PERIODICA POLYTECHNICA SER. CIV. ENG. VOL. 44, NO. 1, PP. 13–32 (2000)

UPPER BOUND OF DENSITY FOR PACKING OF EQUAL CIRCLES IN SPECIAL DOMAINS IN THE PLANE

Zsolt GÁSPÁR and Tibor TARNAI

Department of Structural Mechanics
Technical University of Budapest
H–1521 Budapest, Műegyetem rkp.3., Hungary

Received: April 13, 1999

Abstract

In the paper we will give heuristic upper bounds for the density of packings of non-overlapping equal circles in a square, an equilateral triangle, and a circle. The area of interstices at the boundary of these domains is calculated with greater precision than by other authors, so the obtained upper bounds are sharper than those known before. Because the function int (x) appears in the relationships, the upper bounds are not monotonous functions of the circle number. Not only the formulae of upper bounds of the maximum packing density are given, but their numerical values are listed up to 30 circles.

Keywords: circle packing, packing density, upper bound of density.

1. Introduction

A well-known problem of discrete geometry is: To determine the largest diameter d_n of n equal circles which can be packed in a given convex domain Q in the plane without overlapping; or what is the same, to determine the greatest possible minimum distance d_n between n points which can be distributed in the inner parallel domain of Q at a distance $d_n/2$. (Parallel domain of Q at a distance ρ is a domain bounded by the envelope of circles of radius ρ with centres lying on the boundary of Q [3].)

Another form of this problem is where the convex domain Q and the diameter of the circles are given and the maximum number n of the circles is sought that can be packed in Q without overlapping. Let $A(Q)$ and $P(Q)$ denote the area and the perimeter of Q. FEJES TÓTH [4] has shown that if in a convex domain Q there are packed $n \geq 2$ unit circles, then

$$n\sqrt{12} < A(Q). \qquad (1)$$

GROEMER [8] has sharpened Fejes Tóth's inequality (1) for the number n of the unit circles packed in a convex domain Q, and has proved that

$$n\sqrt{12} \leq A(Q) - \frac{2-\sqrt{3}}{2}P(Q) + \sqrt{12} - \pi(\sqrt{3}-1). \qquad (2)$$

MOLNÁR [17] has also proved this estimate, and showed the cases where equality holds.

On the basis of the results of OLER [19] this upper bound can be sharpened further for polygonal domains. Let Π be a closed convex region bounded by a Jordan polygon, E a finite point set, n the number of points in E such that the vertices of Π belong to E, the set E is contained in the closed region Π, and the distance between any two points in E is not less than 2. Then, due to OLER [19], the following holds:

$$n\sqrt{12} \le A(\Pi) + \frac{\sqrt{3}}{2}P(\Pi) + \sqrt{12}. \tag{3}$$

Let Π_1 be the outer parallel domain of Π at distance 1. Since Π is convex, the area and the perimeter of Π_1 obviously is

$$A(\Pi_1) = A(\Pi) + P(\Pi) + \pi,$$

$$P(\Pi_1) = P(\Pi) + 2\pi.$$

Introducing these expressions into (3) we obtain the same inequality as (2), the only difference is that Π_1 takes place in it, instead of Q.

Density of packing of n equal circles of diameter d_n in a convex domain Q, denoted by D_n, is defined as the ratio of the total area of the circles to the area of the domain: $D_n = nd_n^2\pi/[4A(Q)]$. Exact values of the maximum packing density in special in-plane domains (square, equilateral triangle, circle), where the circle arrangements are proven to be optimal, are known only for few values of n. An up-to-date list of the known exact solutions has been presented by FEJES TÓTH [2]. For other values of n only lower bounds and upper bounds on the maximum density can be given. Lower bounds can most appropriately be given by explicit packing constructions. The results including also packing constructions for special domains are surveyed by CROFT et al. [1] and by MOSER and PACH [18]. To our knowledge the latest results for circle packings in a square have been presented by PEIKERT [20], MELISSEN [14], MANARAS et al. [11], HUJTER [9] and TARNAI and GÁSPÁR [23]; in an equilateral triangle by MELISSEN [12], [13], MELISSEN and SCHUUR [16], GRAHAM and LUBACHEVSKY [6]; in a circle by MELISSEN [15] and GRAHAM et al. [7]. Upper bounds can be derived from the inequalities (1), (2), (3). Let D denote the maximum density of packing of n equal circles in a convex domain. If $n \ge 2$, then Fejes Tóth's inequality (1) results in the following estimate:

$$D < \frac{\pi}{\sqrt{12}}. \tag{4}$$

By (2) and (3), sharper estimates can be obtained for a square, an equilateral triangle and a circle. However, the difference between the known lower and upper bounds is relatively large for different values of n. We want to reduce these differences. The inequality (2) is valid for arbitrary convex domains, and in (3) it is supposed that circles are at all vertices of a polygon, so there is hope to improve the upper bounds using the particular characteristics of the special domains and the occurring interstices along the boundary. However, this has not been realised since the 1960s.

The aim of this paper is to present a heuristic reasoning for further sharpening Groemer's inequality (2) using the special properties of a square, an equilateral triangle and a circle. Thus, sharper upper bounds on the maximum packing density in the case of a square, an equilateral triangle and a circle domain will be given, and not only the formulae but tables containing actual numerical values will be presented.

2. Upper Bounds on Packing Density in a Square

Let D be the maximum density of packing of n equal circles in a square. In this case Groemer's formula (2) results in the following upper estimate of the maximum packing density better than (4):

$$D \leq \frac{n\pi}{\left[2-\sqrt{3}+\sqrt{7-\pi+\sqrt{3}\,(2n-6+\pi)}\right]^2}. \tag{5}$$

Groemer's inequality (5), however, can be sharpened even further due to the special properties of a square.

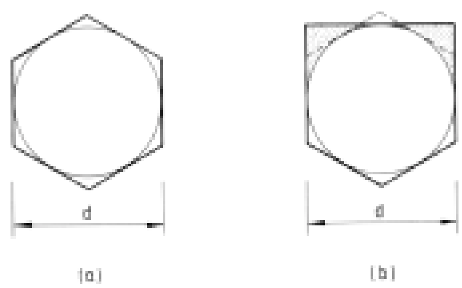

Fig. 1. The Dirichlet cell of a circle (a) in the densest packing in the plane, (b) at the boundary of the square

Consider equal circles of diameter d packed in a square of side length $1+d$. In this square the *Dirichlet cell* is defined as a domain which consists of all points of the square which are nearer to the centre of a particular circle of the packing than to any other centre. In the densest packing of equal circles in the plane, the circles form a hexagonal arrangement in which the Dirichlet cells are regular hexagons of side length $d/\sqrt{3}$ (*Fig. 1a*). The area $(\sqrt{3}/2)d^2$ of such a hexagon can be considered as *space claim* of a circle. If circles touching a side of the square are in a close arrangement, then the Dirichlet cell of such a circle is a pentagon (*Fig. 1b*) whose area is larger than the space claim of the circle by $\frac{d^2}{4}(2-\sqrt{3})$, and this difference appears as the area of an extra interstice corresponding in fact to a semicircle lying in the domain of width $d/2$ around the unit square. The largest density occurs, if as many circles as possible are touching the boundary of the square, that is, the centres of circles are situated along the sides of the unit square with separation d between them, and each vertex of the unit square is the centre of a circle (*Fig. 2*). Along a side of the unit square we cannot put more than $\mathrm{int}(1/d)$ semicircles, and at a vertex we can put additionally at most a quarter of a circle (a half of a semicircle). [Here the symbol $\mathrm{int}\,(x)$ denotes the integer part of the real number x.] If we add the areas of the extra interstices determined above along the boundary of the square, we obtain a lower bound on the real extra interstice area. This bound is even smaller if along the fractional distance $1-\mathrm{int}(1/d)d$, the average extra interstice area is taken into account instead of the actual one. Therefore, a lower bound A_e of the area of the sum of extra interstices is

$$A_e = \left(d + \frac{d^2}{2}\right)\left(2-\sqrt{3}\right).$$

Let A_c be the space claim of n circles:

$$A_c = n\frac{\sqrt{3}}{2}d^2.$$

$A_c + A_e$ cannot be greater than the area of the square of side length $1+d$. Thus, we have the inequality

$$n\frac{\sqrt{3}}{2}d^2 + \left(d + \frac{d^2}{2}\right)\left(2-\sqrt{3}\right) \le (1+d)^2\,.$$

From here d can be expressed, and as $D = nd^2\pi/[4(1+d)^2]$, we obtain an upper bound on the density D:

$$D \le \frac{n\pi}{\left[2-\sqrt{3}+\sqrt{3+2\sqrt{3}(n-1)}\right]^2} \tag{6}$$

which is exact for $n=1$, and which is better than (5) for every n. Interestingly, Oler's inequality (3) results exactly in the same upper estimate (6) of the maximum packing density in a square for $n \ge 4$.

Let us make the upper bound of packing density (6) sharper by calculating the area of the extra gaps along the boundary exactly. Doing so we have to consider also two additional dense arrangements of circles. Therefore, we have three different possibilities for dense packing of the circles along the boundary.

(a) The above-mentioned arrangement, that is, a circle is packed at each vertex of the square, the other circles are closely packed along the sides and a

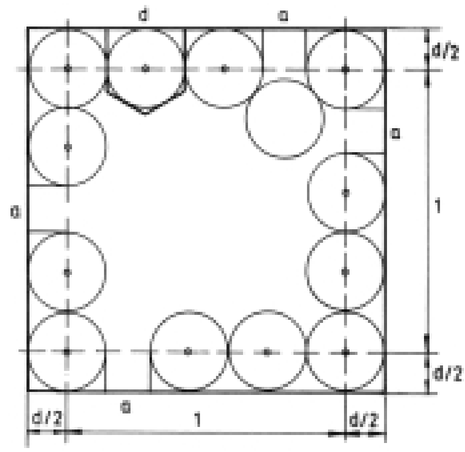

Fig. 2. Arrangement of circles along a side of the square. A gap is at each of the sides

gap of width $a = 1 - \operatorname{int}\left(\frac{1}{d}\right) d$ *appears at each side.* Consider the arrowhead-like domain composed of the rectangle $ABCD$ and the triangle EFG (*Fig. 3*). Let its area be denoted by A_a. Then

$$A_a = a\frac{d}{2} + \frac{a+d}{2}\sqrt{d^2 - \left(\frac{a+d}{2}\right)^2}.$$

The area of the extra gap A_g is obtained if the space claim of a semicircle (the area of the half of a hexagon) is subtracted from the area of the arrowhead-like domain: $A_g = A_a - \frac{\sqrt{3}}{4}d^2$, that is, as a function of a

$$A_g(a) = \frac{ad}{2} + \frac{a+d}{2}\sqrt{d^2 - \left(\frac{a+d}{2}\right)^2} - \frac{\sqrt{3}}{4}d^2. \tag{7}$$

Since on each side of the square there is a gap, the area inequality is obtained for n equal circles as

$$n\frac{\sqrt{3}}{2}d^2 + \left(\operatorname{int}\left(\frac{1}{d}\right) + \frac{1}{2}\right)d^2\left(2 - \sqrt{3}\right) + 4A_g(a) \le (1+d)^2, \; n \ge 5. \tag{8}$$

(b) The gaps appear at two opposite vertices and two adjacent sides such that the gap arrangement is (can be) symmetrical with respect to a diagonal of the square. Let us introduce the circle numbers n_1, n_2 along the sides and distances a_1, a_2, c in *Fig. 4*:

$$n_1 = \operatorname{int}\left(\frac{1}{d}\right),$$

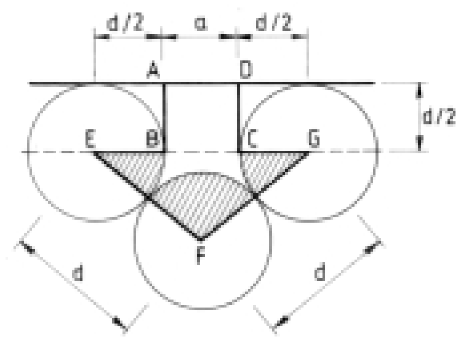

Fig. 3. The gap at the side of the square

$$a_1 = 1 - n_1 \frac{d}{2},$$

$$c = \sqrt{d^2 - a_1^2} - \frac{d}{2},$$

$$n_2 = \mathrm{int}\left(\frac{1 + \frac{d}{2} - c}{d}\right),$$

$$a_2 = 1 + \frac{d}{2} - c - n_2 d.$$

The extra interstice area corresponding to a semicircle at the boundary, as previously obtained, is

$$A_s = \frac{d^2}{4}\left(2 - \sqrt{3}\right).$$

Thus, the area of the extra interstices corresponding to the shaded area in *Fig. 4* is $(n_1 + n_2 + \frac{1}{2})A_s$. The area of the polygon $ABCDEFG$ in *Fig. 5*, at the vertex of the square, is $\frac{d}{2}(a_1 + c) + \frac{a_1}{2}\left(c + \frac{d}{2}\right)$. Subtracting the space claim of a quarter of a circle from that area we obtain the extra gap area A_v at a vertex of the square:

$$A_v(a_1, c) = \frac{d}{2}(a_1 + c) + \frac{a_1}{2}\left(c + \frac{d}{2}\right) - \frac{\sqrt{3}}{8}d^2. \tag{9}$$

The extra gap area corresponding to the gap of width a_2, obtained from the arrowhead-like domain by taking $a = a_2$ in (7), is $A_g(a_2)$. The sum of the space claim of n equal circles and the extra gap area cannot be larger than the area of the square of side $1 + d$:

$$n\frac{\sqrt{3}}{2}d^2 + 2\left[\left(n_1 + n_2 + \frac{1}{2}\right)A_s + A_v(a_1, c) + A_g(a_2)\right] \le (1 + d)^2, \; n \ge 2. \tag{10}$$

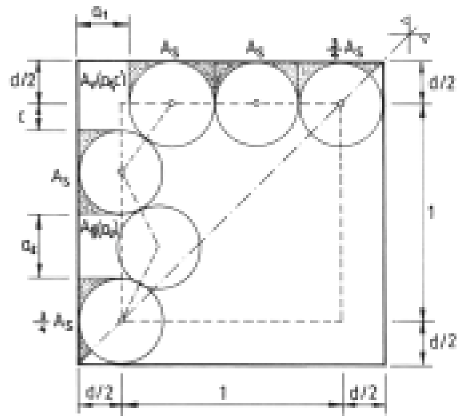

Fig. 4. Arrangement of circles along the sides of the square. Gaps are at two opposite vertices and at two adjacent sides

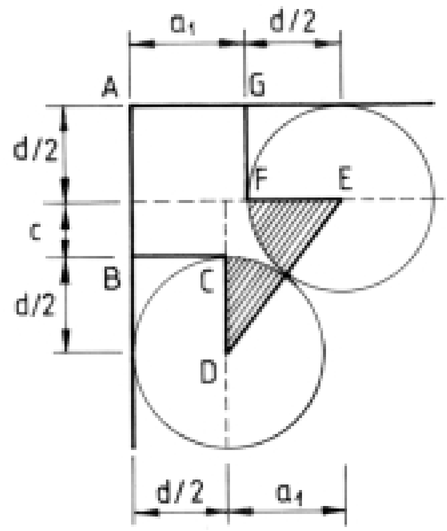

Fig. 5. The gap at a vertex of the square

(c) There are gaps at three vertices and at one of the sides of the square (Fig. 6). Here n_1, a_1, c, n_2, a_2, A_s, A_v (a_1, c) denote the same quantities as in Subsection (b). Let us introduce the circle number n_3 and distances c_2, a_3 in *Fig. 6*:

$$c_2 = \sqrt{d^2 - a_2^2} - \frac{d}{2},$$

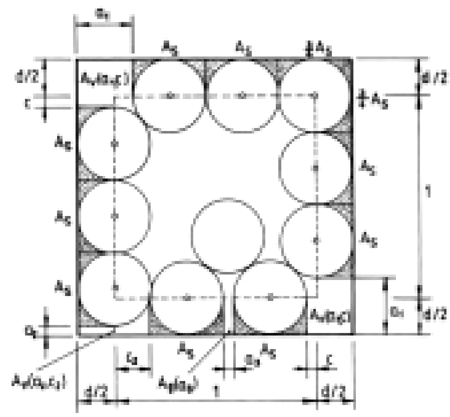

Fig. 6. Arrangement of circles along the sides of the square. Gaps are at three vertices and at a side of the square

$$n_3 = \operatorname{int}\left(\frac{1-c-c_2}{d}\right),$$
$$a_3 = 1 - c - c_2 - n_3 d.$$

Consider the area of the extra gap at a side obtained by substituting a_3 for a in (7), and that at a vertex obtained by substituting a_2 for a_1 and c_2 for c in (9): $A_g\,(a_3)$ and $A_v\,(a_2, c_2)$. The places of these extra gaps are indicated in *Fig. 6*. Then the area inequality takes the form

$$n\frac{\sqrt{3}}{2}d^2 + \left(2n_1 + n_2 + n_3 + \frac{3}{2}\right)A_s + 2A_v\,(a_1, c)$$
$$+ A_g\,(a_3) + A_v\,(a_2, c_2) \le (1+d)^2\,,\ n \ge 3. \tag{11}$$

For a given value of n, from (8), (10), (11) we can determine numerically an upper bound d_a, d_b, d_c, respectively, on the diameter of the circles, whose maximum d_m is an upper bound on the maximum diameter d:

$$d_m = \max\,(d_a, d_b, d_c) \ge d,$$

and we obtain an upper bound of the maximum density D:

$$D \le \frac{n d_m^2 \pi}{4\,(1+d_m)^2} \tag{12}$$

which is better than (6). (The upper bound (12) is always less than the upper bound (6) as if circles can be packed along a side without gaps, then the centres of the circles lie on the side of the unit square, but if the other circles are packed in a regular triangular lattice packing, then circle centres can never lie on the opposite side of the unit square, because the ratio of the side to the altitude of a regular triangle is an irrational number.) This upper bound is exact for $n = 2$ (obtained from (10)), for $n = 3$ (obtained from (11)), and for $n = 5$ (obtained from (8)).

It remains to be shown that, if the gap on a side is not concentrated at one point on the side of the square but it is divided into two parts at two different points of the side, the upper bound of density cannot decrease, that is, for the function $A_g\,(a)$ the inequality

$$A_g\,(a_k + a_l) \le A_g\,(a_k) + A_g\,(a_l)$$

is valid if $a_k + a_l \le d$. For this it is enough to show that $A_g\,(a) \ge 0$ and the second derivative of $A_g(a)$ is not positive in the interval $0 \le a \le d$. Positivity of $A_g(a)$ is obvious. The second derivative of $A_g\,(a)$ is

$$\frac{d^2 A_g}{da^2}$$

$$= -\frac{3}{4}\frac{a+d}{2}\left[d^2 - \left(\frac{a+d}{2}\right)^2\right]^{-\frac{1}{2}} - \frac{1}{4}\left(\frac{a+d}{2}\right)^3\left[d^2 - \left(\frac{a+d}{2}\right)^2\right]^{-\frac{3}{2}} \le 0$$

since both terms in it are ≤ 0. Therefore, the function $A_g\,(a)$ is concave from below in the interval $0 \le a \le d$, that is, we obtain greater density if the gaps at a side are united into a single gap.

In the cases (b) and (c), there are gaps at vertices of the square, and also at some side there is a gap between circles touching the side. If at such a side, the circles are arranged so that the gap here joins the gap at the vertex, then the area of the extra gap calculated from the united gaps is larger than the sum of the areas of the two separate extra gaps. This is so because in the new position, the circle at the apex of the ‘arrowhead’ corresponding to the gap at the side is at a larger distance from the side in question, since the other circle forming a part of the boundary of the gap at the vertex does not touch the side in question. Therefore, we obtain greater density if the gaps at the side and at the vertex are separated.

The upper bounds on the maximum packing density calculated by the formulae (5) and (6) as well as by (12), based on the maxima of the results obtained from (8), (10), (11), are given for up to $n = 30$ in *Table 1.* Unlike the upper bound given by (6) the upper bound given by (12) is not monotonous with n. As to (12), for most values of n, from the three inequalities (8), (10), (11), inequality (8) provides an upper bound on the maximum packing density.

3. Upper Bounds on Packing Density in an Equilateral Triangle

Let D be the maximum density of packing of n equal circles in an equilateral triangle. In this case Groemer’s formula (2) results in the following upper estimate

Table 1. Upper bounds on the maximum density D of packing of n equal circles in a square

n	With Groemer's formula (5)	With average extra interstice area, formula (6)	With exact extra gap area, formulae (8), (10), (11)
2	0.8724125	0.7955012	0.5390121*
3	0.8563422	0.8063246	0.6096448¶
4	0.8519408	0.8146624	0.7854052*
5	0.8509826	0.8211847	0.6737651
6	0.8513033	0.8264434	0.6701081
7	0.8521455	0.8307966	0.7123433*
8	0.8531989	0.8344782	0.7375840
9	0.8543237	0.8376460	0.7930670
10	0.8554535	0.8404109	0.8131862
11	0.8565563	0.8428525	0.7767688
12	0.8576172	0.8450303	0.7641682
13	0.8586298	0.8469892	0.7639387
14	0.8595926	0.8487642	0.7903124*
15	0.8605062	0.8503828	0.7858901
16	0.8613728	0.8518670	0.8066345
17	0.8621948	0.8532347	0.8369299
18	0.8629750	0.8545007	0.8393169
19	0.8637163	0.8556772	0.8229551
20	0.8644212	0.8567744	0.8132350
21	0.8650924	0.8578009	0.8083586
22	0.8657323	0.8587642	0.8072574
23	0.8663430	0.8596706	0.8172430*
24	0.8669266	0.8605255	0.8185761*
25	0.8674850	0.8613338	0.8221864
26	0.8680198	0.8620995	0.8336828
27	0.8685326	0.8628264	0.8508564
28	0.8690249	0.8635176	0.8571610
29	0.8694979	0.8641761	0.8479190
30	0.8699529	0.8648043	0.8411183

*Density is due to (10).
¶Density is due to (11).
All other densities not marked in the last column are due to (8).

of the maximum packing density better than (4):

$$D \leq \frac{n\pi}{\sqrt{3}\left[\sqrt{3} - \frac{3}{2} + \sqrt{\frac{13}{4} - 3\sqrt{3} + \left(1 - \frac{1}{\sqrt{3}}\right)\pi + 2n}\right]^2}. \tag{13}$$

Groemer's inequality (13), however, can be sharpened even further due to the special properties of an equilateral triangle.

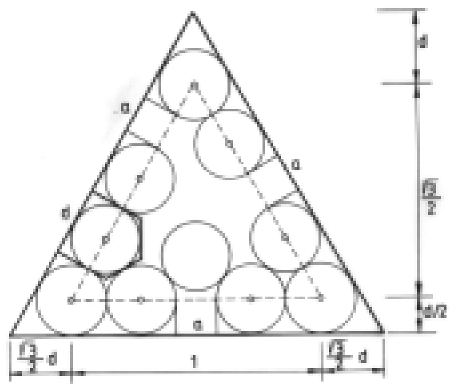

Fig. 7. Arrangement of circles along the sides of the equilateral triangle

Consider equal circles of diameter d packed in an equilateral triangle of side length $1 + \sqrt{3}d$. In this case the centres of the circles are in a closed unilateral triangle (*Fig. 7*). We can repeat the argument applied for packing in a square. The extra interstice area for a circle touching the side of the triangle is $\frac{d^2}{4}(2 - \sqrt{3})$, and that for a circle at the vertex is $d^2/(4\sqrt{3})$. The largest density occurs, if as many circles as possible are touching the boundary of the triangle, that is, the centres of circles are situated along the sides of the unilateral triangle with separation d between them, and each vertex of the unilateral triangle is the centre of a circle. If at each side of the unilateral triangle, along the segments of length $1 - \mathrm{int}\left(\frac{1}{d}\right) d$ the average interstice area is taken into account instead of the actual one, a lower bound A_e of the area of the sum of extra interstices takes place:

$$A_e = \frac{3}{4}\left(2 - \sqrt{3}\right) d + \frac{\sqrt{3}}{4} d^2.$$

The space claim A_c of n circles here also is

$$A_c = n\frac{\sqrt{3}}{2} d^2.$$

$A_c + A_e$ cannot be greater than the area of the equilateral triangle of side length $1+\sqrt{3}d$. Thus, we have the inequality

$$n\frac{\sqrt{3}}{2}d^2 + \frac{3}{4}\left(2-\sqrt{3}\right)d + \frac{\sqrt{3}}{4}d^2 \le \frac{\sqrt{3}}{4}\left(1+\sqrt{3}d\right)^2.$$

From here d can be expressed, and as $D = nd^2\pi/[\sqrt{3}(1+\sqrt{3}d)^2]$, we obtain an upper bound on the density D:

$$D \le \frac{n\pi}{\sqrt{3}\left[\sqrt{3}-\frac{3}{2}+\sqrt{\frac{1}{4}+2n}\right]^2} \tag{14}$$

which is exact for $n = k(k+1)/2$, k is positive integer, and which is better than (13) for every value of n. Interestingly, Oler's inequality (3) results exactly in the same upper estimate (14) of the maximum packing density in an equilateral triangle for $n \ge 3$.

Let us make the upper bound of packing density (14) sharper by calculating the area of the extra gaps along the boundary exactly. Unlike the case of the square, here in fact there is only one possibility for dense packing of the circles along the boundary: *a circle is packed at each vertex of the triangle, the other circles are closely packed along the sides and a gap of width* $a = 1-\mathrm{int}\left(\frac{1}{d}\right)d$ *appears at each side.* The exact area of the extra gap A_g at each side of the triangle is given by (7). So, the area inequality is obtained for n equal circles in the form

$$n\frac{\sqrt{3}}{2}d^2 + \mathrm{int}\left(\frac{1}{d}\right)\cdot\frac{3}{4}\left(2-\sqrt{3}\right)d^2 + \frac{\sqrt{3}}{4}d^2 + 3A_g(a) \le \frac{\sqrt{3}}{4}\left(1+\sqrt{3}d\right)^2. \tag{15}$$

For a given value of n, from (15) we can determine numerically an upper bound d_m on the maximum diameter d, and we obtain an upper bound of the maximum density D:

$$D \le \frac{nd_m^2\pi}{\sqrt{3}\left(1+\sqrt{3}d_m\right)^2} \tag{16}$$

which is exact for $n = k(k+1)/2$, k positive integer.

The upper bounds on the maximum packing density calculated by the formulae (13) and (14) as well as by (15) and (16) are given for up to $n = 30$ in *Table 2*. Unlike (14) the upper bound given by (16) is not monotonous with n.

4. Upper Bounds on Packing Density in a Circle

Let D be the maximum density of packing of n equal circles in a circle. In this case Groemer's formula (2) results in the following upper estimate of the maximum

Table 2. Upper bounds on the maximum density D of packing of n equal circles in an equilateral triangle

n	With Groemer's formula (13)	With average extra interstice area, formula (14)	With exact extra gap area, formulae (15), (16)
2	0.8458039	0.6895766	0.5152682
3	0.8355791	0.7290091	0.7290091
4	0.8342746	0.7528577	0.6045998
5	0.8353128	0.7692306	0.6755717
6	0.8370600	0.7813496	0.7813496
7	0.8389886	0.7907797	0.6982717
8	0.8409060	0.7983842	0.7101487
9	0.8427397	0.8046830	0.7480250
10	0.8444651	0.8100101	0.8100101
11	0.8460779	0.8145913	0.7618431
12	0.8475820	0.8185852	0.7541445
13	0.8489845	0.8221069	0.7644854
14	0.8502940	0.8252425	0.7878198
15	0.8515185	0.8280574	0.8280574
16	0.8526660	0.8306026	0.7998737
17	0.8537436	0.8329186	0.7890594
18	0.8547576	0.8350375	0.7889131
19	0.8557138	0.8369859	0.7967800
20	0.8566173	0.8387852	0.8124710
21	0.8574728	0.8404536	0.8404536
22	0.8582841	0.8420062	0.8230875
23	0.8590550	0.8434557	0.8137333
24	0.8597886	0.8448130	0.8103424
25	0.8604879	0.8460874	0.8118256
26	0.8611555	0.8472870	0.8178400
27	0.8617937	0.8484190	0.8290392
28	0.8624046	0.8494892	0.8494892
29	0.8629900	0.8505033	0.8381867
30	0.8635518	0.8514658	0.8309073

packing density:

$$D \leq \frac{n}{\left[1 - \frac{\sqrt{3}}{2} + \sqrt{\frac{3}{4} + \frac{2\sqrt{3}}{\pi}(n-1)}\right]^2} \tag{17}$$

which is exact for $n = 1$, and which is better than (4) for $n \geq 2$. Groemer's inequality (17), however, can be sharpened even further due to the special properties of a circle.

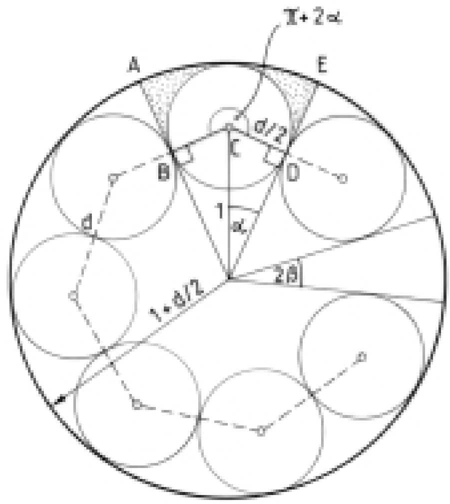

Fig. 8. Arrangement of circles along the boundary of the large circle

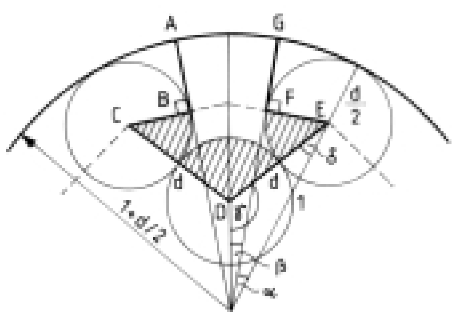

Fig. 9. The gap at the boundary of the large circle

Consider equal circles of diameter d packed in a circle of radius $1 + \frac{d}{2}$. In this case the centres of the circles are in the closed unit circle. If circles touching the boundary of the circle of radius $1 + \frac{d}{2}$ are in a close arrangement, then the area of the part of the Dirichlet cell of such a circle, outside the polygon determined by

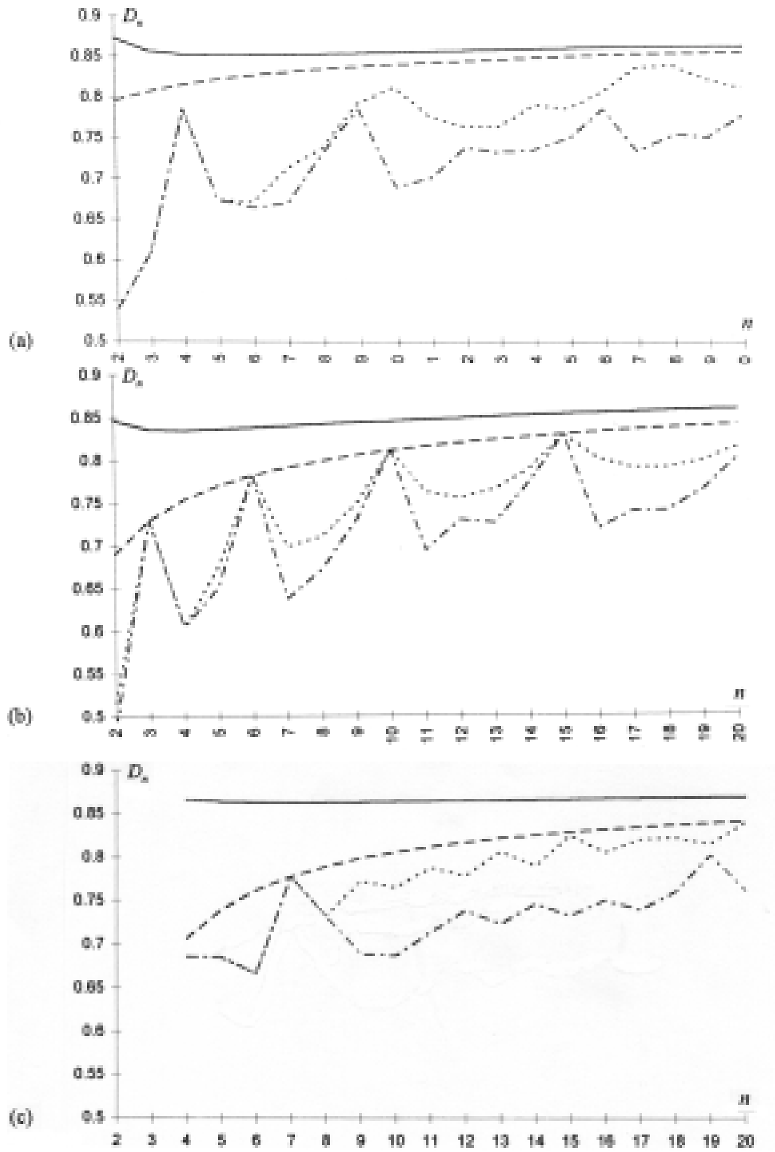

Fig. 10. Upper and lower bounds of the maximum density of packing of *n* equal circles in (a) a square, (b) an equilateral triangle, (c) a circle. The curves represent upper bounds due to Groemer (solid lines), with average extra gap area along the boundary (dashed lines), with exact extra gap area along the boundary (dotted lines), and lower bounds given by constructions (dash-dot lines).

the centres of the circles, is larger than the space claim of the part of the circle lying outside that polygon. Consider such a circle, say that with centre C (*Fig. 8*). Here

$$\alpha = \arcsin \frac{d}{2}$$

and the area of the domain $ABCDE$ is $\left(1+\frac{d}{2}\right)^2 \alpha - \frac{d}{2}\cos\alpha$, and the space claim of the sector of angle $\pi + 2\alpha$ and radius $\frac{d}{2}$ is $\frac{\sqrt{3}}{2} d^2 \frac{\pi+2\alpha}{2\pi}$. Their difference appears as the area of an extra interstice corresponding to that sector. The largest density occurs, if as many circles as possible are touching the boundary of the large circle, that is the centres of circles are situated along the boundary of the unit circle with separation d between them (*Fig. 8*). We cannot put more than $\mathrm{int}\left(\frac{\pi}{\alpha}\right)$ circles. If we add the area of the extra interstices along the boundary of the large circle, we obtain a lower bound on the real extra interstice area. This bound is even smaller if along the fractional arc $2\pi - 2\alpha\,\mathrm{int}\left(\frac{\pi}{\alpha}\right)$, the average extra interstice area is taken into account instead of the actual one. Therefore, a lower bound A_e of the area of the sum of extra interstices is

$$A_e = \left(1+\frac{d}{2}\right)^2 \pi - \frac{d\pi}{2\alpha}\cos\alpha - \frac{d^2}{4}\sqrt{3}\left(2+\frac{\pi}{\alpha}\right).$$

The space claim A_c of n circles is

$$A_c = n\frac{\sqrt{3}}{2}d^2.$$

$A_c + A_e$ cannot be greater than the area of the circle of radius $1+\frac{d}{2}$. This condition yields the inequality:

$$n\sqrt{12} \leq \frac{\pi}{\arcsin\frac{d}{2}}\left(\frac{\sqrt{4-d^2}}{d}+\sqrt{3}\right)+\sqrt{12} \tag{18}$$

from which, with numerical calculation, we can determine an upper bound on d. Let us denote it by d_m. So, we obtain an upper bound on the density D:

$$D \leq \frac{n d_m^2}{(2+d_m)^2} \tag{19}$$

which is exact for $n = 2, 3, 7$, and which is better than (17) for every $n \geq 2$.

Let us make this upper bound of packing density sharper by calculating the area of the extra gap at the boundary exactly. Let us suppose that $d < 1$. For dense packing, the circles are closely packed along the boundary, and a gap of angle 2β appears where

$$\beta = \pi - \alpha\,\mathrm{int}\left(\frac{\pi}{\alpha}\right).$$

Table 3. Upper bounds on the maximum density D of packing of n equal circles in a circle

n	With Groemer's formula (17)	With average extra interstice area, formulae (18), (19)	With exact extra gap area, formulae (22), (19)
2	0.8947269	0.5	
3	0.8736430	0.6461709	
4	0.8666092	0.7064501	0.6862915
5	0.8639636	0.7401823	0.6852102
6	0.8630830	0.7621470	0.6666667
7	0.8630130	0.7777778	0.7777778
8	0.8633427	0.7895698	0.7325021
9	0.8638746	0.7988414	0.7726714
10	0.8645071	0.8063592	0.7660089
11	0.8651847	0.8126023	0.7886108
12	0.8658764	0.8178863	0.7794110
13	0.8665645	0.8224286	0.8071556
14	0.8672391	0.8263840	0.7906235
15	0.8678945	0.8298661	0.8259160
16	0.8685278	0.8329602	0.8063189
17	0.8691378	0.8357317	0.8213498
18	0.8697242	0.8382320	0.8230299
19	0.8702872	0.8405016	0.8158336
20	0.8708277	0.8425731	0.8410926
21	0.8713464	0.8444731	0.8276348
22	0.8718443	0.8462237	0.8257748
23	0.8723226	0.8478430	0.8435271
24	0.8727821	0.8493462	0.8339199
25	0.8732239	0.8507465	0.8333797
26	0.8736489	0.8520546	0.8483184
27	0.8740582	0.8532803	0.8402808
28	0.8744525	0.8544315	0.8389196
29	0.8748327	0.8555154	0.8541269
30	0.8751995	0.8565382	0.8465751

From *Fig. 9* we have:

$$\gamma = \pi - \arcsin \frac{\sin(\alpha + \beta)}{d},$$
$$\delta = \pi - \alpha - \beta - \gamma.$$

Consider the arrowhead-like domain $ABCDEFG$ (*Fig. 9*). Its area A_a is

$$A_a = \left(1 + \frac{d}{2}\right)^2 \beta - d \sin \delta + \frac{d}{2} \cos \alpha.$$

The area A_g of the extra gap is obtained if the space claim of a sector of angle $\pi + 2\alpha$ is subtracted from the area of the arrowhead-like domain:

$$A_g(\beta) = \left(1 + \frac{d}{2}\right)^2 \beta - d \sin \left(\arcsin \frac{\sin(\alpha + \beta)}{d} - \alpha - \beta\right) + \frac{d}{2} \cos \alpha - \frac{d^2}{4}\sqrt{3}\left(1 + 2\frac{\beta}{\pi}\right). \tag{20}$$

The area of the sum of extra interstices (including the extra gap area) and the space claim of n circles together cannot be greater than the area of the circle of radius $1 + \frac{d}{2}$:

$$n\frac{\sqrt{3}}{2}d^2 + \operatorname{int}\left(\frac{\pi}{\alpha}\right)\left[\left(1 + \frac{d}{2}\right)^2 \alpha - \frac{d}{2}\cos\alpha - \frac{d^2}{4}\sqrt{3}\left(1 + 2\frac{\alpha}{\pi}\right)\right] + A_g(\beta) \leq \left(1 + \frac{d}{2}\right)^2 \pi, \tag{21}$$

whence

$$n\sqrt{12} \leq \operatorname{int}\left(\frac{\pi}{\alpha}\right)\left[\frac{2}{d}\cos\alpha + \sqrt{3}\left(1 + 2\frac{\alpha}{\pi}\right)\right] + \frac{2}{d}(2\sin\delta - \cos\alpha) + \sqrt{3}\left(1 + 2\frac{\beta}{\pi}\right), n \geq 4. \tag{22}$$

For a given value of n, from (22) we can determine numerically an upper bound of d which we denote by d_m and by (19) we obtain an upper bound of the maximum density D which is exact for $n = 4, 5, 6, 7$, and which is better than that obtained from (18) for $n \geq 4$, $n \neq 7$.

As in the case of the square, it should be shown also here that if the gap at the boundary is not concentrated at one point at the boundary of the circle domain but it is divided into two parts at two different points of the boundary, the upper bound of the density cannot decrease.

The upper bounds on the maximum packing density calculated by the formulae (17), as well as (18) and (22) with (19) are given for up to $n = 30$ in *Table 3*. Unlike the upper bound given by (18) the upper bound given by (22) is not monotonous with n.

Of course, the presented upper bounds, with increasing n, asymptotically tend to the density of the densest packing of equal circles in the plane: $\pi/\sqrt{12}$.

5. Conclusions

To summarize the results and to provide an overview of them we plotted the obtained upper bounds of the maximum packing density against the circle number n in *Fig. 10*, in the case of a square (a), an equilateral triangle (b) and a circle (c). In order to make a comparison between the upper and lower bounds, the known best lower bounds given by actual packing constructions are also presented there. Their numerical data are taken from [1] and [20] for a square, from [12] and [16] for an equilateral triangle, from [5], [10], [21] and [22] for a circle. The solid lines, dashed lines and dotted lines show the upper bounds obtained with Groemer's inequality, with the average and the exact extra gap areas along the boundary; the dash-dot line shows the lower bounds. We will refer to these curves as 'Groemer', 'average', 'exact' and 'lower'. From the plots in *Fig. 10* we can conclude the following.

(1) The difference between 'Groemer' and 'average' is the greatest for the circle domain, then for the equilateral triangle, and the smallest for the square. This is so because the boundary of a circle domain is curved, and so the interstice area at the boundary is larger than that for a straight line boundary; and the interstice area at a vertex of an equilateral triangle is larger than that for a square.

(2) The difference between 'average' and 'exact' is the smallest in general for the circle domain because along its boundary there is only one gap. For an equilateral triangle, if n is a triangle number, then of course there is no difference as 'average' is exact.

(3) The difference between 'exact' and 'lower' is the smallest in general for the equilateral triangle, because in its inner parallel domain the density approximates $\pi/\sqrt{12}$ better than in the square and the circle.

Acknowledgement

This work was partially supported by OTKA Grant No. T014285 and T0224037 awarded by the Hungarian Scientific Research Foundation, and by FKFP Grant No. 0391/1997 awarded by the Ministry of Education of Hungary.

References

[1] CROFT, H. T. – FALCONER, K. J. – GUY, R. K.: Unsolved Problems in Geometry, Springer, New York, 1991.

[2] FEJES TÓTH, G.: Packing and Covering, Manuscript, 1995.
[3] FEJES TÓTH, L.: Regular Figures, Pergamon-MacMillan, New York, 1964.
[4] FEJES TÓTH, L.: Lagerungen in der Ebene, auf der Kugel und im Raum. Zweite Auflage. Springer, Berlin, 1972.
[5] GOLDBERG, M.: Packing of 14, 16, 17 and 20 Circles in a Circle, *Math. Mag.* Vol. 44, (1971), pp. 134–139.
[6] GRAHAM, R. L. – LUBACHEVSKY, B. D.: Dense Packings of Equal Discs in an Equilateral Triangle: from 22 to 34 and beyond, *Electron. J. Combin.* Vol. 2, (1995), Article 1, approx. 39 pp. (electronic).
[7] GRAHAM, R. L. – LUBACHEVSKY, B. D. – NURMELA, K. J. – ÖSTERGARD P. R. J.: Dense Packings of Congruent Circles in a Circle, *Discrete Math.* Vol. 181, (1998), pp. 139–154.
[8] GROEMER, H.: Über die Einlagerung von Kreisen in einen konvexen Bereich, *Math. Z.* Vol. 73, (1960), pp. 285–294.
[9] HUJTER, M: Some Numerical Problems of Discrete Geometry, Preprint, University of Miskolc, Hungary, 1996.
[10] KRAVITZ, S. Packing Cylinders into Cylindrical Containers, *Math. Mag.* Vol. 40, (1967), pp. 65–71.
[11] MANARAS, C. D. – FLOUDAS, C. A. – PARDALOS, P. M.: New Results in the Packing of Equal Circles in a Square, *Discrete Math.* Vol. 142, (1995), pp. 287–293.
[12] MELISSEN, J. B. H.: Densest Packings of Congruent Circles in an Equilateral Triangle, *Amer. Math. Monthly* Vol. 100, (1993), pp. 916–925.
[13] MELISSEN, J. B. H.: Optimal Packings of Eleven Equal Circles in an Equilateral Triangle, *Acta Math. Hungar.* Vol. 65, (1994), pp. 389–393.
[14] MELISSEN, J. B. H.: Densest Packing of Six Equal Circles in a Square, *Elem. Math.* Vol. 49, (1994), pp. 27–31.
[15] MELISSEN, J. B. H.: Densest Packings of Eleven Congruent Circles in a Circle, *Geometriae Dedicata* Vol. 50, (1994), pp. 15–25.
[16] MELISSEN, J. B. H. – SCHUUR, P. C.: Packing 16, 17 or 18 Circles in an Equilateral Triangle, *Discrete Math.* Vol. 145, (1995), pp. 333–342.
[17] MOLNÁR, J.: On the Packing of Unit Circles in a Convex Domain, *Ann. Univ. Sci. Budapest. Eötvös, Sect. Math.* Vol. 22/23, (1979/80), pp. 113–123.
[18] MOSER, W. – PACH, J.: Research Problems in Discrete Geometry: Packing and Covering, DIMACS Technical Report 93-32, May 1993.
[19] OLER, N.: A Finite Packing Problem, *Canad. Math. Bull.* Vol. 4, (1961), pp. 153–155.
[20] PEIKERT, R.: Dichteste Packungen von gleichen Kreisen in einem Quadrat, *Elem. Math.* Vol. 49, (1994), pp. 16–26.
[21] PIRL, U.: Der Mindestabstand von n in der Einheitskreisscheibe gelegenen Punkten, *Math. Nachr.* Vol. 40, (1969), pp. 111–124.
[22] REIS, G. E.: Dense Packing of Equal Circles within a Circle, *Math. Mag.* Vol. 48, (1975), pp. 33–37.
[23] TARNAI, T. – GÁSPÁR, ZS.: Packing of Equal Circles in a Square, *Acta Technica Acad. Sci. Hung.* Vol. 107, (1995-96), pp. 123–135.