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ILLUSTRATION OF THE INTERACTION OF STRAIN AND DISPLACEMENT NONLINEARITIES IN THE STRUCTURAL TANGENT STIFFNESS

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Abstract

Systematization of complex nonlinearities, the wide-ranging linearization concepts are detailed in [3], [4] related to material, strain, displacement and loading type nonlinearities and their interaction. In this paper, an illustration of the full geometric nonlinearity, the interaction of the strain and displacement nonlinearities are presented, by means of the finite element model of the Timoshenko beam.

Keywords: finite elements, tangent stiffness matrix, strain and displacement nonlinearity.

1. Introduction

As the basis of the nonlinear structural analysis, systematic derivation of the family of tangent stiffness matrices and the possible linearization and approximation aspects were discussed in [3], [4]. The analysis of the tangent stiffness matrix was extended to the effect of nonconvex strain energy functional, namely to material softening, moreover, to convex and nonconvex external potential due to deformation-sensitive loading devices. The aim of the paper [4] was to help in orientation in the large family of the wide-ranging used tangent stiffness matrices of the nonlinear finite element analysis. The term *material tangent modulus* was extended to the effect of deformation-sensitive loading by introducing the term *loading tangent modulus.* An overall approach was presented: from the analytical origin to the finite element discretization. Full structural nonlinearity was assumed: nonlinear material, nonlinear strains, nonlinear displacements and nonlinear loading devices were considered.

In this paper, an illustration of the systematic derivation of the tangent stiffness matrix is presented. For this reason, the basic concepts detailed in [4] are collected here.

2. Derivation of the Tangent Stiffness Matrix in Fully Nonlinear Cases

Let us consider isothermal deformations of a time-independent solid body subject to a quasi-static conservative loading program. Nonlinear material and nonlinear loading program are concerned.

In the Lagrangian description S_{ij} is the second Piola–Kirchhoff stress tensor and E_{ij} is the Lagrange–Green strain tensor. The material is specified by a nonlinear function $S_{ij}\ (E_{mn})$, thus, as the first *linearization condition,* the incrementally linear relation can be established as

$$\mathrm{d}S_{ij} = \frac{\partial S_{ij}\ (E_{mn})}{\partial E_{kl}} \Delta E_{kl} = D^t_{ijkl}\ (E_{mn})\ \Delta E_{kl}, \tag{1}$$

where $D^t_{ijkl}\ (E_{mn})$ is the instantaneous *material tangent modulus tensor.*

Let us consider that in the volume V_0 the body forces F_i, and on a part S_{p0} of the surface S_0 the surface tractions P_i, while on the complementary part S_{u0}, the displacements u_i are specified. Let us assume a scalar loading parameter λ to be varied continuously and infinitely slowly in time. Fundamental classification of loading types is detailed in [1], [2] by distinguishing the term *dead* and *configuration-dependent* load. *Dead type loading device* supposes the applied load to be independent of the occurring deflections. In this case, during a loading process, the load F can be controlled by a scalar load parameter λ, thus $F = \lambda\ F_0$, and $\mathrm{d}F = \mathrm{d}\lambda\ F_0$ (*Fig. 1a*).

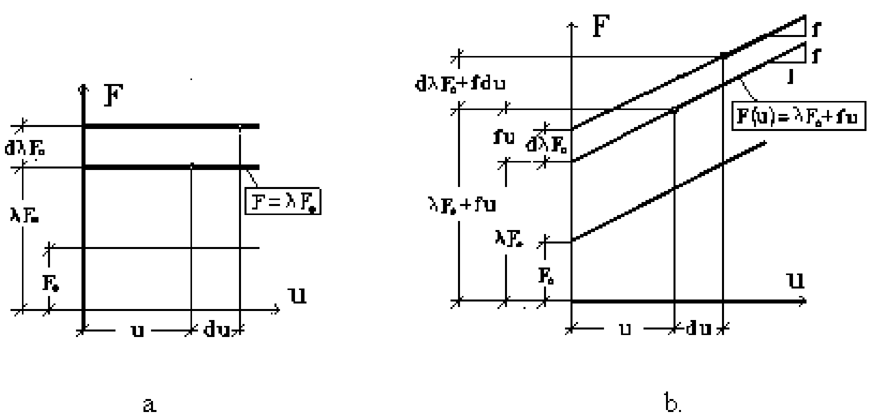

Fig. 1. Configuration-dependent loading

Configuration-dependent loading assumes the applied load to be dependent on the occurring deflections, characterized by a load-deflection diagram $F = F(u) = \lambda\ F_0 + f\,(u)$, which is divided into two parts: the *controllable* part $\lambda\ F_0$ governed by the load parameter λ, and the *deformation-sensitive* part $f\,(u)$ specified as a linear or nonlinear function (*Fig. 1b*). For linear variable load, the loading modulus f is constant.

In the case of *nonlinear variable load*, incrementally linear analysis is needed. Namely, for $F_i = F_i\left(u_j\right) = \lambda\, F_0 + f_i\left(u_j\right)$, the increments $\mathrm{d}F_i = \mathrm{d}\lambda\, F_{i0} + \mathrm{d}\, f_i = \mathrm{d}\lambda\, F_{i0} + M^t_{ij}(u_k)\,\Delta u_j$ contain the second order tensor $M^t_{ij} = \partial f_i/\partial u_j$ being the *loading tangent modulus tensor*

$$\mathrm{d}\, f_i = \frac{\partial f_i}{\partial u_j}\Delta u_j = M^t_{ij}(u_k)\,\Delta u_j, \tag{2}$$

which *linearization condition* is formally the same as in (1).

The tangent stiffness matrix is based on the incremental virtual work, which, completed by the terms of the deformation-sensitive loading is as follows

$$\begin{aligned}\delta\Delta W &= \textstyle\int_{V_0}\left(S_{ij} + \Delta S_{ij}\right)\delta\Delta E_{ij}\,\mathrm{d}V_0 - \\ &- \textstyle\int_{V_0}\left[(\lambda F_{i0} + f_i) + \Delta\,(\lambda F_{i0} + f_i)\right]\delta\Delta u_i\,\mathrm{d}V_0 - \\ &- \textstyle\int_{S_{P0}}\left[(\lambda P_{i0} + p_i) + \Delta\,(\lambda P_{i0} + p_i)\right]\delta\Delta u_i\,\mathrm{d}S_0 = 0,\end{aligned} \tag{3}$$

where $f_i = f_i\left(u_j\right)$, $p_i = p_i\left(u_j\right)$ are the deformation-sensitive parts of the volume and surface loading, respectively [4]. The term Δ represents the total increment and δ indicates the variation. Notice that these forms are the correct versions of the incremental virtual work since here the total increments Δ appear. However, for the stresses in (1), and for the loading in (2) we assumed first order increments, thus, in (3) *further linearization assumptions* $\Delta S_{ij} \cong \mathrm{d}S_{ij}$, $\Delta f_i \cong \mathrm{d}\, f_i$ and $\Delta p_i \cong \mathrm{d}p_i$ can be applied. Obviously, for the scalar parameter λ, $\Delta\lambda = \mathrm{d}\lambda$. Thus, the incremental virtual work (3) yields

$$\begin{aligned}\delta\Delta W &= \textstyle\int_{V_0}\left(S_{ij} + \mathrm{d}S_{ij}\right)\delta\Delta E_{ij}\,\mathrm{d}V_0 - \\ &- \textstyle\int_{V_0}\left[(\lambda F_{i0} + f_i) + \mathrm{d}\,(\lambda F_{i0} + f_i)\right]\delta\Delta u_i\,\mathrm{d}V_0 - \\ &- \textstyle\int_{S_{P0}}\left[(\lambda P_{i0} + p_i) + \mathrm{d}\,(\lambda P_{i0} + p_i)\right]\delta\Delta u_i\,\mathrm{d}S_0 = 0.\end{aligned} \tag{4}$$

Let us consider now the required variational and incremental form of the strains and displacements appearing in the above expression.

In the Lagrange–Green strain tensor

$$E_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right) \tag{5}$$

linear and nonlinear parts can be distinguished in terms of the displacement gradients $u_{i,j}$. In the case of *large displacement gradients, large or finite strains* are considered, while in the case of *small displacement gradients*, that is $u_{i,j} \ll 1$, *small or infinitesimal strains* are distinguished, by neglecting the higher order small term $u_{k,i}u_{k,j}$.

As for the increments and variations of the strain, we can conclude that they depend on both the increments and variations of the displacements $\Delta u_{k,l}$, $\delta u_{k,l}$

and $\delta\Delta u_{k,l}$. However, these terms can be analyzed after the discretization of the displacements only.

The displacement functions *for a single finite element within the body* can be expressed in terms of the geometric and functional coordinates $\mathbf{X}$ and $\mathbf{q}$, respectively, as

$$\underset{(3)}{\mathbf{u}} = \underset{(3)}{\mathbf{u}}(\mathbf{X}, \mathbf{q}) = \begin{bmatrix} u_1(\mathbf{X}, \mathbf{q}_1) \\ u_2(\mathbf{X}, \mathbf{q}_2) \\ u_3(\mathbf{X}, \mathbf{q}_3) \end{bmatrix} = \begin{bmatrix} u_1\,(X_1, X_2, X_3; q_1, q_2, \ldots, q_r) \\ u_2\,(X_1, X_2, X_3; q_1, q_2, \ldots, q_r) \\ u_3\,(X_1, X_2, X_3; q_1, q_2, \ldots, q_r) \end{bmatrix}, \quad (6)$$

where $\mathbf{X}$ are the coordinates of the discretized *geometric space,* and $\mathbf{q}$ are the coordinates of the discretized *function space, r* is the number of generalized coordinates of the elements.

Here we distinguish *small/large displacements,* functions $\mathbf{u}$ to be *linear/nonlinear* in $\mathbf{q}$, respectively. Practically, in the case of *large displacements,* parameters $\mathbf{q}$ contain rotational elements, that is, trigonometrical relations in $\mathbf{u}$. For *small displacements,* functions $\mathbf{u}$ are linear in $\mathbf{q}$, thus, the variables $\mathbf{X}$ and $\mathbf{q}$ in (6) can be separated by the linear combination

$$u_i = \sum_{k=1}^{m} q_i^k \varphi^k(\mathbf{X}), \quad (7)$$

where $\varphi^k(\mathbf{X})$ are the *interpolation or shape functions* corresponding to the nodal points of number m. Expression (7) leads to the classical basic expression of the *linear finite element displacement method*

$$\underset{(3)}{\mathbf{u}(\mathbf{X}, \mathbf{q})} = \underset{(3,r)}{\mathbf{N}(\mathbf{X})}\ \underset{(r)}{\mathbf{q}}\,, \quad (8)$$

where matrix $\mathbf{N}(\mathbf{X})$ contains the shape functions $\varphi^k(\mathbf{X})$ of the classical linear FEM approach.

In the case of *large nonlinear displacements,* the direct separation (8) cannot be applied. In such cases, *incrementally linear analysis* is needed [3], [4], [6].

Let us consider the increments of *large displacements* as $\Delta\mathbf{u} = \mathrm{d}\mathbf{u} + \mathrm{d}^2\mathbf{u}$, where

$$\underset{(3)}{\mathrm{d}\mathbf{u}} = \left.\frac{\partial \mathbf{u}(\mathbf{X}, \mathbf{q})}{\partial q_j}\right|_n \mathrm{d}q_j = \underset{(3,r)}{\mathbf{H}(\mathbf{X}, \mathbf{q}_n)}\, \underset{(r)}{\mathrm{d}\mathbf{q}} = \underset{(3,r)}{\mathbf{H}_n}\, \underset{(r)}{\mathrm{d}\mathbf{q}}, \quad (9)$$

and

$$\underset{(3)}{\mathrm{d}^2\mathbf{u}} = \frac{1}{2}\left.\frac{\partial^2 \mathbf{u}(\mathbf{X}, \mathbf{q})}{\partial q_j \partial q_k}\right|_n \mathrm{d}q_j\, \mathrm{d}q_k = \frac{1}{2}\underset{(r)}{\mathrm{d}\mathbf{q}^T}\, \underset{(r,3,r)}{\mathbf{W}(\mathbf{X}, \mathbf{q}_n)}\, \underset{(r)}{\mathrm{d}\mathbf{q}} = \frac{1}{2}\underset{(r)}{\mathrm{d}\mathbf{q}^T}\, \underset{(r,3,r)}{\mathbf{W}_n}\, \underset{(r)}{\mathrm{d}\mathbf{q}} \quad (10)$$

are the *first and second order increments of the large displacements,* respectively, related to the n-th configuration. Matrix $\mathbf{H}_n$ has $3 \times r$ elements, while matrix $\mathbf{W}_n$ is three dimensional of measure $r \times 3 \times r$. The *incrementally linear relation* (9) can be considered as the basic relation of an iteration process of the *nonlinear finite element displacement method* while matrix $\mathbf{W}_n$ represents the *nonlinear geometry.* The *first variations of the large displacements*

$$\underset{(3)}{\delta\mathbf{u}} = \frac{\partial \mathbf{u}(\mathbf{X},\mathbf{q})}{\partial q_j}\delta q_j = \underset{(3,r)}{\mathbf{H}(\mathbf{X},\mathbf{q})}\,\underset{(r)}{\delta\mathbf{q}} \tag{11}$$

result in a new function since $\mathbf{H}$ contains both $\mathbf{X}$ and $\mathbf{q}$ as variables.

By considering the *variation and increments of large displacements,* table (12) illustrates all the necessary terms in a concise form.

Increments and variations of displacements			
		Small displacements	Large displacements
First variation	$\delta\mathbf{u}$	$\mathbf{N}\delta\mathbf{q}$	$\mathbf{H}\delta\mathbf{q}$
First increment	$\mathrm{d}\mathbf{u}$	$\mathbf{N}\mathrm{d}\mathbf{q}$	$\mathbf{H}_n\mathrm{d}\mathbf{q}$
Second increment	$\mathrm{d}^2\mathbf{u}$	$\mathbf{0}$	$1/2\mathrm{d}\mathbf{q}^T\mathbf{W}_n\mathrm{d}\mathbf{q}$
Total increment	$\Delta\mathbf{u}$	$\mathbf{N}\mathrm{d}\mathbf{q}$	$\mathbf{H}_n\mathrm{d}\mathbf{q} + 1/2\mathrm{d}\mathbf{q}^T\mathbf{W}_n\mathrm{d}\mathbf{q}$
Variation of first increment	$\delta\mathrm{d}\mathbf{u}$	$\mathbf{N}\delta\mathrm{d}\mathbf{q}$	$\mathbf{H}_n\delta\mathrm{d}\mathbf{q}$
Variation of second increment	$\delta\mathrm{d}^2\mathbf{u}$	$\mathbf{0}$	$\mathrm{d}\mathbf{q}^T\mathbf{W}_n\delta\mathrm{d}\mathbf{q}$
Variation of total increment	$\delta\Delta\mathbf{u}$	$\mathbf{N}\delta\mathrm{d}\mathbf{q}$	$\mathbf{H}_n\delta\mathrm{d}\mathbf{q} + \mathrm{d}\mathbf{q}^T\mathbf{W}_n\delta\mathrm{d}\mathbf{q}$

(12)

In these expressions, matrix $\mathbf{N} = \mathbf{N}(\mathbf{X})$ of shape functions of the linear finite element procedure is constant during the total state change analysis, while matrices $\mathbf{H}_n$ and $\mathbf{W}_n$ change during the iteration process. Namely, $\mathbf{H}_n = \mathbf{H}(\mathbf{X},\mathbf{q}_n)$ and $\mathbf{W}_n = \mathbf{W}(\mathbf{X},\mathbf{q}_n)$ are related to the n-th equilibrium configuration, being constant in the n-th iteration sub-step only.

By expressing the nonlinear Green–Lagrange strains in terms of the displacement gradients, for the discrete version we can use the form

$$\underset{(6)}{\mathbf{E}} = \underset{(6)}{\mathbf{E}(\mathbf{u})} = \underset{(6,3)}{\mathbf{A}}\;\underset{(3)}{\mathbf{u}} + \frac{1}{2}\underset{(3)}{\mathbf{u}^T}\;\underset{(3,9)}{\mathbf{B}^T}\;\underset{(9,6,3)}{\mathbf{C}}\;\underset{(3)}{\mathbf{u}}, \tag{13}$$

where $\mathbf{E}$ are in vector arrangement as

$$\mathbf{E}^T = \begin{bmatrix} E_{11} & E_{22} & E_{33} & 2E_{12} & 2E_{13} & 2E_{23} \end{bmatrix},$$

moreover, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are differential operators with respect to $\mathbf{X}$, concerning the *displacement gradients* represented by the *linear* term $\mathbf{Au}$ in the *small (infinitesimal) strains*, and, by the *nonlinear* term $1/2\,\mathbf{u}^T\mathbf{B}^T\mathbf{C}\,\mathbf{u}$ in the case of *large (finite)* *strains.* Matrix **C** is three-dimensional, consisting of six layers of sub-matrices of measure 9×3.

$$\mathbf{A} = \begin{bmatrix} \frac{\partial}{\partial X_1} & 0 & 0 \\ 0 & \frac{\partial}{\partial X_2} & 0 \\ 0 & 0 & \frac{\partial}{\partial X_3} \\ \frac{\partial}{\partial X_2} & \frac{\partial}{\partial X_1} & 0 \\ \frac{\partial}{\partial X_3} & 0 & \frac{\partial}{\partial X_1} \\ 0 & \frac{\partial}{\partial X_3} & \frac{\partial}{\partial X_2} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \frac{\partial}{\partial X_1} & 0 & 0 \\ 0 & \frac{\partial}{\partial X_1} & 0 \\ 0 & 0 & \frac{\partial}{\partial X_1} \\ \frac{\partial}{\partial X_2} & 0 & 0 \\ 0 & \frac{\partial}{\partial X_2} & 0 \\ 0 & 0 & \frac{\partial}{\partial X_2} \\ \frac{\partial}{\partial X_3} & 0 & 0 \\ 0 & \frac{\partial}{\partial X_3} & 0 \\ 0 & 0 & \frac{\partial}{\partial X_3} \end{bmatrix},$$

$$\mathbf{C}_1 = \begin{bmatrix} \frac{\partial}{\partial X_1} & 0 & 0 \\ 0 & \frac{\partial}{\partial X_1} & 0 \\ 0 & 0 & \frac{\partial}{\partial X_1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{C}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial}{\partial X_2} & 0 & 0 \\ 0 & \frac{\partial}{\partial X_2} & 0 \\ 0 & 0 & \frac{\partial}{\partial X_2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{C}_3 = \begin{bmatrix} 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & ccc \\ 0 & 0 & 0 & \\ \frac{\partial}{\partial X_3} & 0 & 0 & \\ 0 & \frac{\partial}{\partial X_3} & 0 & \\ 0 & 0 & \frac{\partial}{\partial X_3} & \end{bmatrix}, \qquad \mathbf{C}_4 = \begin{bmatrix} \frac{\partial}{\partial X_2} & 0 & 0 \\ 0 & \frac{\partial}{\partial X_2} & 0 \\ 0 & 0 & \frac{\partial}{\partial X_2} \\ \frac{\partial}{\partial X_1} & 0 & 0 \\ 0 & \frac{\partial}{\partial X_1} & 0 \\ 0 & 0 & \frac{\partial}{\partial X_1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{C}_5 = \begin{bmatrix} \frac{\partial}{\partial X_3} & 0 & 0 \\ 0 & \frac{\partial}{\partial X_3} & 0 \\ 0 & 0 & \frac{\partial}{\partial X_3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial}{\partial X_1} & 0 & 0 \\ 0 & \frac{\partial}{\partial X_1} & 0 \\ 0 & 0 & \frac{\partial}{\partial X_1} \end{bmatrix}, \qquad \mathbf{C}_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial}{\partial X_3} & 0 & 0 \\ 0 & \frac{\partial}{\partial X_3} & 0 \\ 0 & 0 & \frac{\partial}{\partial X_3} \\ \frac{\partial}{\partial X_2} & 0 & 0 \\ 0 & \frac{\partial}{\partial X_2} & 0 \\ 0 & 0 & \frac{\partial}{\partial X_2} \end{bmatrix}. \tag{14}$$

Let us consider now the different forms of increments and variations of the strains in terms of the displacements in the frame of the FEM analysis, detailed in [4]. Here we emphasize the difference between large and small strains and displacements as the effect of the approximation level, as seen in the following concise form (15) in terms of the matrices **A**, **B** and **C** relating to the geometric space, and **N**, $\mathbf{H}_n$ and $\mathbf{W}_n$ referring to the function space.

Increments and variations of strains			
		Large strains **A**, **B**, **C**	Small strains **A** only
Large displ. **H**, **W**	$\delta\mathbf{E}$	$\left(\mathbf{AH}(\mathbf{q}) + \mathbf{u}(\mathbf{q})^T\mathbf{B}^T\mathbf{CH}\right)\delta\mathbf{q}$	$\mathbf{AH}(\mathbf{q})\delta\mathbf{q}$
	$\mathrm{d}\mathbf{E}$	$\left(\mathbf{AH}_n + \mathbf{u}_n^T\mathbf{B}^T\mathbf{CH}_n\right)\mathrm{d}\mathbf{q}+$ $+1/2\mathrm{d}\mathbf{q}^T\left(\mathbf{AW}_n + \mathbf{u}_n^T\mathbf{B}^T\mathbf{CW}_n\right)\mathrm{d}\mathbf{q}$	$\mathbf{AH}_n\mathrm{d}\mathbf{q} + 1/2\mathrm{d}\mathbf{q}^T\mathbf{AW}_n\mathrm{d}\mathbf{q}$
	$\mathrm{d}^2\mathbf{E}$	$\cong 1/2\mathrm{d}\mathbf{q}^T\mathbf{H}_n^T\mathbf{B}^T\mathbf{CH}_n\mathrm{d}\mathbf{q}$	$\mathbf{0}$
	$\triangle\mathbf{E}$	$\left(\mathbf{AH}_n + \mathbf{u}_n^T\mathbf{B}^T\mathbf{CH}_n\right)\mathrm{d}\mathbf{q}+$ $+1/2\mathrm{d}\mathbf{q}^T\left(\mathbf{AW}_n + \mathbf{u}_n^T\mathbf{B}^T\mathbf{CW}_n+\right.$ $\left.+\,\mathbf{H}_n^T\mathbf{B}^T\mathbf{CH}_n\right)\mathrm{d}\mathbf{q}$	$\mathbf{AH}_n\mathrm{d}\mathbf{q} + 1/2\mathrm{d}\mathbf{q}^T\mathbf{AW}_n\mathrm{d}\mathbf{q}$
	$\delta\mathrm{d}\mathbf{E}$	$\left(\mathbf{AH}_n + \mathbf{u}_n^T\mathbf{B}^T\mathbf{CH}_n\right)\delta\mathrm{d}\mathbf{q}+$ $+\mathrm{d}\mathbf{q}^T\left(\mathbf{AW}_n + \mathbf{u}_n^T\mathbf{B}^T\mathbf{CW}_n\right)\delta\mathrm{d}\mathbf{q}$	$\mathbf{AH}_n\delta\mathrm{d}\mathbf{q} + \mathrm{d}\mathbf{q}^T\mathbf{AW}_n\delta\mathrm{d}\mathbf{q}$
	$\delta\mathrm{d}^2\mathbf{E}$	$\cong \mathrm{d}\mathbf{q}^T\mathbf{H}_n^T\mathbf{B}^T\mathbf{CH}_n\delta\mathrm{d}\mathbf{q}$	$\mathbf{0}$
	$\delta\triangle\mathbf{E}$	$\left(\mathbf{AH}_n + \mathbf{u}_n^T\mathbf{B}^T\mathbf{CH}_n\right)\delta\mathrm{d}\mathbf{q}+$ $+\mathrm{d}\mathbf{q}^T\left(\mathbf{AW}_n + \mathbf{u}_n^T\mathbf{B}^T\mathbf{CW}_n+\right.$ $\left.+\mathbf{H}_n^T\mathbf{B}^T\mathbf{CH}_n\right)\delta\mathrm{d}\mathbf{q}$	$\mathbf{AH}_n\delta\mathrm{d}\mathbf{q} + \mathrm{d}\mathbf{q}^T\mathbf{AW}_n\delta\mathrm{d}\mathbf{q}$
Small displ. **N** only	$\delta\mathbf{E}$	$\mathbf{AN}\delta\mathbf{q} + \mathbf{q}^T\mathbf{N}^T\mathbf{B}^T\mathbf{CN}\delta\mathbf{q}$	$\mathbf{AN}\delta\mathbf{q}$
	$\mathrm{d}\mathbf{E}$	$\mathbf{AN}\mathrm{d}\mathbf{q} + \mathbf{q}_n^T\mathbf{N}^T\mathbf{B}^T\mathbf{CN}\mathrm{d}\mathbf{q}$	$\mathbf{AN}\mathrm{d}\mathbf{q}$
	$\mathrm{d}^2\mathbf{E}$	$1/2\mathrm{d}\mathbf{q}^T\mathbf{N}^T\mathbf{B}^T\mathbf{CN}\mathrm{d}\mathbf{q}$	$\mathbf{0}$
	$\triangle\mathbf{E}$	$\mathbf{AN}\mathrm{d}\mathbf{q} + \mathbf{q}_n^T\mathbf{N}^T\mathbf{B}^T\mathbf{CN}\mathrm{d}\mathbf{q}+$ $+1/2\mathrm{d}\mathbf{q}^T\mathbf{N}^T\mathbf{B}^T\mathbf{CN}\mathrm{d}\mathbf{q}$	$\mathbf{AN}\mathrm{d}\mathbf{q}$
	$\delta\mathrm{d}\mathbf{E}$	$\mathbf{AN}\delta\mathrm{d}\mathbf{q} + \mathbf{q}_n^T\mathbf{N}^T\mathbf{B}^T\mathbf{CN}\delta\mathrm{d}\mathbf{q}$	$\mathbf{AN}\delta\mathrm{d}\mathbf{q}$
	$\delta\mathrm{d}^2\mathbf{E}$	$\mathrm{d}\mathbf{q}^T\mathbf{N}^T\mathbf{B}^T\mathbf{CN}\delta\mathrm{d}\mathbf{q}$	$\mathbf{0}$
	$\delta\triangle\mathbf{E}$	$\left(\mathbf{AN} + \mathbf{q}_n^T\mathbf{N}^T\mathbf{B}^T\mathbf{CN}\right)\delta\mathrm{d}\mathbf{q}+$ $+\mathrm{d}\mathbf{q}^T\mathbf{N}^T\mathbf{B}^T\mathbf{CN}\delta\mathrm{d}\mathbf{q}$	$\mathbf{AN}\delta\mathrm{d}\mathbf{q}$

(15)

The results detailed in tables (12) and (15) are used in the several versions of the tangent stiffness matrix which is derived from the matrix form of the incremental virtual work

$$\begin{aligned}\delta\Delta W &= \int_{V_0}\left(\mathbf{S}^T+\Delta\mathbf{E}^T\mathbf{D}_t\right)\delta\Delta\mathbf{E}\,\mathrm{d}V_0 - \\ &-\int_{V_0}\left[\left(\lambda\,\mathbf{F}_0^T+\mathbf{f}^T\right)+\left(\mathrm{d}\lambda\,\mathbf{F}_0^T+\mathrm{d}\mathbf{f}^T\right)\right]\delta\Delta\mathbf{u}\,\mathrm{d}V_0 - \\ &-\int_{S_{P0}}\left[\left(\lambda\,\mathbf{P}_0^T+\mathbf{p}^T\right)+\left(\mathrm{d}\lambda\,\mathbf{P}_0^T+\mathrm{d}\mathbf{p}^T\right)\right]\delta\Delta\mathbf{u}\,\mathrm{d}S_0 = 0.\end{aligned} \tag{16}$$

Due to the nonlinearity of the state variable functions, caused by the *nonlinearity of the material, loading, strains* and *displacements*, the expression of the incremental virtual work is fully nonlinear, thus, further concepts of linearization are necessary, detailed in [3], [4]. Moreover, expression (16) is still *inhomogeneous* in terms of the increments and variation of the generalized parameters $\mathbf{q}$. For obtaining a *homogeneous* tangent stiffness matrix, further simplifications are needed, detailed in [4].

The *linearized and homogenized form of the incremental virtual work in the case of nonlinear material and loading with large strains and large displacements* yields [4]

$$\begin{aligned}\delta\Delta W = \mathrm{d}\mathbf{q}^T\Big\{&\int_{V_0}\mathbf{H}_n^T\left(\mathbf{A}^T+\mathbf{C}^T\mathbf{B}\mathbf{u}_n\right)\mathbf{D}_t\left(\mathbf{u}_n^T\mathbf{B}^T\mathbf{C}+\mathbf{A}\right)\mathbf{H}_n\,\mathrm{d}V_0 + \\ &+\int_{V_0}\mathbf{S}^T\left(\mathbf{A}\mathbf{W}_n+\mathbf{u}_n^T\mathbf{B}^T\mathbf{C}\mathbf{W}_n+\mathbf{H}_n^T\mathbf{B}^T\mathbf{C}\mathbf{H}_n\right)\mathrm{d}V_0 - \int_{V_0}\mathbf{H}_n^T\mathbf{M}_{ft}\mathbf{H}_n\,\mathrm{d}V_0 - \\ &-\int_{S_0}\mathbf{H}_n^T\mathbf{M}_{pt}\mathbf{H}_n\,\mathrm{d}S_0 - \int_{V_0}\mathbf{f}^T\mathbf{W}_n\,\mathrm{d}V_0 - \int_{S_0}\mathbf{p}^T\mathbf{W}_n\,\mathrm{d}S_0 - \int_{V_0}\lambda\,\mathbf{F}_0^T\mathbf{W}_n\,\mathrm{d}V_0 - \\ &-\int_{S_0}\lambda\,\mathbf{P}_0^T\mathbf{W}_n\mathrm{d}S_0\Big\}\delta\mathrm{d}\mathbf{q} - \mathrm{d}\lambda\left\{\int_{V_0}\mathbf{F}_0^T\mathbf{H}_n\,\mathrm{d}V_0+\int_{S_{P0}}\mathbf{P}_0^T\mathbf{H}_n\,\mathrm{d}S_0\right\}\delta\mathrm{d}\mathbf{q} = 0,\end{aligned} \tag{17}$$

from which, the different forms of the tangent stiffness matrices can be derived. Details on the effect of variable loading on the tangent stiffness can be found in [5], [6], applied to nonsmooth loading functions, too.

The iteration process is based on the *tangent stiffness matrix.* By using the detailed forms of the discrete strains and displacements, different forms of the tangent stiffness matrix can be obtained. In the following table the main versions of the tangent stiffness matrix modified by the different linearization and approximation concepts are summarized.

Structural tangent stiffness		
Nonlinear material, nonlinear loading	Large strains ($\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$)	Small strains ($\mathbf{A}$)
Large displacements ($\mathbf{H}_n$, $\mathbf{W}_n$)	$\int_{V_0} \mathbf{H}_n^T \left(\mathbf{A}^T + \mathbf{C}^T\mathbf{B}\mathbf{u}_n\right)\mathbf{D}_t^n\left(\mathbf{u}_n^T\mathbf{B}^T\mathbf{C}+\mathbf{A}\right)\mathbf{H}_n\mathrm{d}V_0$ $+\int_{V_0}\mathbf{S}_n^T\left(\mathbf{A}\mathbf{W}_n+\mathbf{u}_n^T\mathbf{B}^T\mathbf{C}\mathbf{W}_n+\mathbf{H}_n^T\mathbf{B}^T\mathbf{C}\mathbf{H}_n\right)\mathrm{d}V_0$ $-\int_{V_0}\mathbf{H}_n^T\mathbf{M}_{ft}^n\mathbf{H}_n\mathrm{d}V_0-\int_{S_0}\mathbf{H}_n^T\mathbf{M}_{pt}^n\mathbf{H}_n\mathrm{d}S_0$ $-\int_{V_0}\mathbf{f}_n^T\mathbf{W}_n\mathrm{d}V_0-\int_{S_0}\mathbf{p}_n^T\mathbf{W}_n\mathrm{d}S_0$ $-\int_{V_0}\lambda\mathbf{F}_0^T\mathbf{W}_n\mathrm{d}V_0-\int_{S_0}\lambda\mathbf{P}_0^T\mathbf{W}_n\mathrm{d}S_0$	$\int_{V_0}\mathbf{H}_n^T\mathbf{A}^T\mathbf{D}_t^n\mathbf{A}\mathbf{H}_n\mathrm{d}V_0$ $+\int_{V_0}\mathbf{S}_n^T\mathbf{A}\mathbf{W}_n\mathrm{d}V_0$ $-\int_{V_0}\mathbf{H}_n^T\mathbf{M}_{ft}^n\mathbf{H}_n\mathrm{d}V_0-\int_{S_0}\mathbf{H}_n^T\mathbf{M}_{pt}^n\mathbf{H}_n\mathrm{d}S_0$ $-\int_{V_0}\mathbf{f}_n^T\mathbf{W}_n\mathrm{d}V_0-\int_{S_0}\mathbf{p}_n^T\mathbf{W}_n\mathrm{d}S_0$ $-\int_{V_0}\lambda\mathbf{F}_0^T\mathbf{W}_n\mathrm{d}V_0-\int_{S_0}\lambda\mathbf{P}_0^T\mathbf{W}_n\mathrm{d}S_0$
Small displacements ($\mathbf{N}$)	$\int_{V_0}\mathbf{N}^T\left(\mathbf{A}^T+\mathbf{C}^T\mathbf{B}\mathbf{u}_n\right)\mathbf{D}_t^n\left(\mathbf{u}_n^T\mathbf{B}^T\mathbf{C}+\mathbf{A}\right)\mathbf{N}\mathrm{d}V_0$ $+\int_{V_0}\mathbf{S}_n^T\left(\mathbf{N}^T\mathbf{B}^T\mathbf{C}\mathbf{N}\right)\mathrm{d}V_0$ $-\int_{V_0}\mathbf{N}^T\mathbf{M}_{ft}^n\mathbf{N}\mathrm{d}V_0-\int_{S_0}\mathbf{N}^T\mathbf{M}_{pt}^n\mathbf{N}\mathrm{d}S_0$ ($\mathbf{N}$)	$\int_{V_0}\mathbf{N}^T\mathbf{A}^T\mathbf{D}_t^n\mathbf{A}\mathbf{N}\mathrm{d}V_0$ $-\int_{V_0}\mathbf{N}^T\mathbf{M}_{ft}^n\mathbf{N}\mathrm{d}V_0-\int_{S_0}\mathbf{N}^T\mathbf{M}_{pt}^n\mathbf{N}\mathrm{d}S_0$

(18)

Let us consider now the illustration of the different forms of the tangent stiffness matrix.

3. Illustration of the Tangent Stiffness Matrix in the Case of Combination of Large and Small Strains and Displacements

To illustrate the systematization of the different nonlinearities and the different approximation conditions, the finite element model of Timoshenko beam detailed in [7] will be presented. Some concepts of the nonlinear FEM models are based on [8], [9].

3.1. Beam Element Based on Different Approximations

Let us consider first the so-called Timoshenko beam based on the following kinematic assumptions: each point of the cross sections moves parallel to the xy plane, and the cross sections being initially perpendicular to x remain plane but not necessarily perpendicular to the deformed axis of the beam, seen in *Fig. 2*.

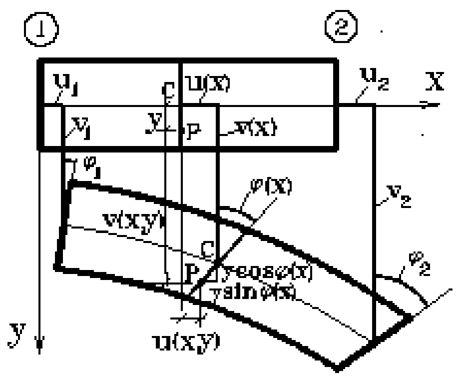

Fig. 2. The Timoshenko beam

3.1.1. Large and Small Displacements

Fig. 2 shows that the displacements $u(x, y)$ and $v(x, y)$ of an arbitrary point $P(x, y)$ are the functions of the displacements $u(x)$ and $v(x)$ of the centroid $C(x, 0)$ and the angular displacement $\varphi(x)$ of the cross section

$$u(x, y) = u(x) - y \sin \varphi(x), \qquad v(x, y) = v(x) - y(1 - \cos \varphi(x)). \tag{19}$$

By introducing the scalar values of the displacements at the nodal points

$$\mathbf{q}^T = \begin{bmatrix} u_1 & v_1 & \varphi_1 & u_2 & v_2 & \varphi_2 \end{bmatrix} \tag{20}$$

as generalized coordinates, thus $r = 6$, moreover, the shape functions

$$N_1 = N_1(x), \qquad N_2 = N_2(x) \tag{21}$$

associated to the nodal points, respectively, by means of the finite element approximation

$$u(x) = u_1 N_1(x) + u_2 N_2(x), \qquad v(x) = v_1 N_1(x) + v_2 N_2(x),$$

$$\varphi(x) = \varphi_1 N_1(x) + \varphi_2 N_2(x) \tag{22}$$

the following *nonlinear displacement function* seen in (6) can be specified:

$$\underset{(2)}{\mathbf{u}} = \begin{bmatrix} u\,(x, y, u_1, \ldots, \varphi_2) \\ v\,(x, y, u_1, \ldots, \varphi_2) \end{bmatrix} = \begin{bmatrix} u(x) - y \sin \varphi(x) \\ v(x) - y(1 - \cos \varphi(x)) \end{bmatrix} =$$

$$= \begin{bmatrix} u_1 N_1 + u_2 N_2 - y \sin\,(\varphi_1 N_1 + \varphi_2 N_2) \\ v_1 N_1 + v_2 N_2 - y\,(1 - \cos\,(\varphi_1 N_1 + \varphi_2 N_2)) \end{bmatrix} = \mathbf{u}(\mathbf{X}, \mathbf{q}) \tag{23}$$

since this function is nonlinear in the generalized coordinates $\mathbf{q}$. In this way, function (23) represents the *large displacements.*

In the case of *small displacements,* the approximations $\sin \varphi \cong \varphi$ and $\cos \varphi \cong 1$ can be applied, thus (19) is simplified to

$$u(x, y) = u(x) - y\,\varphi(x), \qquad v(x, y) = v(x) \tag{24}$$

and (23) changes to the *linear* form

$$\mathbf{u} = \begin{bmatrix} u\,(x, y, u_1, \ldots, \varphi_2) \\ v\,(x, y, u_1, \ldots, \varphi_2) \end{bmatrix} = \begin{bmatrix} u(x) - y\,\varphi(x) \\ v(x) \end{bmatrix} =$$

$$= \begin{bmatrix} u_1 N_1 + u_2 N_2 - y\,(\varphi_1 N_1 + \varphi_2 N_2) \\ v_1 N_1 + v_2 N_2 \end{bmatrix} = \mathbf{u}(\mathbf{X}, \mathbf{q}), \tag{25}$$

which represents the *small displacements.* In this case (25) can be separated with respect to the coordinates of the geometric and function space $\mathbf{X}$ and $\mathbf{q}$, respectively, by the linear combination (8) as

$$\underset{(2)}{\mathbf{u}(\mathbf{X}, \mathbf{q})} = \begin{bmatrix} u_1 N_1 + u_2 N_2 - y\,(\varphi_1 N_1 + \varphi_2 N_2) \\ v_1 N_1 + v_2 N_2 \end{bmatrix} =$$

$$\begin{bmatrix} N_1 & 0 & -yN_1 & N_2 & 0 & -yN_2 \\ 0 & N_1 & 0 & 0 & N_2 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \varphi_1 \\ u_2 \\ v_2 \\ \varphi_2 \end{bmatrix} = \underset{(2,6)}{\mathbf{N}(\mathbf{X})}\ \underset{(6)}{\mathbf{q}}, \tag{26}$$

where the matrix of the shape functions is

$$\underset{(2,6)}{\mathbf{N}(\mathbf{X})} = \begin{bmatrix} N_1 & 0 & -yN_1 & N_2 & 0 & -yN_2 \\ 0 & N_1 & 0 & 0 & N_2 & 0 \end{bmatrix}. \tag{27}$$

However, in the case of large displacements, this direct separation cannot be executed. In this case, the linear combination can be applied to the increments only, related to an equilibrium configuration n

$$\underset{(2)}{\mathrm{d}\mathbf{u}} = \left.\frac{\partial u_i}{\partial q_j}\right|_n \mathrm{d}q_j = \begin{bmatrix} N_1 & 0 & -yN_1\cos\left(\varphi_1^n N_1+\varphi_2^n N_2\right) & N_2 & 0 & -yN_2\cos\left(\varphi_1^n N_1+\varphi_2^n N_2\right) \\ 0 & N_1 & -yN_1\sin\left(\varphi_1^n N_1+\varphi_2^n N_2\right) & 0 & N_2 & -yN_2\sin\left(\varphi_1^n N_1+\varphi_2^n N_2\right) \end{bmatrix} \begin{bmatrix} \mathrm{d}u_1 \\ \mathrm{d}v_1 \\ \mathrm{d}\varphi_1 \\ \mathrm{d}u_2 \\ \mathrm{d}v_2 \\ \mathrm{d}\varphi_2 \end{bmatrix}$$

$$= \underset{(2,6)}{\mathbf{H}\left(\mathbf{X},\mathbf{q}_n\right)}\, \underset{(6)}{\mathrm{d}\mathbf{q}} = \underset{(2,6)}{\mathbf{H}_n}\, \underset{(6)}{\mathrm{d}\mathbf{q}}, \tag{28}$$

where matrix $\mathbf{H}_n$ contains the shape functions and the parameters $\mathbf{q}^n$ known in the configuration n

$$\underset{(2,6)}{\mathbf{H}_n} = \left.\frac{\partial u_i}{\partial q_j}\right|_n = \begin{bmatrix} N_1 & 0 & -yN_1\cos\left(\varphi_1^n N_1+\varphi_2^n N_2\right) & N_2 & 0 & -yN_2\cos\left(\varphi_1^n N_1+\varphi_2^n N_2\right) \\ 0 & N_1 & -yN_1\sin\left(\varphi_1^n N_1+\varphi_2^n N_2\right) & 0 & N_2 & -yN_2\sin\left(\varphi_1^n N_1+\varphi_2^n N_2\right) \end{bmatrix}. \tag{29}$$

In the case of large displacements, the second order increments of the displacements can also be specified for the configuration n, that is

$$\underset{(2)}{\mathrm{d}^2\mathbf{u}} = \frac{1}{2}\left.\frac{\partial^2 u_i}{\partial q_j \partial q_k}\right|_n \mathrm{d}q_j \mathrm{d}q_k = \frac{1}{2}\underset{(6)}{\mathrm{d}\mathbf{q}^T}\, \underset{(6,2,6)}{\mathbf{W}\left(\mathbf{X},\mathbf{q}_n\right)}\, \underset{(6)}{\mathrm{d}\mathbf{q}}, \tag{30}$$

where the three-dimensional matrix $\mathbf{W}(\mathbf{X},\mathbf{q}_n)$ containing also the shape functions and the known parameters $\mathbf{q}^n$ consists of two layers as follows

$$\underset{(6,2,6)}{(\mathbf{W}_n)^1} = \left.\frac{\partial^2 u_1}{\partial q_j \partial q_k}\right|_n = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & yN_1^2\sin\left(\varphi_1^n N_1+\varphi_2^n N_2\right) & 0 & 0 & yN_1N_2\sin\left(\varphi_1^n N_1+\varphi_2^n N_2\right) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & yN_1N_2\sin\left(\varphi_1^n N_1+\varphi_2^n N_2\right) & 0 & 0 & yN_2^2\sin\left(\varphi_1^n N_1+\varphi_2^n N_2\right) \end{bmatrix} \tag{31}$$

and

$$\underset{(6,2,6)}{(\mathbf{W}_n)^2} = \left.\frac{\partial^2 u_2}{\partial q_j \partial q_k}\right|_n = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -yN_1^2\cos\left(\varphi_1^n N_1+\varphi_2^n N_2\right) & 0 & 0 & -yN_1N_2\cos\left(\varphi_1^n N_1+\varphi_2^n N_2\right) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -yN_1N_2\cos\left(\varphi_1^n N_1+\varphi_2^n N_2\right) & 0 & 0 & -yN_2^2\cos\left(\varphi_1^n N_1+\varphi_2^n N_2\right) \end{bmatrix}. \tag{32}$$

Matrices $\mathbf{H}_n(\mathbf{X},\mathbf{q}_n)$ and $\mathbf{W}_n(\mathbf{X},\mathbf{q}_n)$ are used in all forms of variations and increments of the displacements and strains as the basis of the iteration process.

3.2. Large Strains with Large Displacements

In the case of large strains with large displacements, in the first two terms of the tangent stiffness matrix, the expressions $\mathbf{AH}_n$ and $\mathbf{u}_n^T\mathbf{B}^T\mathbf{CH}_n$, moreover, $\mathbf{AW}_n$, $\mathbf{u}_n^T\mathbf{B}^T\mathbf{CW}_n$ and $\mathbf{H}_n^T\mathbf{B}^T\mathbf{CH}_n$ are needed. These terms can be obtained by using the differential operator matrices **A**, **B** and **C**, which, in our two-dimensional geometrical space $\mathbf{X}^T = [\, x \;\; y \,]$, by considering the strains $\mathbf{E}^T = [\, E_{xx} \;\; 2E_{xy} \,]$ and the stresses $\mathbf{S}^T = [\, S_{xx} \;\; S_{xy} \,]$, take the form

$$\mathbf{A} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & 0 \\ 0 & \frac{\partial}{\partial y} \end{bmatrix},$$

$$\mathbf{C}_1 = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial x} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{C}_4 = \begin{bmatrix} \frac{\partial}{\partial y} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial x} \end{bmatrix}. \tag{33}$$

By introducing the abbreviations related to the fixed equilibrium configuration n

$$u_1^n N_1 + u_2^n N_2 = u^n, \quad v_1^n N_1 + v_2^n N_2 = v^n, \quad \varphi_1^n N_1 + \varphi_2^n N_2 = \varphi^n, \tag{34}$$

$$u_1^n \frac{\partial N_1}{\partial x} + u_2^n \frac{\partial N_2}{\partial x} = u_{,x}^n, \qquad v_1^n \frac{\partial N_1}{\partial x} + v_2^n \frac{\partial N_2}{\partial x} = v_{,x}^n,$$
$$\varphi_1^n \frac{\partial N_1}{\partial x} + \varphi_2^n \frac{\partial N_2}{\partial x} = \varphi_{,x}^n, \tag{35}$$

$$\sin\left(\varphi_1^n N_1 + \varphi_2^n N_2\right) = S^n, \qquad \cos\left(\varphi_1^n N_1 + \varphi_2^n N_2\right) = C^n, \tag{36}$$

$$\frac{\partial N_1}{\partial x} = N_{1,x}, \qquad \frac{\partial N_2}{\partial x} = N_{2,x} \tag{37}$$

which are kept constant during the sub-cycles between the configurations n and $n+1$ of the total iteration process. The symmetric tangent stiffness matrix $\mathbf{k}_t^n$ associated with the n-th configuration can be obtained in the following blocks

$$\underset{(6,6)}{\mathbf{k}_t^n} = \begin{bmatrix} \underset{(3,3)}{\mathbf{k}_t^n} & \underset{(3,3)}{\mathbf{k}_t^n} \\ \underset{(3,3)}{\mathbf{k}_t^n} & \underset{(3,3)}{\mathbf{k}_t^n} \end{bmatrix}. \tag{38}$$

First we express the terms $\mathbf{AH}_n$ and $\mathbf{u}_n^T\mathbf{B}^T\mathbf{CH}_n$ containing the material moduli. Thus,

$$\underset{(6,2)}{\mathbf{H}_n^T}\ \underset{(2,2)}{\mathbf{A}^T} = \begin{bmatrix} N_{1,x} & 0 \\ 0 & N_{1,x} \\ N_1 y\varphi_{,x}^n S^n - N_{1,x} yC^n & -N_1\left(1+y\varphi_{,x}^n\right)C^n - N_{1,x} yS^n \\ N_{2,x} & 0 \\ 0 & N_{2,x} \\ N_2 y\varphi_{,x}^n S^n - N_{2,x} yC^n & -N_2\left(1+y\varphi_{,x}^n\right)C^n - N_{2,x} yS^n \end{bmatrix}, \tag{39}$$

while the term $\mathbf{H}_n^T\mathbf{C}^T\mathbf{Bu}_n$ is as follows

$$\underset{(6,2)}{\mathbf{H}_n^T}\ \underset{(2,2,4)}{\mathbf{C}^T}\ \underset{(4,2)}{\mathbf{B}}\ \underset{(2)}{\mathbf{u}_n} = \begin{bmatrix} N_{1,x}\left(u_{,x}^n - y\varphi_{,x}^n C^n\right) & -N_{1,x} S^n \\ N_{1,x}\left(v_{,x}^n - y\varphi_{,x}^n S^n\right) & -N_{1,x}(1-C^n) \\ N_1 y\varphi_{,x}^n\left(u_{,x}^n S^n - v_{,x}^n C^n\right) - & -N_1\left\{\left(u_{,x}^n - y\varphi_{,x}^n\right)C^n + v_{,x}^n S^n\right\} + \\ -N_{1,x} y\left(u_{,x}^n C^n + v_{,x}^n S^n - y\varphi_{,x}^n\right) & +N_{1,x} yS^n \\ N_{2,x}\left(u_{,x}^n - y\varphi_{,x}^n C^n\right) & -N_{2,x} S^n \\ N_{2,x}\left(v_{,x}^n - y\varphi_{,x}^n S^n\right) & -N_{2,x}(1-C^n) \\ N_2 y\varphi_{,x}^n\left(u_{,x}^n S^n - v_{,x}^n C^n\right) - & -N_2\left\{\left(u_{,x}^n - y\varphi_{,x}^n\right)C^n + v_{,x}^n S^n\right\} + \\ b - N_{2,x} y\left(u_{,x}^n C^n + v_{,x}^n S^n - y\varphi_{,x}^n\right) & +N_{2,x} yS^n \end{bmatrix}, \tag{40}$$

in this way, the sum of (39) and (40) is

$$\underset{(6,2)}{\mathbf{H}_n^T}\left(\underset{(2,2)}{\mathbf{A}^T} + \underset{(2,2,4)}{\mathbf{C}^T}\ \underset{(4,2)}{\mathbf{B}}\ \underset{(2)}{\mathbf{u}_n}\right) = \begin{bmatrix} N_{1,x}\left(1+u_{,x}^n - y\varphi_{,x}^n C^n\right) & -N_{1,x} S^n \\ N_{1,x}\left(v_{,x}^n - y\varphi_{,x}^n S^n\right) & -N_{1,x} C^n \\ N_1 y\varphi_{,x}^n\left\{\left(1+u_{,x}^n\right)S^n - v_{,x}^n C^n\right\} - & -N_1\left\{\left(1+u_{,x}^n\right)C^n + v_{,x}^n S^n\right\} \\ -N_{1,x} y\left\{\left(1+u_{,x}^n\right)C^n + v_{,x}^n S^n - y\varphi_{,x}^n\right\} & \\ N_{2,x}\left(1+u_{,x}^n - y\varphi_{,x}^n C^n\right) & -N_{2,x} S^n \\ N_{2,x}\left(v_{,x}^n - y\varphi_{,x}^n S^n\right) & N_{2,x} C^n \\ N_2 y\varphi_{,x}^n\left\{\left(1+u_{,x}^n\right)S^n - v_{,x}^n C^n\right\} - & -N_2\left\{\left(1+u_{,x}^n\right)C^n + v_{,x}^n S^n\right\} \\ -N_{2,x} y\left\{\left(1+u_{,x}^n\right)C^n + v_{,x}^n S^n - y\varphi_{,x}^n\right\} & \end{bmatrix}. \tag{41}$$

Consequently, since we assumed uniaxial material functions represented by the diagonal construction of the matrix of the material constants

$$\mathbf{D}_t^n = \begin{bmatrix} D_{xx}^n & 0 \\ 0 & D_{yy}^n \end{bmatrix} = \begin{bmatrix} E_t^n J(x) & 0 \\ 0 & G_t^n \overline{A}(x) \end{bmatrix} \tag{42}$$

containing the Young and shear moduli, and the moment of inertia and the reduced area of the cross section, respectively, the blocks (38) of the *symmetric material tangent stiffness matrix* $\left(\mathbf{k}_{tang}^{mat}\right)^n$ of (18) related to the n-th equilibrium configuration, as a component of the total system gradient matrix $\mathbf{k}_t^n$ of (18) are as follows

$$\left(\mathbf{k}^{mat}_{tang}\right)^n_{1,1} = \int_{V_0} \left(\mathbf{H}^T_n \left(\mathbf{A}^T + \mathbf{C}^T\mathbf{B}\mathbf{u}_n\right)\mathbf{D}^n_t \left(\mathbf{u}^T_n\mathbf{B}^T\mathbf{C} + \mathbf{A}\right)\mathbf{H}_n\right)_{1,1} \mathrm{d}V_0 =$$

$$\int_{V_0} \begin{bmatrix} \begin{array}{l} D^n_{xx}\left(N_{1,x}\right)^2\left(1+u^n_{,x}-y\varphi^n_{,x}C^n\right)^2 + \\ +D^n_{yy}\left(N_{1,x}\right)^2\left(S^n\right)^2 \end{array} & \begin{array}{l} D^n_{xx}\left\{\left(N_{1,x}\right)^2\left(1+u^n_{,x}-y\varphi^n_{,x}C^n\right)\cdot \right. \\ \left. \cdot\left(v^n_{,x}-y\varphi^n_{,x}S^n\right)\right\} - \\ -D^n_{yy}\left(N_{1,x}\right)^2\left(S^nC^n\right) \end{array} & \begin{array}{l} D_{xx}\left\{N_1N_{1,x}y\varphi^n_{,x}\left(1+u^n_{,x}-y\varphi^n_{,x}C^n\right)\cdot\right. \\ \cdot\left(\left(1+u^n_{,x}\right)S^n-v^n_{,x}C^n\right) - \\ -\left(N_{1,x}\right)^2 y\left(1+u^n_{,x}-y\varphi^n_{,x}C^n\right)\cdot \\ \left.\cdot\left(\left(1+u^n_{,x}\right)C^n+v^n_{,x}S^n-y\varphi^n_{,x}\right)\right\} + \\ +D^n_{yy}\left\{N_1N_{1,x}S^n\left(\left(1+u^n_{,x}\right)C^n+v^n_{,x}S^n\right)\right\} \end{array} \\ \begin{array}{l} D^n_{xx}\left\{\left(N_{1,x}\right)^2\left(1+u^n_{,x}-y\varphi^n_{,x}C^n\right)\cdot\right. \\ \left.\cdot\left(v^n_{,x}-y\varphi^n_{,x}S^n\right)\right\} - \\ -D^n_{yy}\left(N_{1,x}\right)^2\left(S^nC^n\right) \end{array} & \begin{array}{l} D^n_{xx}\left(N_{1,x}\right)^2\left(v^n_{,x}-y\varphi^n_{,x}S^n\right)^2 + \\ +D^n_{yy}\left(N_{1,x}\right)^2\left(C^n\right)^2 \end{array} & \begin{array}{l} D^n_{xx}\left\{N_1N_{1,x}y\varphi^n_{,x}\left(v^n_{,x}-y\varphi^n_{,x}S^n\right)\cdot\right. \\ \cdot\left(\left(1+u^n_{,x}\right)S^n-v^n_{,x}C^n\right) - \\ -\left(N_{1,x}\right)^2 y\left(v^n_{,x}-y\varphi^n_{,x}S^n\right)\cdot \\ \left.\cdot\left(\left(1+u^n_{,x}\right)C^n+v^n_{,x}S^n-y\varphi^n_{,x}\right)\right\} - \\ -D^n_{yy}N_1N_{1,x}C^n\left(\left(1+u^n_{,x}\right)C^n+v^n_{,x}S^n\right) \end{array} \\ \begin{array}{l} D^n_{xx}\left\{N_1N_{1,x}y\varphi^n_{,x}\left(1+u^n_{,x}-y\varphi^n_{,x}C^n\right)\cdot\right. \\ \cdot\left(\left(1+u^n_{,x}\right)S^n-v^n_{,x}C^n\right) - \\ -\left(N_{1,x}\right)^2 y\left(1+u^n_{,x}-y\varphi^n_{,x}C^n\right)\cdot \\ \left.\cdot\left(\left(1+u^n_{,x}\right)C^n+v^n_{,x}S^n-y\varphi^n_{,x}\right)\right\} + \\ +D^n_{yy}\left\{N_1N_{1,x}S^n\left(\left(1+u^n_{,x}\right)C^n+v^n_{,x}S^n\right)\right\} \end{array} & \begin{array}{l} D^n_{xx}\left\{N_1N_{1,x}y\varphi^n_{,x}\left(v^n_{,x}-y\varphi^n_{,x}S^n\right)\cdot\right. \\ \cdot\left(\left(1+u^n_{,x}\right)S^n-v^n_{,x}C^n\right) - \\ -\left(N_{1,x}\right)^2 y\left(v^n_{,x}-y\varphi^n_{,x}S^n\right)\cdot \\ \left.\cdot\left(\left(1+u^n_{,x}\right)C^n+v^n_{,x}S^n-y\varphi^n_{,x}\right)\right\} - \\ -D^n_{yy}\left\{N_1N_{1,x}C^n\left(\left(1+u^n_{,x}\right)C^n+v^n_{,x}S^n\right)\right\} \end{array} & \begin{array}{l} D^n_{xx}\left\{N_1y\varphi^n_{,x}\left(\left(1+u_{,x}\right)S^n-v^n_{,x}C^n\right)\right. \\ \left.-N_{1,x}y\left(\left(1+u^n_{,x}\right)C^n+v^n_{,x}S^n-y\varphi^n_{,x}\right)\right\}^2 + \\ +D^n_{yy}\left\{N_1\left(\left(1+u^n_{,x}\right)C^n+v^n_{,x}S^n\right)\right\}^2 \end{array} \end{bmatrix} \mathrm{d}V_0 \tag{43}$$

and

$$\left(\mathbf{k}_{tang}^{mat}\right)_{1,2}^{n} = \int_{V_0} \left(\mathbf{H}_n^T \left(\mathbf{A}^T + \mathbf{C}^T \mathbf{B} \mathbf{u}_n\right) \mathbf{D}_t^n \left(\mathbf{u}_n^T \mathbf{B}^T \mathbf{C} + \mathbf{A}\right) \mathbf{H}_n\right)_{1,2} \mathrm{d}V_0 =$$

$$\int_{V_0} \begin{bmatrix}
\begin{array}{l} D_{xx}^n N_{1,x} N_{2,x} \left(1 + u_{,x}^n - y\varphi_{,x}^n C^n\right)^2 + \\ + D_{yy}^n N_{1,x} N_{2,x} \left(S^n\right)^2 \end{array} &
\begin{array}{l} D_{xx}^n \left\{N_{1,x} N_{2,x} \left(1 + u_{,x}^n - y\varphi_{,x}^n C^n\right) \cdot \right. \\ \left. \cdot \left(v_{,x}^n - y\varphi_{,x}^n S^n\right)\right\} - \\ - D_{yy}^n N_{1,x} N_{2,x} S^n C^n \end{array} &
\begin{array}{l} D_{xx} \left\{N_1 N_{2,x} y \varphi_{,x}^n \left(1 + u_{,x}^n - y\varphi_{,x}^n C^n\right) \cdot \right. \\ \cdot \left(\left(1 + u_{,x}^n\right) S^n - v_{,x}^n C^n\right) - \\ - N_{1,x} N_{2,x} y \left(1 + u_{,x}^n - y\varphi_{,x}^n C^n\right) \cdot \\ \left. \cdot \left(\left(1 + u_{,x}^n\right) C^n + v_{,x}^n S^n - y\varphi_{,x}^n\right)\right\} + \\ + D_{yy}^n \left\{N_1 N_{2,x} S^n \left(\left(1 + u_{,x}^n\right) C^n + v_{,x}^n S^n\right)\right\} \end{array} \\
\begin{array}{l} D_{xx}^n \left\{N_{1,x} N_{2,x} \left(1 + u_{,x}^n - y\varphi_{,x}^n C^n\right) \cdot \right. \\ \left. \cdot \left(v_{,x}^n - y\varphi_{,x}^n S^n\right)\right\} - \\ - D_{yy}^n N_{1,x} N_{2,x} S^n C^n \end{array} &
\begin{array}{l} D_{xx}^n N_{1,x} N_{2,x} \left(v_{,x}^n - y\varphi_{,x}^n S^n\right)^2 + \\ + D_{yy}^n N_{1,x} N_{2,x} \left(C^n\right)^2 \end{array} &
\begin{array}{l} D_{xx}^n \left\{N_1 N_{2,x} y \varphi_{,x}^n \left(v_{,x}^n - y\varphi_{,x}^n S^n\right) \cdot \right. \\ \cdot \left(\left(1 + u_{,x}^n\right) S^n - v_{,x}^n C^n\right) - \\ - N_{1,x} N_{2,x} y \left(v_{,x}^n - y\varphi_{,x}^n S^n\right) \cdot \\ \left. \cdot \left(\left(1 + u_{,x}^n\right) C^n + v_{,x}^n S^n - y\varphi_{,x}^n\right)\right\} - \\ - D_{yy}^n N_1 N_{2,x} C^n \left(\left(1 + u_{,x}^n\right) C^n + v_{,x}^n S^n\right) \end{array} \\
\begin{array}{l} D_{xx}^n \left\{N_1 N_{2,x} y \varphi_{,x}^n \left(1 + u_{,x}^n - y\varphi_{,x}^n C^n\right) \cdot \right. \\ \cdot \left(\left(1 + u_{,x}^n\right) S^n - v_{,x}^n C^n\right) - \\ - \left(N_{1,x}\right)^2 y \left(1 + u_{,x}^n - y\varphi_{,x}^n C^n\right) \cdot \\ \left. \cdot \left(\left(1 + u_{,x}^n\right) C^n + v_{,x}^n S^n - y\varphi_{,x}^n\right)\right\} + \\ + D_{yy}^n \left\{N_1 N_{1,x} S^n \left(\left(1 + u_{,x}^n\right) C^n + v_{,x}^n S^n\right)\right\} \end{array} &
\begin{array}{l} D_{xx}^n \left\{N_1 N_{1,x} y \varphi_{,x}^n \left(v_{,x}^n - y\varphi_{,x}^n S^n\right) \cdot \right. \\ \cdot \left(\left(1 + u_{,x}^n\right) S^n - v_{,x}^n C^n\right) - \\ - \left(N_{1,x}\right)^2 y \left(v_{,x}^n - y\varphi_{,x}^n S^n\right) \cdot \\ \left. \cdot \left(\left(1 + u_{,x}^n\right) C^n + v_{,x}^n S^n - y\varphi_{,x}^n\right)\right\} - \\ - D_{yy}^n \left\{N_1 N_{1,x} C^n \left(\left(1 + u_{,x}^n\right) C^n + v_{,x}^n S^n\right)\right\} \end{array} &
\begin{array}{l} D_{xx}^n \left\{N_1 N_2 \left(y\varphi_{,x}^n \left(\left(1 + u_{,x}^n\right) S^n - v_{,x}^n C^n\right)\right)^2 \right. \\ - \left(N_{1,x} N_2 + N_{2,x} N_1\right) \cdot \\ \cdot y^2 \varphi_{,x}^n \left(\left(1 + u_{,x}^n\right) S^n - v_{,x}^n C^n\right) \cdot \\ \cdot \left(\left(1 + u_{,x}\right) C^n + v_{,x}^n S^n - y\varphi_{,x}^n\right) + \\ \left. + N_{1,x} N_{2,x} \left(\left(1 + u_{,x}^n\right) C^n + v_{,x} S^n - y\varphi_{,x}^n\right)^2\right\} + \\ + D_{yy}^n \left\{N_1 N_2 \left(\left(1 + u_{,x}^n\right) C^n + v_{,x}^n S^n\right)^2\right\}^2 \end{array}
\end{bmatrix} \mathrm{d}V_0 \tag{44}$$

and

$$\left(\mathbf{k}^{mat}_{tang}\right)^n_{2,2} = \int_{V_0} \left(\mathbf{H}^T_n \left(\mathbf{A}^T + \mathbf{C}^T\mathbf{B}\mathbf{u}_n\right) \mathbf{D}^n_t \left(\mathbf{u}^T_n \mathbf{B}^T \mathbf{C} + \mathbf{A}\right) \mathbf{H}_n\right)_{2,2} \mathrm{d}V_0 =$$

$$\int_{V_0} \begin{bmatrix}
\begin{array}{l} D^n_{xx}\left(N_{2,x}\right)^2\left(1+u^n_{,x}-y\varphi^n_{,x}C^n\right)^2+ \\ +D^n_{yy}\left(N_{2,x}\right)^2\left(S^n\right)^2 \end{array} &
\begin{array}{l} D^n_{xx}\left\{\left(N_{2,x}\right)^2\left(1+u^n_{,x}-y\varphi^n_{,x}C^n\right)\cdot \right. \\ \left. \cdot\left(v^n_{,x}-y\varphi^n_{,x}S^n\right)\right\}- \\ -D^n_{yy}\left(N_{2,x}\right)^2\left(S^nC^n\right) \end{array} &
\begin{array}{l} D_{xx}\left\{N_2N_{2,x}y\varphi^n_{,x}\left(1+u^n_{,x}-y\varphi^n_{,x}C^n\right)\cdot \right. \\ \cdot\left(\left(1+u^n_{,x}\right)S^n-v^n_{,x}C^n\right)- \\ -\left(N_{2,x}\right)^2y\left(1+u^n_{,x}-y\varphi^n_{,x}C^n\right)\cdot \\ \left. \cdot\left(\left(1+u^n_{,x}\right)C^n+v^n_{,x}S^n-y\varphi^n_{,x}\right)\right\}+ \\ +D^n_{yy}\left\{N_2N_{2,x}S^n\left(\left(1+u^n_{,x}\right)C^n+v^n_{,x}S^n\right)\right\} \end{array} \\
\begin{array}{l} D^n_{xx}\left\{\left(N_{2,x}\right)^2\left(1+u^n_{,x}-y\varphi^n_{,x}C^n\right)\cdot \right. \\ \left. \cdot\left(v^n_{,x}-y\varphi^n_{,x}S^n\right)\right\}+ \\ +D^n_{yy}\left(N_{2,x}\right)^2\left(S^nC^n\right) \end{array} &
\begin{array}{l} D^n_{xx}\left(N_{2,x}\right)^2\left(v^n_{,x}-y\varphi^n_{,x}S^n\right)^2+ \\ +D^n_{yy}\left(N_{2,x}\right)^2\left(C^n\right)^2 \end{array} &
\begin{array}{l} D_{xx}\left\{N_2N_{2,x}y\varphi^n_{,x}\left(v^n_{,x}-y\varphi^n_{,x}S^n\right)\cdot \right. \\ \cdot\left(\left(1+u^n_{,x}\right)S^n-v^n_{,x}C^n\right)- \\ -\left(N_{2,x}\right)^2y\left(v^n_{,x}-y\varphi^n_{,x}S^n\right)\cdot \\ \left. \cdot\left(\left(1+u^n_{,x}\right)C^n+v^n_{,x}S^n-y\varphi^n_{,x}\right)\right\}- \\ -D^n_{yy}N_2N_{2,x}C^n\left(\left(1+u^n_{,x}\right)C^n+v^n_{,x}S^n\right) \end{array} \\
\begin{array}{l} D^n_{xx}\left\{N_2N_{2,x}y\varphi^n_{,x}\left(1+u^n_{,x}-y\varphi^n_{,x}C^n\right)\cdot \right. \\ \cdot\left(\left(1+u^n_{,x}\right)S^n-v^n_{,x}C^n\right)- \\ -\left(N_{2,x}\right)^2y\left(1+u^n_{,x}-y\varphi^n_{,x}C^n\right)\cdot \\ \left. \cdot\left(\left(1+u^n_{,x}\right)C^n+v^n_{,x}S^n-y\varphi^n_{,x}\right)\right\}+ \\ +D^n_{yy}\left\{N_2N_{2,x}S^n\left(\left(1+u^n_{,x}\right)C^n+v^n_{,x}S^n\right)\right\} \end{array} &
\begin{array}{l} D^n_{xx}\left\{N_2N_{2,x}y\varphi^n_{,x}\left(v^n_{,x}-y\varphi^n_{,x}S^n\right)\cdot \right. \\ \cdot\left(\left(1+u^n_{,x}\right)S^n-v^n_{,x}C^n\right)- \\ -\left(N_{2,x}\right)^2y\left(v^n_{,x}-y\varphi^n_{,x}S^n\right)\cdot \\ \left. \cdot\left(\left(1+u^n_{,x}\right)C^n+v^n_{,x}S^n-y\varphi^n_{,x}\right)\right\}- \\ -D^n_{yy}\left\{N_2N_{2,x}C^n\left(\left(1+u^n_{,x}\right)C^n+v^n_{,x}S^n\right)\right\} \end{array} &
\begin{array}{l} D^n_{xx}\left\{yN_2\varphi^n_{,x}\left(\left(1+u^n_{,x}\right)S^n-v^n_{,x}C^n\right)\right. \\ \left. yN_{2,x}\left(\left(1+u^n_{,x}\right)C^n+v^n_{,x}S^n-y\varphi^n_{,x}\right)\right\}+ \\ +D^n_{yy}\left\{N_2\left(\left(1+u^n_{,x}\right)C^n+v^n_{,x}S^n\right)^2\right\}^2 \end{array}
\end{bmatrix} \mathrm{d}V_0 \tag{45}$$

in which, through the abbreviations (34) – (37), also the terms S^n, C^n, $\varphi^n_{,x}$, $u^n_{,x}$, $v^n_{,x}$ contain the shape functions N_1, N_2.

Let us consider now the terms $\mathbf{AW}_n$, $\mathbf{u}_n^T\mathbf{B}^T\mathbf{CW}_n$ and $\mathbf{H}_n^T\mathbf{B}^T\mathbf{CH}_n$ concerning the *initial stresses or geometric tangent stiffness* $\left(\mathbf{k}_{geom}^{stress}\right)^n$ of (18) as the part of the total tangent stiffness matrix (18). The two layers of the three-dimensional $\mathbf{AW}_n$ are

$$\left(\underset{(2,2)}{\mathbf{A}}\ \underset{(2,6,6)}{\mathbf{W}_n}\right)^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \begin{matrix} N_1^2 y\varphi^n_{,x}C^n + \\ +2N_1N_{1,x}yS^n \end{matrix} & 0 & 0 & \begin{matrix} N_1N_2 y\varphi^n_{,x}C^n + \\ +\left(N_2N_{1,x}+N_1N_{2,x}\right)yS^n \end{matrix} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \begin{matrix} N_1N_2\varphi^n_{,x}yC^n + \\ +\left(N_2N_{1,x}+N_1N_{2,x}\right)yS^n \end{matrix} & 0 & 0 & \begin{matrix} N_2^2\varphi^n_{,x}yC^n + \\ +2N_2N_{2,x}yS^n \end{matrix} \end{bmatrix} \tag{46}$$

and

$$\left(\underset{(2,2)}{\mathbf{A}}\ \underset{(2,6,6)}{\mathbf{W}_n}\right)^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \begin{matrix} N_1^2\left(1+y\varphi^n_{,x}\right)S^n - \\ -2N_1N_{1,x}yC^n \end{matrix} & 0 & 0 & \begin{matrix} N_1N_2\left(1+y\varphi^n_{,x}\right)S^n - \\ -\left(N_1N_{2,x}+N_2N_{1,x}\right)yC^n \end{matrix} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \begin{matrix} N_1N_2\left(1+y\varphi^n_{,x}\right)S^n - \\ -\left(N_1N_{2,x}+N_2N_{1,x}\right)yC^n \end{matrix} & 0 & 0 & \begin{matrix} N_2^2\left(1+y\varphi^n_{,x}\right)S^n - \\ -2N_2N_{2,x}yC^n \end{matrix} \end{bmatrix}, \tag{47}$$

while the two layers of the three-dimensional $\mathbf{u}_n^T\mathbf{B}^T\mathbf{CW}_n$ are

$$\left(\underset{(2)}{\mathbf{u}_n^T}\ \underset{(2,4)}{\mathbf{B}^T}\ \underset{(4,2,2)}{\mathbf{C}}\ \underset{(2,6,6)}{\mathbf{W}_n}\right)^1 =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \begin{array}{l} N_1^2 y\varphi^n_{,x}\left(u^n_{,x}C^n + v^n_{,x}S^n - y\varphi^n_{,x}\right) + \\ +2N_1N_{1,x}y\left(u^n_{,x}S^n - v^n_{,x}C^n\right) \end{array} & 0 & 0 & \begin{array}{l} N_1N_2y\varphi^n_{,x}\left(u^n_{,x}C^n + v^n_{,x}S^n - y\varphi^n_{,x}\right) + \\ +\left(N_1N_{2,x} + N_2N_{1,x}\right)y\left(u^n_{,x}S^n - v^n_{,x}C^n\right) \end{array} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \begin{array}{l} N_1N_2y\varphi^n_{,x}\left(u^n_{,x}C^n + v^n_{,x}S^n - y\varphi^n_{,x}\right) + \\ +\left(N_1N_{2,x} + N_2N_{1,x}\right)y\left(u^n_{,x}S^n - v^n_{,x}C^n\right) \end{array} & 0 & 0 & \begin{array}{l} N_2^2 y\varphi^n_{,x}\left(u^n_{,x}C^n + v^n_{,x}S^n - y\varphi^n_{,x}\right) + \\ +2N_2N_{2,x}y\left(u^n_{,x}S^n - v^n_{,x}C^n\right) \end{array} \end{bmatrix} \tag{48}$$

$$\left(\underset{(2)}{\mathbf{u}_n^T}\ \underset{(2,3)}{\mathbf{B}^T}\ \underset{(3,2,2)}{\mathbf{C}}\ \underset{(2,6,6)}{\mathbf{W}_n}\right)^2 =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \begin{array}{l} N_1^2\left(u^n_{,x}S^n - v^n_{,x}C^n - y\varphi^n_{,x}S^n\right) - \\ -2N_1N_{1,x}y\left(1 - C^n\right) \end{array} & 0 & 0 & \begin{array}{l} N_1N_2\left(u^n_{,x}S^n - v^n_{,x}C^n - y\varphi^n_{,x}S^n\right) - \\ -\left(N_2N_{1,x} + N_1N_{2,x}\right)y\left(1 - C^n\right) \end{array} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \begin{array}{l} N_1N_2\left(u^n_{,x}S^n - v^n_{,x}C^n - y\varphi^n_{,x}S^n\right) - \\ -\left(N_2N_{1,x} + N_1N_{2,x}\right)y\left(1 - C^n\right) \end{array} & 0 & 0 & \begin{array}{l} N_2^2\left(u^n_{,x}S^n - v^n_{,x}C^n - y\varphi^n_{,x}S^n\right) - \\ -2N_2N_{2,x}y\left(1 - C^n\right) \end{array} \end{bmatrix} \tag{49}$$

moreover, the two layers of the three-dimensional $\mathbf{H}_n^T\mathbf{B}^T\mathbf{C}\mathbf{H}_n$ are as follows

$$= \left(\underset{(6,2)}{\mathbf{H}_n^T} \ \underset{(2,4)}{\mathbf{B}^T} \ \underset{(4,3,2)}{\mathbf{C}} \ \underset{(2,6)}{\mathbf{H}_n} \right)^1 =$$

$$\begin{bmatrix}
(N_{1,x})^2 & 0 & \begin{array}{c} -\left(N_{1,x}\right)^2 yC^n + \\ +N_1 N_{1,x} y\varphi_{,x}^n S^n \end{array} & N_{1,x} N_{2,x} & 0 & \begin{array}{c} -N_{1,x} N_{2,x} yC^n + \\ +N_2 N_{1,x} y\varphi_{,x}^n S^n \end{array} \\
0 & (N_{1,x})^2 & \begin{array}{c} -\left(N_{1,x}\right)^2 yS^n + \\ +N_1 N_{1,x} y\varphi_{,x}^n C^n \end{array} & 0 & N_{1,x} N_{2,x} & \begin{array}{c} -N_{1,x} N_{2,x} yS^n + \\ +N_2 N_{1,x} y\varphi_{,x}^n C^n \end{array} \\
\begin{array}{c} -\left(N_{1,x}\right)^2 yC^n + \\ +N_1 N_{1,x} y\varphi_{,x}^n S^n \end{array} & \begin{array}{c} -\left(N_{1,x}\right)^2 yS^n + \\ +N_1 N_{1,x} y\varphi_{,x}^n C^n \end{array} & \begin{array}{c} -\left(N_{1,x}\right)^2 y^2 + \\ +N_1^2 \left(\varphi_{,x}^n\right)^2 y^2 \end{array} & \begin{array}{c} -N_{1,x} N_{2,x} yC^n + \\ +N_1 N_{2,x} y\varphi_{,x}^n S^n \end{array} & \begin{array}{c} -N_{1,x} N_{2,x} yS^n + \\ +N_1 N_{2,x} y\varphi_{,x}^n C^n \end{array} & \begin{array}{c} N_{1,x} N_{2,x} y^2 + \\ +N_1 N_2 \left(\varphi_{,x}^n\right)^2 y^2 \end{array} \\
x N_{1,x} N_{2,x} & 0 & \begin{array}{c} -N_{1,x} N_{2,x} yC^n + \\ +N_1 N_{2,x} y\varphi_{,x}^n S^n \end{array} & (N_{2,x})^2 & 0 & \begin{array}{c} -\left(N_{2,x}\right)^2 yC^n + \\ +N_2 N_{2,x} y\varphi_{,x}^n S^n \end{array} \\
0 & N_{1,x} N_{2,x} & \begin{array}{c} -N_{1,x} N_{2,x} yS^n + \\ +N_1 N_{2,x} y\varphi_{,x}^n C^n \end{array} & 0 & (N_{2,x})^2 & \begin{array}{c} -\left(N_{2,x}\right)^2 yS^n + \\ +N_2 N_{2,x} y\varphi_{,x}^n C^n \end{array} \\
\begin{array}{c} -N_{1,x} N_{2,x} yC^n + \\ +N_2 N_{1,x} y\varphi_{,x}^n S^n \end{array} & \begin{array}{c} -N_{1,x} N_{2,x} yS^n + \\ +N_2 N_{1,x} y\varphi_{,x}^n C^n \end{array} & \begin{array}{c} N_{1,x} N_{2,x} y^2 + \\ +N_1 N_2 \left(\varphi_{,x}^n\right)^2 y^2 \end{array} & \begin{array}{c} -\left(N_{2,x}\right)^2 yC^n + \\ +N_2 N_{2,x} y\varphi_{,x}^n S^n \end{array} & \begin{array}{c} -\left(N_{2,x}\right)^2 yS^n + \\ +N_2 N_{2,x} y\varphi_{,x}^n C^n \end{array} & \begin{array}{c} \left(N_{2,x}\right)^2 y^2 + \\ +N_2^2 \left(\varphi_{,x}\right)^2 y^2 \end{array}
\end{bmatrix} \tag{50}$$

and $\left(\underset{(6,2)}{\mathbf{H}_n^T}\;\underset{(2,4)}{\mathbf{B}^T}\;\underset{(4,3,2)}{\mathbf{C}}\;\underset{(2,6)}{\mathbf{H}_n}\right)^2 =$

$$\begin{bmatrix} 0 & 0 & -N_1 N_{1,x} C^n & 0 & 0 & -N_2 N_{1,x} C^n \\ 0 & 0 & -N_1 N_{1,x} S^n & 0 & 0 & -N_2 N_{1,x} S^n \\ -N_1 N_{1,x} C^n & -N_1 N_{1,x} S^n & 2N_1 N_{1,x} y & -N_1 N_{2,x} C^n & -N_1 N_{2,x} S^n & \left(N_1 N_{2,x} + N_2 N_{1,x}\right) y \\ 0 & 0 & -N_1 N_{2,x} C^n & 0 & 0 & -N_2 N_{2,x} C^n \\ 0 & 0 & -N_1 N_{2,x} S^n & 0 & 0 & -N_2 N_{2,x} S^n \\ -N_2 N_{1,x} C^n & -N_2 N_{1,x} S^n & \left(N_1 N_{2,x} + N_2 N_{1,x}\right) y & -N_2 N_{2,x} C^n & -N_2 N_{2,x} S^n & 2N_2 N_{2,x} y \end{bmatrix} \tag{51}$$

By considering the stresses $\mathbf{S}^T = [\, S_{xx} \;\; S_{xy} \,]$, the blocks of the *symmetric initial stress stiffness matrix or geometric stiffness matrix* related to the n-th equilibrium configuration, as a component of the total system gradient matrix take the form

$$\left(\mathbf{k}_{geom}^{stress}\right)_{1,1}^n = \int_{V_0} \left(\mathbf{S}^T \left(\mathbf{A}\mathbf{W}_n + \mathbf{u}_n^T \mathbf{B}^T \mathbf{C}\mathbf{W}_n + \mathbf{H}_n^T \mathbf{B}^T \mathbf{C}\mathbf{H}_n\right)\right)_{1,1} \mathrm{d}V_0 =$$

$$\int_V \begin{bmatrix} S_{xx}^n \left(N_{1,x}\right)^2 & 0 & \begin{array}{l} -S_{xx}^n \left(\left(N_{1,x}\right)^2 y C^n - \right. \\ \left. -\; N_1 N_{1,x} y \varphi_{,x}^n S^n\right) - \\ -S_{xy}^n N_1 N_{1,x} C^n \end{array} \\ 0 & S_{xx}^n \left(N_{1,x}\right)^2 & \begin{array}{l} -S_{xx}^n \left(\left(N_{1,x}\right)^2 y S^n + \right. \\ \left. +\; N_1 N_{1,x} y \varphi_{,x}^n C^n\right) - \\ -S_{xy}^n N_1 N_{1,x} S^n \end{array} \\ \begin{array}{l} -S_{xx}^n \left\{\left(N_{1,x}\right)^2 y C^n - \right. \\ \left. -\; N_1 N_{1,x} y \varphi_{,x}^n S^n\right\} - \\ -S_{xy}^n N_1 N_{1,x} C^n \end{array} & \begin{array}{l} -S_{xx}^n \left\{\left(N_{1,x}\right)^2 y S^n + \right. \\ \left. +\; N_1 N_{1,x} y \varphi_{,x}^n C^n\right\} - \\ -S_{xy}^n N_1 N_{1,x} S^n \end{array} & \begin{array}{l} S_{xx}^n \left(N_{1,x}\right)^2 y^2 + \\ +S_{xx}^n N_1^2 y \varphi_{,x}^n \left\{\left(1 + u_{,x}^n\right) C^n + v_{,x}^n S^n\right\} + \\ +S_{xx}^n 2N_1 N_{1,x} y \left\{\left(1 + u_{,x}^n\right) S^n - v_{,x}^n C^n\right\} + \\ +S_{xy}^n N_1^2 \left\{\left(1 + u_{,x}^n\right) S^n - v_{,x}^n C^n\right\} \end{array} \end{bmatrix} \tag{52}$$

and $\left(\mathbf{k}_{geom}^{stress}\right)_{1,2}^n = \int_{V_0} \left(\mathbf{S}^T \left(\mathbf{A}\mathbf{W}_n + \mathbf{u}_n^T \mathbf{B}^T \mathbf{C}\mathbf{W}_n + \mathbf{H}_n^T \mathbf{B}^T \mathbf{C}\mathbf{H}_n\right)\right)_{1,2} \mathrm{d}V_0 =$

$$\int_{V_0} \begin{bmatrix} S_{xx}^n N_{1,x} N_{2,x} & 0 & \begin{array}{l} -S_{xx}^n \left(N_{1,x} N_{2,x} y C^n - \right. \\ \left. -\; N_2 N_{1,x} y \varphi_{,x}^n S^n\right) - \\ -S_{xy}^n N_2 N_{1,x} C^n \end{array} \\ 0 & S_{xx}^n N_{1,x} N_{2,x} & \begin{array}{l} -S_{xx}^n \left(N_{1,x} N_{2,x} y S^n + \right. \\ \left. +\; N_2 N_{1,x} y \varphi_{,x}^n C^n\right) - \\ -S_{xy}^n N_2 N_{1,x} S^n \end{array} \\ \begin{array}{l} -S_{xx}^n \left(N_{1,x} N_{2,x} y C^n + \right. \\ \left. +\; N_1 N_{2,x} y \varphi_{,x}^n S^n\right) - \\ -S_{xy}^n N_1 N_{2,x} C^n \end{array} & \begin{array}{l} -S_{xx}^n \left(N_{1,x} N_{2,x} y S^n + \right. \\ \left. +\; N_1 N_{2,x} y \varphi_{,x}^n C^n\right) - \\ -S_{xy}^n N_1 N_{2,x} S^n \end{array} & \begin{array}{l} S_{xx}^n N_{1,x} N_{2,x} y^2 \\ S_{xx}^n N_1 N_2 y \varphi_{,x}^n \left(\left(1 + u_{,x}^n\right) C^n + v_{,x}^n S^n\right) \\ S_{xx}^n \left(N_1 N_{2,x} + N_2 N_{1,x}\right) y \cdot \\ \cdot \left(\left(1 + u_{,x}^n\right) S^n - v_{,x}^n C^n\right) \\ S_{xy}^n N_1 N_2 \left(\left(1 + u_{,x}^n\right) S^n - v_{,x}^n C^n\right) \end{array} \end{bmatrix}, \tag{53}$$

$$\left(\mathbf{k}_{geom}^{stress}\right)_{2,2}^{n} = \int_{V_0} \left(\mathbf{S}^T\left(\mathbf{AW}_n + \mathbf{u}_n^T\mathbf{B}^T\mathbf{CW}_n + \mathbf{H}_n^T\mathbf{B}^T\mathbf{CH}_n\right)\right)_{2,2} \mathrm{d}V_0 =$$

$$\int_{V_0} \begin{bmatrix} S_{xx}^n \left(N_{2,x}\right)^2 & 0 & \begin{array}{l} -S_{xx}^n \left\{\left(N_{2,x}\right)^2 yC^n - \right. \\ \left. - \; N_2 N_{3,x} y\varphi_{,x}^n S^n\right\} - \\ -S_{xy}^n N_2 N_{2,x} C^n \end{array} \\ 0 & S_{xx}^n \left(N_{2,x}\right)^2 & \begin{array}{l} -S_{xx}^n \left\{\left(N_{2,x}\right)^2 yS^n + \right. \\ \left. + \; N_2 N_{2,x} y\varphi_{,x}^n C^n\right) - \\ -S_{xy}^n N_2 N_{2,x} S^n \end{array} \\ \begin{array}{l} -S_{xx}^n \left\}\left(N_{2,x}\right)^2 yC^n - \right. \\ \left. - \; N_2 N_{2,x} y\varphi_{,x}^n S^n\right\} - \\ -S_{xy}^n N_2 N_{2,x} C^n \end{array} & \begin{array}{l} -S_{xx}^n \left\{\left(N_{2,x}\right)^2 yS^n + \right. \\ \left. + \; N_2 N_{2,x} y\varphi_{,x}^n C^n\right\} - \\ -S_{xy}^n N_2 N_{2,x} S^n \end{array} & \begin{array}{l} S_{xx}^n y^2 \left(N_{2,x}\right)^2 + \\ +S_{xx}^n N_2^2 y\varphi_{,x}^n \left\{\left(1+u_{,x}^n\right) C^n + v_{,x}^n S^n\right\} + \\ +S_{xx}^n 2N_2 N_{2,x} y \left\{\left(1+u_{,x}^n\right) S^n - v_{,x}^n C^n\right\} + \\ +S_{xy}^n N_2^2 \left\{\left(1+u_{,x}^n\right) S^n - v_{,x}^n C^n\right\} \end{array} \end{bmatrix} \tag{54}$$

in which, through the abbreviations (34) – (39), the terms S^n, C^n, $\varphi_{,x}^n$, $u_{,x}^n$, $v_{,x}^n$ also contain the shape functions N_1, N_2.

Let us consider now the *loading tangent stiffness* with $\mathbf{H}_n^T \mathbf{M}_t \mathbf{H}_n$ of the configuration-dependent loading containing the loading tangent modulus $\mathbf{M}_t$. By assuming uni-axial loading program represented by the diagonal matrix

$$\mathbf{M}_t^n = \begin{bmatrix} M_{xx}^n & 0 \\ 0 & M_{yy}^n \end{bmatrix} \tag{55}$$

related to the configuration n, the concerning part in the system gradient matrix reads

$$\left(\mathbf{k}_{tang}^{load}\right)^n = \int_{V_0} \underset{(6,2)}{\mathbf{H}_n^T} \; \underset{(2,2)}{\mathbf{M}_t^n} \; \underset{(2,6)}{\mathbf{H}_n} \, \mathrm{d}V_0 =$$

$$\int_{V_0} \begin{bmatrix} M_{xx}^n N_1^2 & 0 & -M_{xx}^n N_1^2 yC^n & M_{xx}^n N_1 N_2 & 0 & -M_{xx}^n N_1 N_2 yC^n \\ 0 & M_{yy}^n N_1^2 & -M_{yy}^n N_1^2 yS^n & 0 & M_{yy}^n N_1 N_2 & -M_{yy}^n N_1 N_2 yS^n \\ -M_{xx}^n N_1^2 yC^n & -M_{yy}^n N_1^2 yS^n & \begin{array}{c} M_{xx}^n N_1^2 \left(yC^n\right)^2 + \\ +M_{yy}^n N_1^2 \left(yS^n\right)^2 \end{array} & -M_{xx}^n N_1 N_2 yC^n & -M_{yy}^n N_1 N_2 yS^n & \begin{array}{c} M_{xx}^n N_1 N_2 \left(yC^n\right)^2 + \\ +M_{yy}^n N_1 N_2 \left(yS^n\right)^2 \end{array} \\ M_{xx}^n N_1 N_2 & 0 & -M_{xx}^n N_1 N_2 yC^n & M_{xx}^n N_2^2 & 0 & -M_{xx}^n N_2^2 yC^n \\ 0 & M_{yy}^n N_1 N_2 & -M_{yy}^n N_1 N_2 yS^n & 0 & M_{yy}^n N_2^2 & -M_{yy}^n N_2^2 yS^n \\ -M_{xx}^n N_1 N_2 yC^n & -M_{yy}^n N_1 N_2 yS^n & \begin{array}{c} M_{xx}^n N_1 N_2 \left(yC^n\right)^2 + \\ +M_{yy}^n N_1 N_2 \left(yS^n\right)^2 \end{array} & -M_{xx}^n N_2^2 yC^n & -M_{yy}^n N_2^2 yS^n & \begin{array}{c} M_{xx}^n N_2^2 \left(yC^n\right)^2 + \\ +M_{yy}^n N_2^2 \left(yS^n\right)^2 \end{array} \end{bmatrix} \tag{56}$$

moreover, by considering the initial volume load $\mathbf{f}_n^T = [\, f_1^n \;\; f_2^n \,]$ and controllable load $\lambda^n \mathbf{F}_0^T = \lambda^n [\, F_1^0 \;\; F_2^0 \,]$ in the configuration n, the symmetric terms can be obtained, by using the two layers of (31) and (32) as

$$\left(\mathbf{k}_{geom}^{load}\right)^n = \int_{V_0} \mathbf{f}_n^T \mathbf{W}_n \,\mathrm{d}V_0 =$$

$$\int_{V_0} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f_1^n N_1^2 y S^n - f_2^n N_1^2 y C^n & 0 & 0 & f_1^n N_1 N_2 y S^n - f_2^n N_1 N_2 y C^n \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f_1^n N_1 N_2 y S^n - f_2^n N_1 N_2 y C^n & 0 & 0 & f_1^n N_2^2 y S^n - f_2^n N_2^2 y C^n \end{bmatrix} \mathrm{d}V_0 \tag{57}$$

and $\left(\mathbf{k}_{dead}^{load}\right)^n = \int_{V_0} \lambda^n \mathbf{F}_0^T \mathbf{W}_n \,\mathrm{d}V_0 =$

$$\int_{V_0} \lambda^n \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & F_1^0 N_1^2 y S^n - F_2^0 N_1^2 y C^n & 0 & 0 & F_1^0 N_1 N_2 y S^n - F_2^0 N_1 N_2 y C^n \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & F_1^0 N_1 N_2 y S^n - F_2^0 N_1 N_2 y C^n & 0 & 0 & F_1^0 N_2^2 y S^n - F_2^0 N_2^2 y C^n \end{bmatrix} \mathrm{d}V_0 \tag{58}$$

In this way, from the above detailed matrices, *the total system gradient matrix*

$$\mathbf{k}_t^n = \left(\mathbf{k}_{tang}^{mat}\right)^n + \left(\mathbf{k}_{geom}^{stress}\right)^n - \left(\mathbf{k}_{tang}^{load}\right)^n - \left(\mathbf{k}_{geom}^{load}\right)^n - \left(\mathbf{k}_{dead}^{load}\right)^n$$

containing equally the material, geometry, variable and dead loading terms, can be obtained. This total matrix is not detailed here.

3.3. Small Strains with Large Displacements

In the case of *small strains with large displacements,* the above forms are simplified. Namely, by using (39) only, the blocks of the *material tangent stiffness* (42) – (45) are reduced to

$$\left(\mathbf{k}_{tang}^{mat}\right)_{1,1}^{n} = \int_{V_0} \left(\mathbf{H}_n^T \mathbf{A}^T \mathbf{D}_t^n \mathbf{A} \mathbf{H}_n\right)_{1,1} \mathrm{d}V_0 =$$

$$\int_{V_0} \begin{bmatrix} D_{xx}^n \left(N_{1,x}\right)^2 & 0 & D_{xx}^n N_{1,x} \left(N_1 y \varphi_{,x}^n S^n - N_{1,x} y C^n\right) \\ 0 & D_{yy}^n \left(N_{1,x}\right)^2 & -D_{yy}^n N_{1,x} \left\{N_1 \left(1 + y\varphi_{,x}^n\right) C^n + N_{1,x} y S^n\right\} \\ \begin{array}{l} D_{xx}^n N_{1,x} \left(N_1 y \varphi_{,x}^n S^n - \right. \\ \left. - N_{1,x} y C^n\right) \end{array} & \begin{array}{l} -D_{yy}^n N_{1,x} \left\{N_1 \left(1 + y\varphi_{,x}^n\right) C^n + \right. \\ \left. + N_{1,x} y S^n\right\} \end{array} & \begin{array}{l} D_{xx}^n \left\{N_1 y \varphi_{,x}^n S^n - N_{1,x} y C^n\right\}^2 \\ D_{yy}^n \left\{-N_1 \left(1 + y\varphi_{,x}^n\right) C^n - N_{1,x} y S^n\right\}^2 \end{array} \end{bmatrix} \mathrm{d}V_0 \tag{59}$$

$$\left(\mathbf{k}_{tang}^{mat}\right)_{1,2}^{n} = \int_{V_0} \left(\mathbf{H}_n^T \mathbf{A}^T \mathbf{D}_t^n \mathbf{A} \mathbf{H}_n\right)_{1,2} \mathrm{d}V_0 =$$

$$\int_{V_0} \begin{bmatrix} D_{xx}^n N_{1,x} N_{2,x} & 0 & \begin{array}{l} D_{xx}^n N_{1,x} \left(N_2 y \varphi_{,x}^n S^n - \right. \\ \left. - N_{2,x} y C^n\right) \end{array} \\ 0 & D_{yy}^n N_{1,x} N_{2,x} & \begin{array}{l} -D_{yy}^n N_{1,x} \left\{N_2 \left(1 + y\varphi_{,x}^n\right) C^n + \right. \\ \left. + N_{2,x} y S^n\right\} \end{array} \\ \begin{array}{l} D_{xx}^n N_{1,x} \left\{N_2 y \varphi_{,x}^n S^n - \right. \\ \left. - N_{2,x} y C^n\right\} \end{array} & \begin{array}{l} -D_{yy}^n N_{1,x} \left\{N_2 \left(1 + y\varphi_{,x}^n\right) C^n - \right. \\ \left. - N_{2,x} y S^n\right\} \end{array} & \begin{array}{l} D_{xx}^n \left(N_1 y \varphi_{,x}^n S^n - N_{1,x} y C^n\right) \cdot \\ \cdot \left(N_2 y \varphi_{,x}^n S^n - N_{2,x} y C^n\right) + \\ + D_{yy}^n \left\{\left(-N_1 \left(1 + y\varphi_{,x}^n\right) C^n - N_{1,x} y S^n\right) \cdot \right. \\ \left. \cdot \left(-N_2 \left(1 + y\varphi_{,x}^n\right) C^n - y N_{2,x} S^n\right)\right\} \end{array} \end{bmatrix} \mathrm{d}V_0 \tag{60}$$

and

$$\left(\mathbf{k}_{tang}^{mat}\right)_{2,2}^{n} = \int_{V_0} \left(\mathbf{H}_n^T \mathbf{A}^T \mathbf{D}_t^n \mathbf{A} \mathbf{H}_n\right)_{2,2} \mathrm{d}V_0 =$$

$$\int_{V_0} \begin{bmatrix} D_{xx}^n \left(N_{2,x}\right)^2 & 0 & \begin{array}{l} D_{xx}^n N_{2,x} \left\{N_2 y \varphi_{,x}^n S^n - \right. \\ \left. - N_{2,x} y C^n\right\} \end{array} \\ 0 & D_{yy}^n \left(N_{2,x}\right)^2 & \begin{array}{l} -D_{yy}^n N_{2,x} \left\{N_2 \left(1 + y\varphi_{,x}^n\right) C^n + \right. \\ \left. + N_{2,x} y S^n\right\} \end{array} \\ \begin{array}{l} D_{xx}^n N_{2,x} \left\{N_2 y \varphi_{,x}^n S^n - \right. \\ \left. - N_{2,x} y C^n\right\} \end{array} & \begin{array}{l} -D_{yy}^n N_{2,x} \left\{N_2 \left(1 + y\varphi_{,x}^n\right) C^n + \right. \\ \left. + N_{2,x} y S^n\right\} \end{array} & \begin{array}{l} D_{xx}^n \left\{N_2 y \varphi_{,x}^n S^n - N_{2,x} y C^n\right\}^2 + \\ + D_{yy}^n \left\{-N_2 \left(1 + y\varphi_{,x}^n\right) C^n - \right. \\ \left. - N_2 y S^n\right\}^2 \end{array} \end{bmatrix} \mathrm{d}V_0 \tag{61}$$

moreover, the *initial stress or geometric stiffness* matrices in (52) – (54) are simplified, too, namely

$$\left(\mathbf{k}_{geom}^{stress}\right)^{n} = \int_{V_0} \mathbf{S}^T \mathbf{A} \mathbf{W}_n \mathrm{d}V_0 =$$

$$\int_{V_0} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \begin{matrix} S^n_{xx}\left\{N_1^2 y\varphi^n_{,x} C^n + \right. \\ \left. +2N_1 N_{1,x} y S^n\right\} + \\ +S^n_{xy}\left\{N_1^2\left(1+y\varphi^n_{,x}\right)S^n - \right. \\ \left. -2N_1 N_{1,x} y C^n\right\} \end{matrix} & 0 & 0 & \begin{matrix} S^n_{xx}\left\{N_1 N_2\left(1+y\varphi^n_{,x}\right)C^n + \right. \\ \left. +\left(N_1 N_{2,x}+N_2 N_{1,x}\right)yS^n\right\} + \\ +S^n_{xy}\left\{N_1 N_2 S^n - \right. \\ \left. -\left(N_1 N_{2,x}+N_2 N_{1,x}\right)yC^n\right\} \end{matrix} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \begin{matrix} S^n_{xx}\left\{N_1 N_2\left(1+y\varphi^n_{,x}\right)C^n + \right. \\ \left. +\left(N_1 N_{2,x}+N_2 N_{1,x}\right)yS^n\right\} + \\ +S^n_{xy}\left\{N_1 N_2 S^n - \right. \\ \left. -\left(N_1 N_{2,x}+N_2 N_{1,x}\right)yC^n\right\} \end{matrix} & 0 & 0 & \begin{matrix} S^n_{xx}\left\{N_2^2 y\varphi^n_{,x} C^n + \right. \\ \left. +2N_2 N_{2,x} y S^n\right\} + \\ +S^n_{xy}\left\{N_2^2\left(1+y\varphi^n_{,x}\right)S^n - \right. \\ \left. -2N_2 N_{2,x} y C^n\right\} \end{matrix} \end{bmatrix} \mathrm{d}V_0 \tag{62}$$

Finally, the *loading stiffness matrices* are the same as detailed in (56) – (58).

3.4. Large Strains with Small Displacements

In the case of small displacements, the approximations $\sin\varphi \cong \varphi$ and $\cos\varphi \cong 1$ are applied, thus $\mathbf{W} = \mathbf{0}$, and we have the configuration-independent matrix $\mathbf{H}_n = \mathbf{N}$ of (27) in the expressions (43) – (58). In this case $\mathbf{AH}_n = \mathbf{AN}$, that is, (39) changes to the configuration-independent form

$$\underset{(6,2)}{\mathbf{N}^T}\ \underset{(2,2)}{\mathbf{A}^T} = \begin{bmatrix} N_{1,x} & 0 \\ 0 & N_{1,x} \\ -yN_{1,x} & -N_1 \\ N_{2,x} & 0 \\ 0 & N_{2,x} \\ -yN_{2,x} & -N_2 \end{bmatrix}. \tag{63}$$

Let us observe that this matrix concerns the first order strains in the point P in *Fig. 2*. The first column of this matrix represents the first order normal strain in the direction x, since in *Fig. 2* for small displacements $u(x, y) = u(x) - y\varphi(x)$, thus

$$E_x = \frac{\partial u(x, y)}{\partial x} = \frac{\partial u(x)}{\partial x} - y\frac{\partial \varphi(x)}{\partial x}, \tag{64}$$

while the second column represents the shear strain defined by Timoshenko as

$$2E_{xy} = \frac{\partial v(x, y)}{\partial x} + \frac{\partial u(x, y)}{\partial y} = \frac{\partial v(x)}{\partial x} - \varphi(x). \tag{65}$$

Moreover, the higher order strains represented by the two layers of the three-dimensional matrix $\mathbf{H}_n^T\mathbf{B}^T\mathbf{C}\mathbf{H}_n = \mathbf{N}^T\mathbf{B}^T\mathbf{C}\mathbf{N}$ are also configuration-independent as follows

$$\left(\underset{(6,2)}{\mathbf{N}^T}\;\underset{(2,4)}{\mathbf{B}^T}\;\underset{(4,2,2)}{\mathbf{C}}\;\underset{(2,6)}{\mathbf{N}}\right)^1 = \begin{bmatrix} (N_{1,x})^2 & 0 & -(N_{1,x})^2 y & N_{1,x}N_{2,x} & 0 & -N_{1,x}N_{2,x}y \\ 0 & (N_{1,x})^2 & 0 & 0 & N_{1,x}N_{2,x} & 0 \\ -(N_{1,x})^2 y & 0 & (N_{1,x})^2 y^2 & -N_{1,x}N_{2,x}y & 0 & N_{1,x}N_{2,x}y^2 \\ N_{1,x}N_{2,x} & 0 & -N_{1,x}N_{2,x}y & (N_{2,x})^2 & 0 & -(N_{2,x})^2 y \\ 0 & N_{1,x}N_{2,x} & 0 & 0 & (N_{2,x})^2 & 0 \\ -N_{1,x}N_{2,x}y & 0 & N_{1,x}N_{2,x}y^2 & -(N_{2,x})^2 y & 0 & (N_{2,x})^2 y^2 \end{bmatrix} \tag{66}$$

concerns the normal strain in the direction x in the point P, while the second layer

$$\left(\underset{(6,2)}{\mathbf{N}^T}\;\underset{(2,4)}{\mathbf{B}^T}\;\underset{(4,2,2)}{\mathbf{C}}\;\underset{(2,6)}{\mathbf{N}}\right)^2 = \begin{bmatrix} 0 & 0 & -N_1N_{1,x} & 0 & 0 & -N_2N_{1,x} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -N_1N_{1,x} & 0 & 2N_1N_{1,x}y & -N_1N_{2,x} & 0 & (N_1N_{2,x}+N_2N_{1,x})y \\ 0 & 0 & -N_1N_{2,x} & 0 & 0 & -N_2N_{2,x} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -N_2N_{1,x} & 0 & (N_1N_{2,x}+N_2N_{1,x})y & -N_2N_{2,x} & 0 & 2N_2N_{2,x}y \end{bmatrix} \tag{67}$$

concerns the shear strain in the direction xy in the point P.

For the *material tangent stiffness* each layer (66) and (67) is multiplied by the actual value of $\mathbf{q}_n$ and $\mathbf{D}_t^n$, thus, the matrix

$$\left(\mathbf{N}^T\mathbf{A}^T + \mathbf{N}^T\mathbf{C}^T\mathbf{B}\mathbf{N}\mathbf{q}_n\right)\mathbf{D}_t^n\left(\mathbf{q}_n^T\mathbf{N}^T\mathbf{B}^T\mathbf{C}\mathbf{N} + \mathbf{A}\mathbf{N}\right)$$

in the tangent stiffness becomes configuration-dependent, that is, associated with the configuration n. The term $\mathbf{N}^T\mathbf{C}^T\mathbf{B}\mathbf{N}\mathbf{q}_n$ is as follows

$$\underset{(6,2)}{\mathbf{N}^T}\;\underset{(2,2,4)}{\mathbf{C}^T}\;\underset{(4,2)}{\mathbf{B}}\;\underset{(2,6)}{\mathbf{N}}\;\underset{(6)}{\mathbf{q}_n} = \begin{bmatrix} N_{1,x}(u_{,x}^n - y\varphi_{,x}^n) & -N_{1,x}\varphi^n \\ N_{1,x}v_{,x}^n & 0 \\ -N_{1,x}y(u_{,x}^n - y\varphi_{,x}^n) & -N_1(u_{,x}^n - y\varphi_{,x}^n) + N_{1,x}\varphi^n \\ N_{2,x}(u_{,x}^n - y\varphi_{,x}^n) & -N_{2,x}\varphi^n \\ N_{2,x}v_{,x}^n & 0 \\ -N_{2,x}y(u_{,x}^n - y\varphi_{,x}^n) & -N_2(u_{,x}^n - y\varphi_{,x}^n) + N_{2,x}\varphi^n \end{bmatrix} \tag{68}$$

and the sum of (39) and (40) changes to the sum of (63) and (68)

$$\underset{(6,2)}{\mathbf{N}^T}\left(\underset{(2,2)}{\mathbf{A}^T} + \underset{(2,2,4)}{\mathbf{C}^T}\;\underset{(4,2)}{\mathbf{B}}\;\underset{(2,6)}{\mathbf{N}}\;\underset{(6)}{\mathbf{q}_n}\right) = \begin{bmatrix} N_{1,x}(1+u_{,x}^n - y\varphi_{,x}^n) & -N_{1,x}\varphi^n \\ N_{1,x}v_{,x}^n & N_{1,x} \\ N_{1,x}y(1+u_{,x}^n - y\varphi_{,x}^n) & -N_1(1+u_{,x}^n) + yN_{1,x}\varphi^n \\ N_{2,x}(1+u_{,x}^n - y\varphi_{,x}^n) & -N_{2,x}\varphi^n \\ N_{2,x}v_{,x}^n & N_{2,x} \\ -N_{2,x}y(1+u_{,x}^n - y\varphi_{,x}^n) & -N_2(1+u_{,x}^n - y\varphi_{,x}^n) + yN_{2,x}\varphi^n \end{bmatrix}. \tag{69}$$

In the case of small displacements and large strains, the first block of the *material tangent stiffness* related to the configuration n takes the form as follows

$$\left(\mathbf{k}^{mat}_{tang}\right)^n_{1,1} = \int_{V_0} \left(\mathbf{N}^T \left(\mathbf{A}^T + \mathbf{C}^T\mathbf{B}\mathbf{q}_n\right) \mathbf{D}^n_t \left(\mathbf{q}^T_n \mathbf{B}^T \mathbf{C} + \mathbf{A}\right) \mathbf{N}\right)_{1,1} \mathrm{d}V_0 =$$

$$\int_{V_0} \left[\begin{array}{lll}
\begin{array}{l} D^n_{xx}\left(N_{1,x}\right)^2 \cdot \\ \cdot\left(1+u^n_{,x}-y\varphi^n_{,x}\right)^2 + \\ +D^n_{yy}\left(N_{1,x}\right)^2\left(\varphi^n\right)^2 \end{array} &
\begin{array}{l} D^n_{xx}\left(N_{1,x}\right)^2 \cdot \\ \cdot\left(1+u^n_{,x}-y\varphi^n_{,x}\right) v^n_{,x} - \\ -D^n_{yy}\left(N_{1,x}\right)^2 \varphi^n \end{array} &
\begin{array}{l} -D^n_{xx}\left(N_{1,x}\right)^2 y \cdot \\ \cdot\left(1+u^n_{,x}-y\varphi^n_{,x}\right)^2 + \\ +D^n_{yy}\left\{N_1 N_{1,x}\varphi^n \cdot\right. \\ \cdot\left(1+u^n_{,x}-y\varphi^n_{,x}\right) - \\ \left.-\left(N_{1,x}\right)^2 y\left(\varphi^n\right)^2\right\} \end{array} \\
\begin{array}{l} D^n_{xx}\left(N_{1,x}\right)^2 \cdot \\ \cdot\left(1+u^n_{,x}-y\varphi^n_{,x}\right) v^n_{,x} - \\ -D^n_{yy}\left(N_{1,x}\right)^2 \varphi^n \end{array} &
\begin{array}{l} D^n_{xx}\left(N_{1,x}\right)^2\left(v^n_{,x}\right)^2 + \\ +D^n_{yy}\left(N_{1,x}\right)^2 \end{array} &
\begin{array}{l} -D^n_{xx}\left(N_{1,x}\right)^2 y \cdot \\ \cdot\left(v^n_{,x}-y\varphi^n_{,x}S^n\right) v^n_{,x} - \\ -D^n_{yy}\left\{N_1 N_{1,x} \cdot\right. \\ \cdot\left(1+u^n_{,x}-y\varphi^n_{,x}\right) - \\ \left.-\left(N_{1,x}\right)^2 y\varphi^n\right\} \end{array} \\
\begin{array}{l} -D^n_{xx}\left(N_{1,x}\right)^2 y \cdot \\ \cdot\left(1+u^n_{,x}-y\varphi^n_{,x}\right)^2 + \\ +D^n_{yy}\left\{N_1 N_{1,x}\varphi^n \cdot\right. \\ \cdot\left(1+u^n_{,x}-y\varphi^n_{,x}\right) - \\ \left.-\left(N_{1,x}\right)^2 y\left(\varphi^n\right)^2\right\} \end{array} &
\begin{array}{l} -D^n_{xx}\left(N_{1,x}\right)^2 y \cdot \\ \cdot\left(v^n_{,x}-y\varphi^n_{,x}S^n\right) v^n_{,x} - \\ -D^n_{yy}\left\{N_1 N_{1,x} \cdot\right. \\ \cdot\left(1+u^n_{,x}-y\varphi^n_{,x}\right) - \\ \left.-\left(N_{1,x}\right)^2 y\varphi^n\right\} \end{array} &
\begin{array}{l} D^n_{xx}\left\{N_{1,x} y \cdot\right. \\ \left.\cdot\left(1+u^n_{,x}-y\varphi^n_{,x}\right)\right\}^2 + \\ +D^n_{yy}\left\{-N_1 \cdot\right. \\ \cdot\left(1+u^n_{,x}-y\varphi^n_{,x}\right) + \\ \left.+N_{1,x} y\varphi^n\right\}^2 \end{array}
\end{array}\right] \mathrm{d}V_0 \tag{70}$$

in which, through the abbreviations (34) – (37), also the terms S^n, C^n, φ^n, $\varphi^n_{,x}$, $u^n_{,x}$, $v^n_{,x}$ associated with the configuration n contain the shape functions N_1, N_2 . The other blocks of the matrix can similarly be obtained.

In the case of small displacements and large strains the *initial stress or geometric stiffness* matrix is simplified to the matrix $\mathbf{S}^T_n\left(\mathbf{N}^T\mathbf{B}^T\mathbf{C}\mathbf{N}\right)$ only, thus

$$\left(\mathbf{k}^{stress}_{geom}\right)^n = \int_{V_0} \mathbf{S}^T_n \mathbf{N}^T \mathbf{B}^T \mathbf{C}\mathbf{N}\, \mathrm{d}V_0 =$$

$$\int_{V_0} \begin{bmatrix} S^n_{xx}\left(N_{1,x}\right)^2 & 0 & -S^n_{xx}y\left(N_{1,x}\right)^2 - -S^n_{xy}N_1N_{1,x} & S^n_{xx}N_{1,x}N_{2,x} & 0 & -S^n_{xx}yN_{1,x}N_{2,x} - -S^n_{xy}N_2N_{1,x} \\ 0 & S^n_{xx}\left(N_{1,x}\right)^2 & 0 & 0 & S^n_{xx}N_{1,x}N_{2,x} & 0 \\ -S^n_{xx}y\left(N_{1,x}\right)^2 - -S^n_{xy}N_1N_{1,x} & 0 & S^n_{xx}y^2\left(N_{1,x}\right)^2 + +S^n_{xy}2yN_1N_{1,x} & -S^n_{xx}yN_{1,x}N_{2,x} - -S^n_{xy}N_1N_{2,x} & 0 & S^n_{xx}y^2N_{1,x}N_{2,x} + +S^n_{xy}y\left(N_1N_{2,x} + +N_2N_{1,x}\right) \\ S^n_{xx}N_{1,x}N_{2,x} & 0 & -S^n_{xx}yN_{1,x}N_{2,x} - -S^n_{xy}N_1N_{2,x} & S^n_{xx}\left(N_{2,x}\right)^2 & 0 & -S^n_{xx}y\left(N_{2,x}\right)^2 - -S^n_{xy}N_2N_{2,x} \\ 0 & S^n_{xx}N_{1,x}N_{2,x} & 0 & 0 & S^n_{xx}\left(N_{2,x}\right)^2 & 0 \\ -S^n_{xx}yN_{1,x}N_{2,x} - -S^n_{xy}N_2N_{1,x} & 0 & S^n_{xx}y^2N_{1,x}N_{2,x} + +S^n_{xy}y\left(N_1N_{2,x} + +N_2N_{1,x}\right) & -S^n_{xx}y\left(N_{2,x}\right)^2 - -S^n_{xy}N_2N_{2,x} & 0 & S^n_{xx}y^2\left(N_{2,x}\right)^2 + +S^n_{xy}2yN_2N_{2,x} \end{bmatrix} \mathrm{d}V_0 \tag{71}$$

in which, only the stresses depend on the configuration n.

Let us consider now the *loading tangent stiffness* which, in the case of small displacements consists of a single term only, related to the deformation-sensitive part of loading, containing the loading tangent moduli $\mathbf{M}_t^n$. Here the configuration-independent matrix $\mathbf{H}_n = \mathbf{N}$ is used, that is

$$\left(\mathbf{k}^{load}_{tang}\right)^n = \int_{V_0} \underset{(6,2)}{\mathbf{N}^T} \; \underset{(2,2)}{\mathbf{M}_t^n} \; \underset{(2,6)}{\mathbf{N}} \; \mathrm{d}V_0 =$$

$$\int_{V_0} \begin{bmatrix} M^n_{xx}N_1^2 & 0 & -M^n_{xx}N_1^2y & M^n_{xx}N_1N_2 & 0 & -M^n_{xx}N_1N_2y \\ 0 & M^n_{yy}N_1^2 & 0 & 0 & M^n_{yy}N_1N_2 & 0 \\ -M^n_{xx}N_1^2y & 0 & M^n_{xx}N_1^2y^2 & -M^n_{xx}N_1N_2y & 0 & M^n_{xx}N_1N_2y^2 \\ M^n_{xx}N_1N_2 & 0 & -M^n_{xx}N_1N_2y & M^n_{xx}N_2^2 & 0 & -M^n_{xx}N_2^2y \\ 0 & M^n_{yy}N_1N_2 & 0 & 0 & M^n_{yy}N_2^2 & 0 \\ -M^n_{xx}N_1N_2y & 0 & M^n_{xx}N_1N_2y^2 & -M^n_{xx}N_2^2y & 0 & M^n_{xx}N_2^2y^2 \end{bmatrix} \mathrm{d}V_0 \tag{72}$$

namely, in this matrix only the loading tangent moduli depend on the configuration n.

Finally, the first block of the *total system gradient matrix* $\mathbf{k}_t^n$ for small strains and large displacements reads

$$\left(\mathbf{k}_t\right)^n_{1,1} = \left(\mathbf{k}^{mat}_{tang} + \mathbf{k}^{stress}_{geom} - \mathbf{k}^{load}_{tang}\right)^n_{1,1} =$$

$$
\int_{V_0}
\begin{bmatrix}
\begin{array}{l}
D^n_{xx}\left(N_{1,x}\right)^2\cdot\\
\cdot\left(1+u^n_{,x}-y\varphi^n_{,x}\right)^2+\\
+D^n_{yy}\left(N_{1,x}\right)^2(\varphi^n)^2+\\
+S^n_{xx}\left(N_{1,x}\right)^2-\\
-M^n_{xx}N_1^2
\end{array}
&
\begin{array}{l}
D^n_{xx}\left(N_{1,x}\right)^2\cdot\\
\cdot\left(1+u^n_{,x}-y\varphi^n_{,x}\right)v^n_{,x}-\\
-D^n_{yy}\left(N_{1,x}\right)^2\varphi^n
\end{array}
&
\begin{array}{l}
-D^n_{xx}\left(N_{1,x}\right)^2 y\cdot\\
\cdot\left(1+u^n_{,x}-y\varphi^n_{,x}\right)^2+\\
+D^n_{yy}\big\{N_1N_{1,x}\varphi^n\cdot\\
\cdot\left(1+u^n_{,x}-y\varphi^n_{,x}\right)-\\
-\left(N_{1,x}\right)^2y(\varphi^n)^2\big\}\\
-S^n_{xx}y\left(N_{1,x}\right)^2-\\
-S^n_{xy}N_1N_{1,x}+\\
+M^n_{xx}N_1^2y
\end{array}
\\
\begin{array}{l}
D^n_{xx}\left(N_{1,x}\right)^2\cdot\\
\cdot\left(1+u^n_{,x}-y\varphi^n_{,x}\right)v^n_{,x}-\\
-D^n_{yy}\left(N_{1,x}\right)^2\varphi^n
\end{array}
&
\begin{array}{l}
D^n_{xx}\left(N_{1,x}\right)^2\left(v^n_{,x}\right)^2+\\
+D^n_{yy}\left(N_{1,x}\right)^2+\\
+S^n_{xx}\left(N_{1,x}\right)^2-\\
-M^n_{yy}N_1^2
\end{array}
&
\begin{array}{l}
-D^n_{xx}\left(N_{1,x}\right)^2 y\cdot\\
\cdot\left(v^n_{,x}-y\varphi^n_{,x}S^n\right)v^n_{,x}-\\
-D^n_{yy}\big\{N_1N_{1,x}\cdot\\
\cdot\left(1+u^n_{,x}-y\varphi^n_{,x}\right)-\\
-\left(N_{1,x}\right)^2y\varphi^n\big\}
\end{array}
\\
\begin{array}{l}
-D^n_{xx}\left(N_{1,x}\right)^2 y\cdot\\
\cdot\left(1+u^n_{,x}-y\varphi^n_{,x}\right)^2+\\
+D^n_{yy}\big\{N_1N_{1,x}\varphi^n\cdot\\
\cdot\left(1+u^n_{,x}-y\varphi^n_{,x}\right)-\\
-\left(N_{1,x}\right)^2y(\varphi^n)^2\big\}-\\
-S^n_{xx}y\left(N_{1,x}\right)^2-\\
-S^n_{xy}N_1N_{1,x}+\\
+M^n_{xx}N_1^2y
\end{array}
&
\begin{array}{l}
-D^n_{xx}\left(N_{1,x}\right)^2 y\cdot\\
\cdot\left(v^n_{,x}-y\varphi^n_{,x}S^n\right)v^n_{,x}-\\
-D^n_{yy}\big\{N_1N_{1,x}\cdot\\
\cdot\left(1+u^n_{,x}-y\varphi^n_{,x}\right)-\\
-\left(N_{1,x}\right)^2y\varphi^n\big\}
\end{array}
&
\begin{array}{l}
D^n_{xx}\big\{N_{1,x}y\cdot\\
\cdot\left(1+u^n_{,x}-y\varphi^n_{,x}\right)\big\}^2+\\
+D^n_{yy}\{-N_1\cdot\\
\cdot\left(1+u^n_{,x}-y\varphi^n_{,x}\right)+\\
+\;N_{1,x}y\varphi^n\}^2+\\
+S^n_{xx}y^2\left(N_{1,x}\right)^2+\\
+S^n_{xy}2N_1N_{1,x}y-\\
-M^n_{xx}N_1^2y^2
\end{array}
\end{bmatrix}
\mathrm{d}V_0
\tag{73}
$$

from which the interaction of the material and loading characteristics can be studied.

3.5. *Small Strains with Small Displacements*

In the case of small displacements and small strains, the system gradient matrix consists of two terms only, the terms of the material and loading tangent moduli. The *material tangent stiffness* related to the configuration n takes the form as follows

$$
\left(\mathbf{k}^{mat}_{tang}\right)^n=\int_{V_0}\mathbf{N}^T\mathbf{A}^T\mathbf{D}^n_t\mathbf{A}\mathbf{N}\,\mathrm{d}V_0=
$$

$$
\int_{V_0}
\begin{bmatrix}
D^n_{xx}\left(N_{1,x}\right)^2 & 0 & -D^n_{xx}y\left(N_{1,x}\right)^2 & D^n_{xx}N_{1,x}N_{2,x} & 0 & -D^n_{xx}yN_{1,x}N_{2,x} \\
0 & D^n_{yy}\left(N_{1,x}\right)^2 & -D^n_{yy}N_1N_{1,x} & 0 & D^n_{yy}N_{1,x}N_{2,x} & -D^n_{yy}N_2N_{1,x} \\
-D^n_{xx}y\left(N_{1,x}\right)^2 & -D^n_{yy}N_1N_{1,x} & \begin{array}{l} D^n_{xx}y^2\left(N_{1,x}\right)^2+ \\ +D^n_{yy}N_1^2 \end{array} & -D^n_{xx}yN_{1,x}N_{2,x} & -D^n_{yy}N_1N_{2,x} & \begin{array}{l} D^n_{xx}y^2N_{1,x}N_{2,x} \\ D^n_{yy}yN_1N_2 \end{array} \\
D^n_{xx}N_{1,x}N_{2,x} & 0 & -D^n_{xx}yN_{1,x}N_{2,x} & D^n_{xx}\left(N_{2,x}\right)^2 & 0 & -D^n_{xx}y\left(N_{2,x}\right)^2 \\
0 & D^n_{yy}N_{1,x}N_{2,x} & -D^n_{yy}N_1N_{2,x} & 0 & D^n_{yy}\left(N_{2,x}\right)^2 & -D^n_{yy}N_2N_{2,x} \\
-D^n_{xx}yN_{1,x}N_{2,x} & -D^n_{yy}N_2N_{1,x} & \begin{array}{l} D^n_{xx}y^2N_{1,x}N_{2,x}+ \\ +D^n_{yy}yN_1N_2 \end{array} & \begin{array}{l} -S^n_{xx}y\left(N_{2,x}\right)^2- \\ -S^n_{xy}N_2N_{2,x} \end{array} & -D^n_{yy}N_2N_{2,x} & \begin{array}{l} D^n_{xx}y^2\left(N_{2,x}\right)^2 \\ D^n_{yy}N_2^2 \end{array}
\end{bmatrix}
\mathrm{d}V_0 \tag{74}
$$

where only the material moduli depend on the configuration n. In the case of dead loading program, this matrix is the total tangent stiffness matrix in itself since the *initial stress or geometric stiffness*, is based on the nonlinear strains. In the case of deformation-sensitive loading, the *loading tangent stiffness* is equal to (72), thus, the *total system gradient matrix* is as follows

$$
\mathbf{k}^n_t = \mathbf{k}^{mat}_{tang} - \mathbf{k}^{load}_{tang} =
$$

$$
\int_{V_0}
\begin{bmatrix}
\begin{array}{l} D^n_{xx}\left(N_{1,x}\right)^2- \\ -M^n_{xx}N_1^2 \end{array} & 0 & \begin{array}{l} -D^n_{xx}y\left(N_{1,x}\right)^2+ \\ +M^n_{xx}yN_1^2 \end{array} & \begin{array}{l} D^n_{xx}N_{1,x}N_{2,x}- \\ -M^n_{xx}N_1N_2 \end{array} & 0 & \begin{array}{l} -D^n_{xx}yN_{1,x}N_{2,x} \\ M^n_{xx}yN_1N_2 \end{array} \\
0 & \begin{array}{l} D^n_{yy}\left(N_{1,x}\right)^2- \\ -M^n_{yy}N_1^2 \end{array} & -D^n_{yy}N_1N_{1,x} & 0 & \begin{array}{l} D^n_{yy}N_{1,x}N_{2,x}- \\ -M^n_{yy}N_1N_2 \end{array} & -D^n_{yy}N_2N_{1,x} \\
\begin{array}{l} -D^n_{xx}y\left(N_{1,x}\right)^2+ \\ +M^n_{xx}yN_1^2 \end{array} & -D^n_{yy}N_1N_{1,x} & \begin{array}{l} D^n_{xx}y^2\left(N_{1,x}\right)^2+ \\ +D^n_{yy}N_1^2- \\ -M^n_{xx}y^2N_1^2 \end{array} & \begin{array}{l} -D^n_{xx}yN_{1,x}N_{2,x}+ \\ +M^n_{xx}yN_1N_2 \end{array} & -D^n_{yy}yN_1N_{1,x} & \begin{array}{l} D^n_{xx}y^2N_{1,x}N_{2,x} \\ D^n_{yy}yN_1N_2- \\ -M^n_{xx}y^2N_1N_2 \end{array} \\
\begin{array}{l} D^n_{xx}N_{1,x}N_{2,x}- \\ -M^n_{xx}N_1N_2 \end{array} & 0 & \begin{array}{l} -D^n_{xx}yN_{1,x}N_{2,x}+ \\ +M^n_{xx}yN_1N_2 \end{array} & \begin{array}{l} D^n_{xx}\left(N_{2,x}\right)^2- \\ -M^n_{xx}N_2^2 \end{array} & 0 & \begin{array}{l} -D^n_{xx}y\left(N_{2,x}\right)^2 \\ M^n_{xx}yN_2^2 \end{array} \\
0 & \begin{array}{l} D^n_{yy}N_{1,x}N_{2,x}- \\ -M^n_{yy}N_1N_2 \end{array} & -D^n_{yy}N_1N_{2,x} & 0 & \begin{array}{l} D^n_{yy}\left(N_{2,x}\right)^2- \\ -M^n_{yy}N_2^2 \end{array} & -D^n_{yy}N_2N_{2,x} \\
\begin{array}{l} -D^n_{xx}yN_{1,x}N_{2,x}+ \\ +M^n_{yy}yN_1N_2 \end{array} & -D^n_{yy}N_2N_{1,x} & \begin{array}{l} D^n_{xx}y^2N_{1,x}N_{2,x}+ \\ +D^n_{yy}yN_1N_2- \\ -M^n_{xx}yN_1N_2 \end{array} & \begin{array}{l} -S^n_{xx}y\left(N_{2,x}\right)^2- \\ -S^n_{xy}N_2N_{2,x}+ \\ +M^n_{xx}yN_2 \end{array} & -D^n_{yy}N_2N_{2,x} & \begin{array}{l} D^n_{xx}y^2\left(N_{2,x}\right)^2 \\ D^n_{yy}N_2^2- \\ -M^n_{xx}y^2N_2^2 \end{array}
\end{bmatrix}
\mathrm{d}V_0 \tag{75}
$$

by means of which the opposite effect of the material and loading characteristics on the tangent stiffness can be made evident. Further illustrations of the effect of configuration-dependent loading on the structural tangent stiffness can be seen in [3] – [6].

4. Conclusions

The effect of different linearization or approximation conditions is in the focus of the paper. The aim of this paper was to illustrate the different versions of the tangent stiffness matrices obtained by the possible combinations of the structural nonlinearities and linearities detailed in [4] where full nonlinearity is supposed: nonlinear material, strain and displacement, moreover, nonlinear loading device is considered. As a consequence of the latter, the tangent stiffness matrix is extended to the effect of the loading terms.

Through the example of the Timoshenko beam displacement nonlinearity was represented. At the same time, strain nonlinearity was also considered. In a systematic way, all the possible versions of the tangent stiffness matrices were presented, from the most complicated fully nonlinear one to the most simple fully linear version.

Analytical and numerical aspects of linearization were distinguished, by constructing the complete fully nonlinear discrete model. The difference between the nonlinearity of the strains and displacements was strictly distinguished and emphasized.

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References

[1] BAZANT, Z. P. – CEDOLIN, L. (1991): Stability of Structures. Elastic, Inelastic, Fracture and Damage Theories, Oxford University Press, New York, Oxford.

[2] THOMPSON, J. M. T. – HUNT, G. W. (1973): A General Theory of Elastic Stability, Wiley, London.

[3] KURUTZ, M. (1996): Modification of the Structural Tangent Stiffness Due to Nonlinear Configuration-dependent Conservative Loading, *Computer Assisted Mechanics and Engineering Sciences,* Vol. 3, 4, pp. 67–85.

[4] KURUTZ, M. (1997): A Survey of Structural Tangent Stiffness in Fully Nonlinear and Nonconvex Cases Including Material Softening, *J. of Mechanics of Structures and Machines* (in press).

[5] KURUTZ, M. (1995): Modification of the Classical Postbifurcation Equilibrium Paths Due to Variable Load, *Proc. of the Fourth International Conference on Stability of Steel Structures,* Final Report, Budapest, Hungary.

[6] KURUTZ, M. (1996): Postbifurcation Equilibrium Paths Due to Nonlinear Configuration-dependent Loading by Using Nonsmooth Analysis, *J. of Mechanics of Structures and Machines* (in press).

[7] DE VILLE, V. (1986): Analyse nonlineaire des structures, *Lecture Notes, manuscript,* Liège.

[8] CHEN, W. F. – MIZUNO, E. (1990): Nonlinear Analysis in Soil Mechanics, Theory and Implementation, Elsevier, Amsterdam, Oxford, New York, Tokyo.

[9] GADALA, M. S. – DOKAINISH, M. A. – ORAVAS, G. A. E. (1984): Formulation Methods of Geometric and Material Nonlinearity Problems,. *Int J. Numerical Methods in Engineering,* Vol. 20, pp. 887–914.