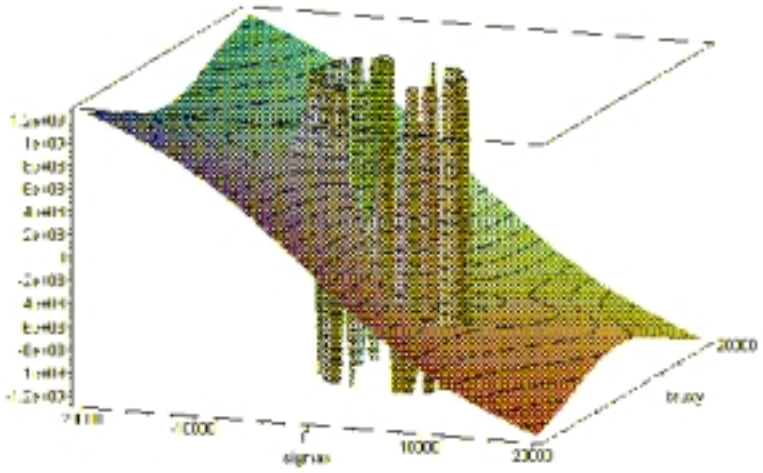


Fig. 3. The inner part of the tube is determined by the Drucker postulate.

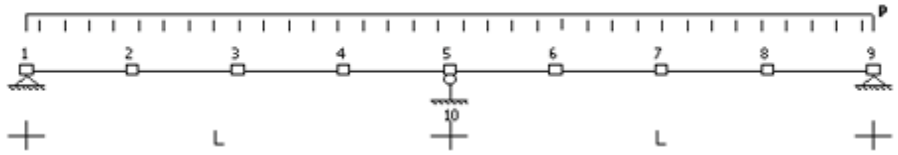
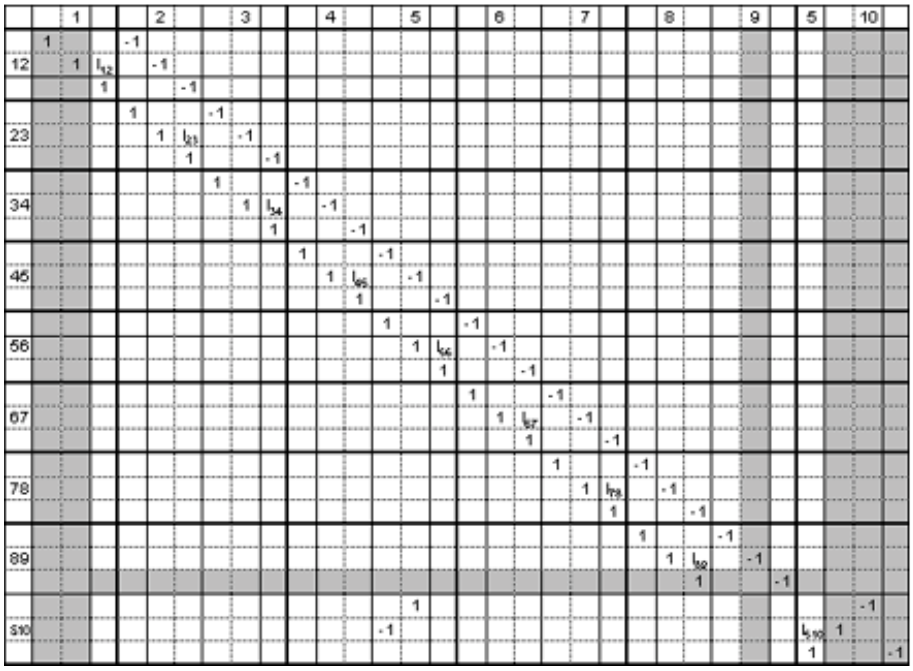


Fig. 4. The beam structure

and the stress state of each node is described by the three-element vector

$$\sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}.$$

The modified geometric matrix ($\tilde{\mathbf{B}}$), concerning the boundary conditions (shaded rows and columns) in the global coordinate system is as follows:



The vector of external forces (**P**) can be formed by concentrating the distributed load to the nodes.

The material law described in chapter 3 is now simply:

$$\begin{aligned} \varepsilon_x &= \Phi_1 + \Phi_2\sigma_x + \Phi_3(\sigma_x^2 + \tau_{xy}^2), \\ \gamma_{xy} &= \Phi_2\tau_{xy} + \Phi_3\sigma_x\tau_{xy}. \end{aligned}$$

The complementary virtual energy can be written as the function of the stresses:

$$\Omega(\sigma,\tau) = \int_0^{\sigma}\int_0^{\tau} \varepsilon_x \,\mathrm{d}\tau_{xy}\,\mathrm{d}\sigma_x + \int_0^{\sigma}\int_0^{\tau} \gamma_{xy}\,\mathrm{d}\tau_{xy}\,\mathrm{d}\sigma_x,$$

where the upper limits of the integrals are unknowns. Using three-point Gauss–Legendre integration scheme:

$$\begin{aligned} \Omega(\sigma,\tau) &= \frac{\sigma}{2}\frac{\tau}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} c_i c_j \varepsilon_x\left(\frac{\sigma}{2}(\xi_i+1), \frac{\tau}{2}(\xi_i+1)\right) + \\ &\quad + \frac{\sigma}{2}\frac{\tau}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} c_i c_j \gamma_{xy}\left(\frac{\sigma}{2}(\xi_i+1), \frac{\tau}{2}(\xi_i+1)\right). \end{aligned}$$

The total complementary strain energy of the structure can be obtained by the volume integration along the two elements:

$$\Omega_t = \sum_{n=1}^{2} \frac{l_n}{2} \sum_{i=1}^{3} c_i \Omega(\sigma_i, \tau_i).$$

Again, a three-point Gauss–Legendre integration scheme was used for the volume integration. σ_i and τ_i denote here the stress state in the i-th Gauss point.

The non-linear mathematical programming problem now can be stated as

$$\begin{aligned} &\min\{\Omega_t\}, && \text{(objective function)} \\ &\tilde{\mathbf{B}}^* \cdot \langle\rho\rangle\sigma + \mathbf{P} = 0, && \text{(equality conditions)} \\ &\Delta\sigma^* \cdot \mathbf{H}' \cdot \Delta\sigma > 0. && \text{(inequality conditions)} \end{aligned}$$

The unknowns are the stress vectors in the Gauss–Legendre integration points and the three material parameters. The structure has one degree of statical indetermination and therefore, eliminating the equality conditions, one variable remains free. Altogether, there are 4 independent variables, and two inequality constraints for each integration point.

The problem was run on a non-linear programming solver [4] written in Fortran 77 in a Silicon Graphics workstation environment. Since there was not any defined yield function, the Drucker postulate was the active criterion at the stop of the iteration.

The example data were the following:

Distributed load	$p = 5$ kN/m
Span	$L = 2$ m
Cross-sectional area	$F = 125$ cm^2
Inertia / cross-section extremum distance	$K = 12500$ cm^3
The solutions were	$\Phi_2 = 5$, $\Phi_3 = 0.01$
Moment at the middle support:	-8.98 kNm
Energy of the structure:	88460 kNm

The efficient (convex) domain of Φ_2 and Φ_3 is

$$-2 < \Phi_2 < 8.8, \qquad 3.34 \cdot 10^{-3} < \Phi_3 < 9.9.$$

The numerical computation has shown clearly that this formulation gives the desired results and thus it can be the basis of future extensions.

5. Conclusion

A mathematical programming formulation has been shown for the calculation of statically indeterminate structures in the framework of virtual force and virtual displacement theorems. A Green-type hyperelastic material model has been used with three unknown parameters to be optimized. This simple model has shown the applicability of the mathematical programming formulation.

This may serve as a basis to a further extension which includes the use of more sophisticated material models, the yield condition and plasticity features, and the consideration of time dependency.

Acknowledgement

This research was supported by the Hungarian Scientific Research Foundation (OTKA) under project number T029638.

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