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OPTIMAL DESIGN OF BOUNDARY CONDITIONS

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Abstract

The presented numerical analysis is capable of determining the minimal weight design of a structure together with the optimal place, type and number of the undefined boundary conditions.

The presented theoretical analysis gives the basic ideas of the applied finite element optimization method.

Keywords: boundaries, calculus of variations, FEM, minimal weight design.

1. Introduction

Structural optimization may be defined as the rational establishment of a structural design that is the best of all possible designs within a prescribed objective and a given set of geometrical and/or behavioural limitations.

Design variables may describe the configuration of a structure by element quantities like cross-sections, etc., and physical properties of the material.

The possible design variables had been divided into six different classes. One of them contains the *supports* and loadings. In that discussion the idea is proposed but results are not mentioned [3]. ROZVANY et al. dealt with boundary problems by the help of reciprocity theorem [5]. DEMS and MÓRZ analyze a boundary value problem with mixed Dirichlet and Neumann boundary conditions [2].

In this presentation the boundary conditions and the dimension of the cross-sectional area are the design variables. The structure is discretized and the possible places of boundaries are the nodes.

The 2$^{\text{nd}}$ section of this presentation shows the results of the mathematical boundary analysis. The solution of the minimum problem of strain energy by variational method is used to determine the possible types and optimal place of the supports.

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The 3rd section presents the optimal weight design. The structure is supported at any indeterminate place, which are design variables and determined during the calculation. Number of nodes determines number and place of the possible supports.

To present this idea a computer program has been written using the finite element method and mathematical programming. The results are shown in this section.

2. Analytical Task of Boundary Determination

2.1. The Minimum Problem of Strain Energy

The object of the research is a beam. The material and cross-sectional designs are given. The beam is supported at the two ends. The types of the supports are unknown.

Fig. 1. The model of the structure without supports

The variational problem of that beam is

$$\pi = \int_0^l \left[\frac{EI}{2} (u''(x))^2 - P(x)u(x) \right] \mathrm{d}x = \min, \tag{1}$$

where
E : Young modulus
I : moment of inertia
$P(x)$: load function
$u(x)$: displacement function and
l : length of the beam [6].

The solution of the integral minimum problem by variational method is:

$$\int_0^l [\underbrace{EIu^{IV}(x) - P(x)}_{\text{Euler's equation}}]\, \eta(x)\, \mathrm{d}x + [EIu''(x)\eta'(x)]_0^l - [EIu'''(x)\eta(x)]_0^l = \emptyset, \tag{2}$$

residual (physical) boundary conditions

essential (geometric) boundary conditions

where η is an arbitrary function and $\eta \in C^1$ [3].

2.2. The Places of Boundaries

If the boundary conditions of the calculated beam are not given at the end points, the structure must be divided into elements at the points where boundaries are possible. In this case in the Euler's differential equation the function of displacement $u(x)$ depends on the location of the supports. It gives the possibility of having that location as an unknown variable [4].

2.3. The Types of Boundaries

The variational calculus defines the natural boundary conditions (2).

The solution function $u(x)$ can be determined by solving the Euler's equation. Consequently $u(x)$ depends on the constants of integration so that can be written as $u(x, c_i)$ $i = 1, \ldots, 4$. The function $\eta(x)$ is arbitrary and satisfies the boundaries, so it can be chosen as $u(x, c_i)$. Finally there is only one function with 4 constants of integration as parameters.

Substituting them into the boundary conditions there is an equation system including four equations and four unknown integration constants as unknown parameters.

The number of the possible solutions is ten:

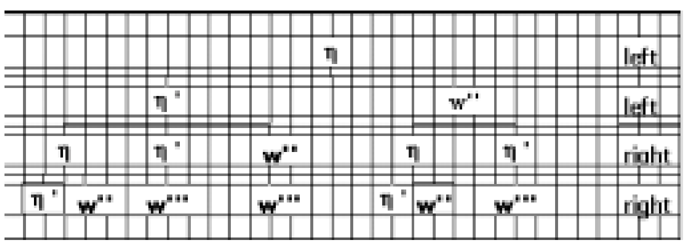

Fig. 2. Support possibilities

The graph shows the 7 possibilities for the supports from the left to the right. The two signed are the symmetrical ones. Mirroring it from right to left there are other 7 possibilities, where 4 are repetitions [4].

3. Task of Boundary Optimization

3.1. FEM

In the FEM the boundary conditions are defined in vector ρ containing the spring coefficients. This vector is added to the main diagonal of the stiffness matrix

$$K = K + E\rho, \tag{3}$$

where E is the unit matrix.

The state equation system is:

$$[K + E\rho]w = q, \tag{4}$$

where w is the displacement vector and q is the external load vector [1].

The number of the possible support types are determined in the case of a simple beam as can be seen in *Fig.* 2 [4].

From it follows that the finite element analysis allows to determine the types of the undefined boundaries.

The nodes of the FEM mesh the possible places and the possible number of supports.

3.2. The Non-Linear Optimization Problem

The implementation of a sequential quadratic programming method for solving non-linear optimization problems:

Minimize	X_{2n+1}	cross-sectional dimension
Subject to	$(K(X_{2n+1}) + EX_i)X_j - q_i = \emptyset$	equality equations
	$-10^{18} \le X_i \le 10^{19}$	boundary limits
	$LL_j \le X_j \le UL_j$	displacement and rotation limits,
	$LL_{2n+1} \le X_{2n+1} \le UL_{2n+1}$	cross-section limits,
	$i = 1, \dots, n, \quad j = n + 1, \dots, 2n$	

(5)

where X is the vector of the unknown variables, where the first n elements (X_i), $i = 1, \dots, n$ are the variables of spring coefficients (ρ), the second n elements (X_j), $j = n + 1 \dots, 2n$ are the displacements (w) and the last one (X_{2n+1}) is the width of the cross-section, LL and UL are the vectors of lower and upper limits of the variables, q is the vector of external loads, K is the stiffness matrix whose elements contain the unknown width and its different powers, etc. in the inertia, E is the unit matrix, and n is the number of the nodes multiplied by the degree of their displacement freedoms.

The examples are solved by a sequential quadratic programming method.

3.3. The Optimization Problem of a Beam

3.3.1. Problem formulation of beam

The object of the research is a beam, supported at both ends. In the center point there is a unit force. The types of the supports are unknown.

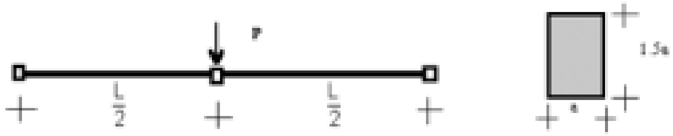

Fig. 3. The model of the discretized structure and its cross-section

The stiffness matrix of the structure contains the cross-sectional area and inertia, $K(a^2, a^4)$, where a is the unknown variable.

The lower and upper limits of the displacements are:

$$
\begin{array}{ll}
-10^{18} \le X_i \le 10^{19}, \quad i = 1, \ldots, 9 & \text{boundary limits} \\
-0.05 \le X_{9+3*i+j} \le 0.05, \quad i = 0, \ldots, 2, \quad j = 1, \ldots, 2 & \text{displacement limits} \\
-0.10 \le X_{9+3*i} \le 0.01, \quad i = 1, \ldots, 3 & \text{rotation limits} \\
0.05 \le X_{19} \le 0.50 & \text{cross-section limits}
\end{array}
\tag{6}
$$

3.3.2. Example I

The structure is fixed at the nodes 1 and 3. The hardness of the supports is the upper limit. The width of the cross-section is 0.31 m. Solving the equation $Kw = q$ we have the starting deflections. The displacement of the 2 node in direction y is greater than the limit. The lower and upper limits of the support variables of the 2nd node are equal to zero.

The solution: None of the supports has been changed. The displacement of the middle node went to the limit value. The optimal width of the cross-sectional area is 0.3489 m.

3.3.3. Example II

The structure is fixed at node 1 and the rotation of node 3 is tied. The hardness of the supports is the upper limit. The width of the cross-section is 0.31 m. More of the calculated displacements are out of limits. The lower and upper limits of the support variables of the 2nd node are equal to zero.

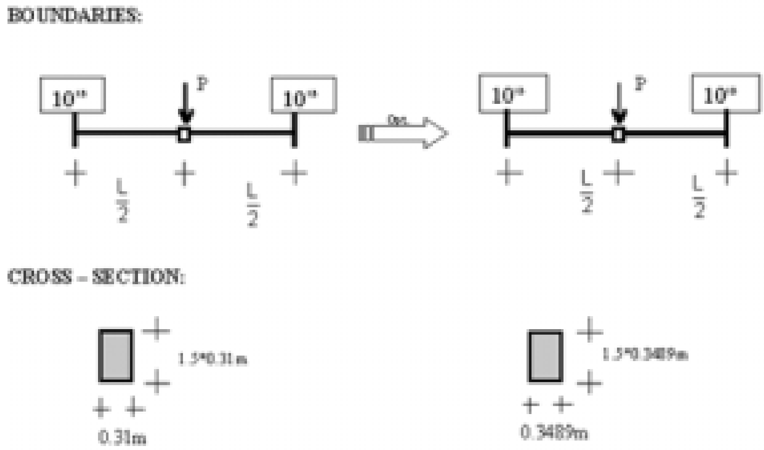

Fig. 4. Starting and optimal values of example I

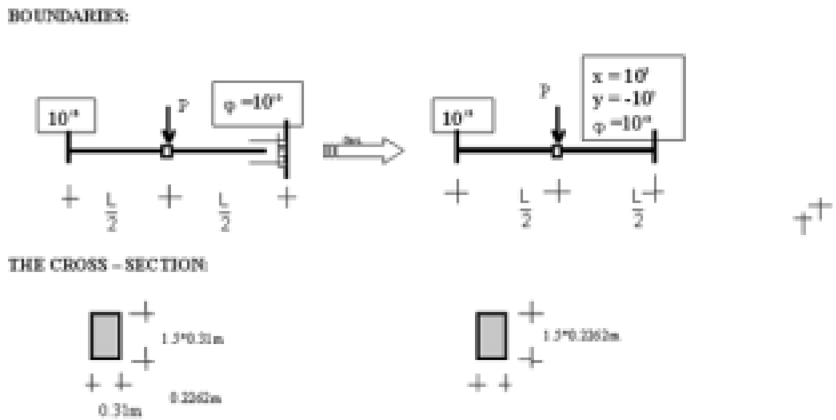

Fig. 5. Starting and optimal values of example II

The solution: The changeable unfixed support variables of the 3rd node have got an elastic value. The displacement of the nodes went to or under the limit values. The optimal width of the cross-sectional area is 0.2262 m.

3.3.4. Example III

The structure is fixed at node 1 and the rotation of the node 3 is hindered. The value of the starting support variable is less, 10^{13}. The width of the cross-section is 0.31 m. More of the calculated displacements are out of limits. The lower and upper limits of the support variables of the 2nd node are equal to the other ones, as written in (6).

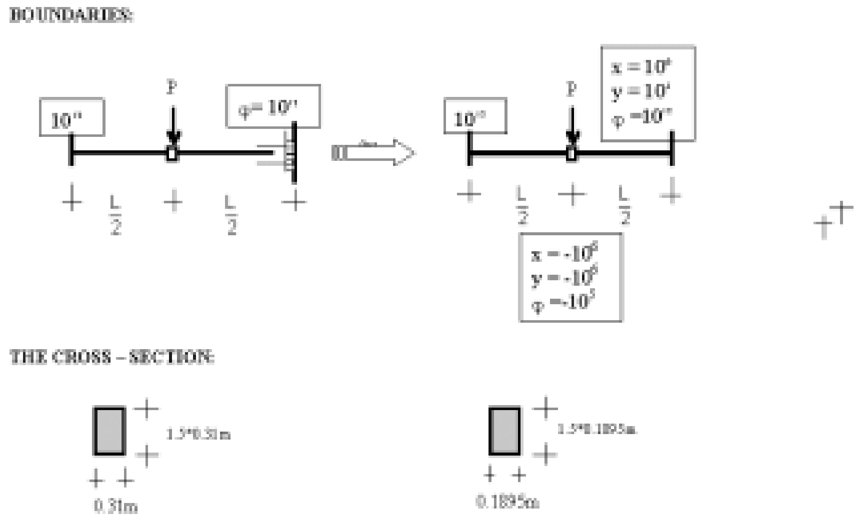

Fig. 6. Starting and optimal values of example III

The solution: All of the undefined supports have been changed. The changeable unfixed support variables of the 3rd node have got an elastic value. In the middle node the supports have the elastic values in opposite direction. The displacement of the nodes went to or under the limit values. The optimal width of the cross-sectional area is 0.1895 m. The result structure is not symmetrical.

3.4. *The Optimization Problem of a Ring*

3.4.1. *Problem formulation of a ring*

The object of the research is a ring. The structure is divided symmetrically into eight elements (*Fig.* 7).

The types, numbers and places of the supports and the width of the cross-sectional area are unknown variables. The material of the ring is steel, $E = 206.000$ N/mm^2 is the Young modulus and $R = 2$ m is the radius.

The stiffness matrix of the structure contains the cross-sectional area and inertia, $K(a, a^3)$, where a is the unknown variable.
The loads:

1. Uniform external load presses the upper ring and pulls the lower ring.
2. Inside there is a gas pressure.
3. Outside there is an asymmetrical wind load.

The lower and upper limits of the displacements are:

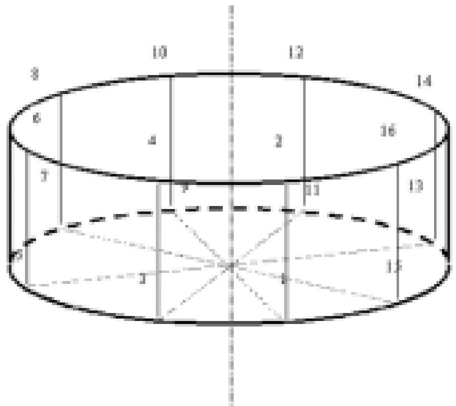

Fig. 7. The model of discretized ring

$$
\begin{array}{lll}
-10^{18} \leq X_i \leq 10^{19}, \quad i = 1, \ldots, 96 & \text{boundary limits} & \\
-0.05 \leq X_{96+6^{*}i+j} \leq 0.05, \quad i = 0, \ldots, 15, \; j = 1, \ldots, 3 & \text{displacement limits (m)} & \\
-0.10 \leq X_{96+6^{*}+i+3+j} \leq 0.10, \quad i = 0, \ldots, 15, \; j = 1, \ldots, 3 & \text{rotation limits (rad)} & \\
0.005 \leq X_{193} \leq 0.04 & \text{cross-section limits (m)} & (7)
\end{array}
$$

3.4.2. Example I

All of the boundary values of the input data are equal to zero. The width of the cross-section is 0.035 m. The structure is internally statically determinate. Therefore the displacements can be calculated. More of them are out of limits.

The solution: All of the supports have been changed equally to the value of $-0.25E+04$. The width of the cross-sectional area has not changed, it is 0.03499 m. The displacement of the nodes went to or under the limit values.

3.4.3. Example II

The structure is externally and internally statically determinate. The ring is fixed at node 3. The width of the cross-section is 0.035 m. More of the calculated displacements are out of limits.

The solution: There is no change. The width of the cross-sectional area continues to be at 0.035 m, the boundaries are at node 3.

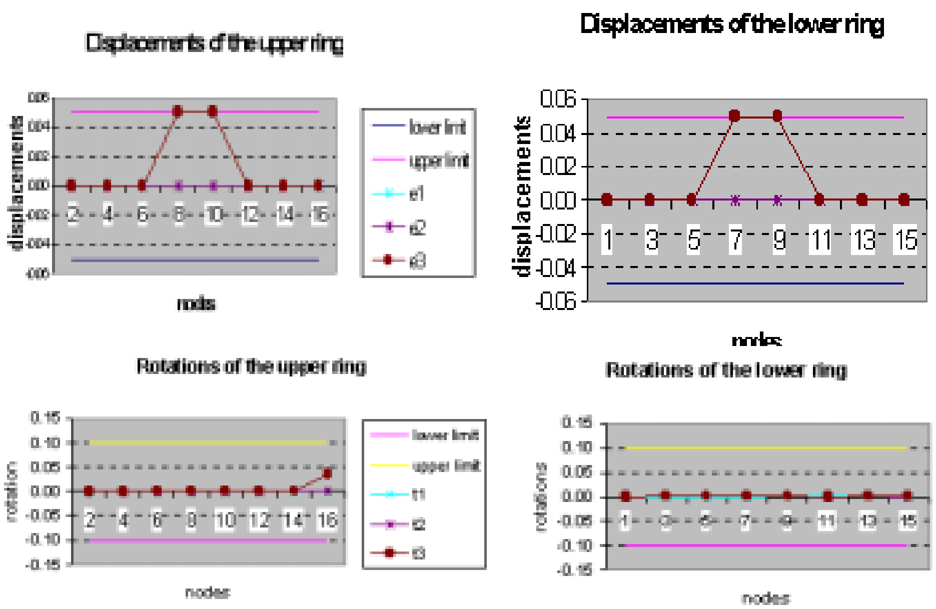

Fig. 8. The diagrams of the displacements

3.4.4. Example III

The structure is externally statically indeterminate and internally statically determinate. The width of the cross-section is 0.035 m. The boundary values of the starting structure are in *Table* 1.

The hardness of the supports is the upper limit. The calculated displacements are within the limits.

The solution: The optimum width of the cross-sectional area is 0.0314 m. All of the boundary values have been changed. Those ones, which were on the upper limit, went to 0.215E+10, to be softer and the others to 0.114E-09, which is different from zero mathematically. The displacement of the nodes went to or under the limit values.

3.4.5. Example IV

The structure is externally statically indeterminate and internally statically determinate. The width of the cross-section is 0.035 m. The boundary values of the starting structure are in *Table* 2.

The hardness of the supports is below the upper limit. Some of the calculated displacements are out of limits.

Table 1.

Node	e1	e2	e3	t1	t2	t3
1	0.0E+00	0.0E+00	0.0E+00	0.0E+00	**0.1E+19**	0.0E+00
2	0.0E+00	0.0E+00	0.0E+00	**0.1E+19**	0.0E+00	0.0E+00
3	**0.1E+19**	**0.1E+19**	**0.1E+19**	0.0E+00	0.0E+00	0.0E+00
4	0.0E+00	**0.1E+19**	0.0E+00	0.0E+00	0.0E+00	0.0E+00
5	0.0E+00	0.0E+00	0.0E+00	0.0E+00	0.0E+00	0.0E+00
6	**0.1E+19**	0.0E+00	0.0E+00	0.0E+00	0.0E+00	**0.1E+19**
7	**0.1E+19**	0.0E+00	**0.1E+19**	**0.1E+19**	0.0E+00	**0.1E+19**
8	0.0E+00	0.0E+00	**0.1E+19**	0.0E+00	0.0E+00	**0.1E+19**
9	0.0E+00	0.0E+00	0.0E+00	**0.1E+19**	0.0E+00	**0.1E+19**
10	0.0E+00	0.0E+00	0.0E+00	0.0E+00	**0.1E+19**	0.0E+00
11	0.0E+00	0.0E+00	0.0E+00	**0.1E+19**	0.0E+00	0.0E+00
12	0.0E+00	0.0E+00	0.0E+00	0.0E+00	0.0E+00	**0.1E+19**
13	**0.1E+19**	**0.1E+19**	**0.1E+19**	0.0E+00	0.0E+00	**0.1E+19**
14	0.0E+00	0.0E+00	0.0E+00	0.0E+00	0.0E+00	**0.1E+19**
15	0.0E+00	0.0E+00	0.0E+00	0.0E+00	**0.1E+19**	0.0E+00
16	0.0E+00	0.0E+00	0.0E+00	0.0E+00	**0.1E+19**	0.0E+00

Table 2.

Node	e1	e2	e3	t1	t2	t3
1	0.0E+00	0.0E+00	0.0E+00	0.0E+00	**0.1E+13**	0.0E+00
2	0.0E+00	0.0E+00	0.0E+00	**0.1E+13**	0.0E+00	0.0E+00
3	**0.1E+13**	**0.1E+13**	**0.1E+13**	0.0E+00	0.0E+00	0.0E+00
4	0.0E+00	**0.1E+13**	0.0E+00	0.0E+00	0.0E+00	0.0E+00
5	0.0E+00	0.0E+00	0.0E+00	0.0E+00	0.0E+00	0.0E+00
6	**0.1E+13**	0.0E+00	0.0E+00	0.0E+00	0.0E+00	**0.1E+13**
7	**0.1E+13**	0.0E+00	**0.1E+13**	**0.1E+13**	0.0E+00	**0.1E+13**
8	0.0E+00	0.0E+00	**0.1E+13**	0.0E+00	0.0E+00	**0.1E+13**
9	0.0E+00	0.0E+00	0.0E+00	**0.1E+13**	0.0E+00	**0.1E+13**
10	0.0E+00	0.0E+00	0.0E+00	0.0E+00	**0.1E+13**	0.0E+00
11	0.0E+00	0.0E+00	0.0E+00	**0.1E+13**	0.0E+00	0.0E+00
12	0.0E+00	0.0E+00	0.0E+00	0.0E+00	0.0E+00	**0.1E+13**
13	**0.1E+13**	**0.1E+13**	**0.1E+13**	0.0E+00	0.0E+00	**0.1E+13**
14	0.0E+00	0.0E+00	0.0E+00	0.0E+00	0.0E+00	**0.1E+13**
15	0.0E+00	0.0E+00	0.0E+00	0.0E+00	**0.1E+13**	0.0E+00
16	0.0E+00	0.0E+00	0.0E+00	0.0E+00	**0.1E+13**	0.0E+00

The solution: The optimal width of the cross-sectional area is 0.0314 m. All of the boundary values have been changed. Those ones, which were bounded, went to 0.215E+10, to be softer and the others to 0.114E-09. The displacement of the nodes went to or under the limit values. This result is the same as the result of example III.

4. Conclusion

The presented finite element optimization method solves the minimal weight design of a structure. During the calculations the dimensions of the cross-section and the optimal supports are determined.

The results of the optimization show that the different starting values of design variables present different solutions:

1. Out of input boundary values of an internally statically determinate structure the optimal design can be calculated.
2. Starting with the values of an externally statically indeterminate structure from external or internal point of the feasible set, there is an optimal solution.
3. A statical determinate structure is an optimal solution itself.

The cross-sectional area and inertia cause the nonconvexity of the feasible set, in this way different local extremal values are found by the computer algorithm.

The presented method and results show that the structural optimization with boundary optimization is a solvable problem, but the global optimum cannot be guaranteed.

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