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APPLICATION OF INTERNAL VARIABLES IN CASE OF TIME-DEPENDENT LOADING FOR ANALYSIS OF STRUCTURES WITH DAMPING

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Abstract

A new approach is presented for the analysis of structures with time-dependent loading based on mathematical programming in the function space L^2. The solution occurred in the vector space. In this paper the computational model of the structures with damping is detailed by the use of internal variables. The energy dissipation is taken into account. A comparison between the conventional and this new model can be read.

Keywords: internal variable, time-dependent loading, mathematical programming, damping.

1. Introduction

In structural mechanics and in engineering practice the time-dependent loading plays an important role and mathematically it is necessary to apply complex knowledge. Our purpose is to analyze the effect of the time-dependent loading and to take the energy dissipation into consideration in the case of damping. The presented models include the theoretical results of the earlier studies where the damping effect was neglected. Furthermore we will point out the limitation of the classic mechanical models.

In Chapter 2 the mathematical background is discussed shortly. The non-equilibrium state and the properties of the internal variables are dealt with in Chapter 3. The model of nonholonomous system of time-dependent loading of the structures is presented and the inertia forces are taken into consideration as well in Chapter 4. The damping models can be found later in this chapter. At first the Coulomb-type friction model is discussed and later the application of internal variables are detailed in the case of damping. This paper is closed by a comparison of the classic and the newly introduced model in Chapter 5.

2. Mathematical Background

The structure is discretized in the usual way. The state variables depend on the time continuously ($\mathbf{u}(x,t)$) and they are approximated by:

$$\mathbf{u}(x,t) \approx \sum_{i=1}^{n} \mathbf{x}(t) \cdot N(\xi), \tag{1}$$

where $N(\xi)$ denotes the function system of approximation depending on the position vector (ξ) and the coefficients $\mathbf{x}(t)$ depend on time. Within the context of the small displacement theory the position vector does not depend on time. The state variables are given in both local and global coordinate systems as vector–vector functions which elements depend on time. These elements are described in a function space, which is determined on the local coordinate axes. It means that the state variables can be given as a vector with function elements on the local coordinate axes in a function subspace. The state variables ($\mathbf{x}(t)$) can be expressed by the generalized Fourier series according to the basis of the function subspace:

$$\mathbf{x}^{\ell}(t) = \sum_{j=1}^{s} \left[\sum_{i=1}^{\infty} \alpha_{ij} P_i(t) \right] \mathbf{e}_j^{\ell}, \quad \alpha_{ij} \in \Re, \quad P_i(t) \in L^2, \quad t \in [t_1, t_2], \tag{2}$$

where $\mathbf{e}^{\ell}$ ($j = 1, \ldots, s$): j-th unit vector of the local coordinate system ordered to the ℓ-th node,
s: number of degrees of freedom at the nodes,
$P_i(t)$: i-th element of basis of the function subspace (orthonormal polynomial system in the interval $[t_1, t_2]$),.

The state variables are described in every node in the following space: $F = L_1^2 \times L_2^2 \times \ldots \times L_s^2$ and on the whole structure

$$F^n = (L_2^2 \times L_3^2 \times \ldots \times L_s^2)^n, \tag{3}$$

where n is the number of the nodes.

To describe the process a nonlinear mathematical programming problem is created in space F^n where the variables are Fourier coefficients α_{ij}. It has been proved [9] that the Fritz-John theorem and Wolfe's duality [1],[3] are valid in space F^n.

The solution of the nonlinear programming problem is a stationary curve. The limitation of the presented model is:

- at least one continuous component has to be assumed,
- the small displacement theory is valid,
- stability problems are neglected.

For the numerical solution it is necessary to transform the problem into the ℓ^2 space where the Fritz-John theorem is valid [3] and the results are mapped back to the space F^n.

3. Non-Equilibrium State

In a real system there are different processes at the same time. The system can reach its equilibrium only in limit state and very often the local equilibrium states are not stationary. Due to interactions such dissipative changes are created in the microstructure of the material of the structures which have influence on the macroscopic behaviour of the phenomena. In the case of macroscopic analysis of the structures it is not possible to take these microstructural changes into consideration at micro level. Even if we have not enough information and deep knowledge of the processes in the phenomena of the structure far from the equilibrium state, the mathematical description of the process can be performed by the use of internal variables.

The structure is in non-equilibrium state between the local equilibrium states and the non-equilibrium state can be described by the use of internal variables. The definition of the internal variables is described very clearly by KESTIN [4] and this definition can be used generally. Concerning the thermodynamical state variables we distinguish between external and internal variables. External state variables like stress or temperature are measurable and controllable. Internal state variables are measurable but not separately controllable. Their changes depend on the history of external variables.

The equilibrium equations are inequalities in the case of non-equilibrium state and the equilibrium equations can be defined by the use of internal variables and state variables.

Properties of the internal variables are [5] :

- the value of the internal variable is zero in the case of local equilibrium states,
- generally they are symmetrical tensors and their order depends on the problem,
- the internal variables are of deformation type (extensive internal variables),
- as a generalized force an intensive variable (ξ) can be created to each deformation type internal variable ($[\Xi]$) as a pair and the dissipative energy can be expressed by the following formula:

$$\tilde{E}_d(t) = T \cdot \xi \cdot [\Xi] = \sum_{i=1}^{n} T(r_i, t) \sum_{j=1}^{s} \sum_{k=1}^{f} \sum_{v=1}^{f} \xi_v(r_i^j, t) L_{vk} \xi_k(r_i^j, t). \quad (4)$$

The change in time of the above formula 4 can be written as follows:

$$\frac{\mathrm{d}\tilde{E}_d}{\mathrm{d}t} = T\xi \frac{\mathrm{d} \cdot [\Xi]}{\mathrm{d}t}. \quad (5)$$

Such a connection is possible with the neighborhood which can transfer the generalized force (ξ) and these forces which belong to the deformation type internal variables can be taken into consideration in the equilibrium equations.

Neglecting the generalized forces (ξ) the system reaches such a state which is proportional to velocity $\frac{\mathrm{d}[\Xi]^k(r_i^j,t)}{\mathrm{d}t}$ and any limitation exists in it according to the k-th internal variable component.

Each process can be determined by the instationarity relaxation time of the deformation type internal variables $[\Xi]$.

A process can be modelled by the use of one or more internal variables depending on type of the problem. If several internal variables are used which have different instationary relaxation time, the local equilibrium states can exist in the subset of internal variables having the same instationary relaxation time.

An intensive internal variable such as the pair of the deformation type internal variable exists in the equilibrium equations if any part of the structure is in plastic state. The value of the intensive internal variable should be bounded. If the intensive internal variable reaches this given limit, the plastic region of that plastic point involves the total cross-section and irreversible plastic deformations occur.

4. Analytical Model for Time-Dependent Loading

In the dynamic problem the strain and the kinetic energies are taken into consideration. First the problem without damping is discussed. The continuous functions of displacements and stresses can be determined by the help of mathematical tools (Chapter 2), it means the resonance cannot be handled in this way. In the mathematical programming model bounds are given for the stresses to cut the non continuous part of the stress functions.

It is assumed that

- the structure is in equilibrium state at the beginning of loading,
- the external load is decomposed in two parts, the ratio of them is unknown and it is changed during the process:
  1. the first part causes elastic deformations,
  2. the second part of the external load is moving the structure.
- Resonance is not examined. The bound values are given for stresses, if this limit is reached, the process reaches the resonance state.

4.1. Dynamic Problems without Damping

The coenergy minimum theorem is used to write the mathematical primal problem [5]. The strain and the kinetic energies are taken into consideration in the form deducted by [6]. For the sake of simplicity linear elastic material is assumed. The transfer matrix is given in the global coordinate system.

$$\min : \left\{ \sum_{k=1}^{G} \rho \left(\frac{1}{2} \sigma_e^*(t)[\mathbf{F}]\sigma_e(t) + \right.\right.$$
$$+ \frac{1}{2} \left(\int_0^t (\sigma_d(\tau))\, \mathrm{d}\tau \right)^* \times \qquad \text{(coenergy function)} \qquad (6.a)$$
$$\left.\left. \times [\mathbf{M}]^{-1} \left(\int_0^t (\sigma_d(\tau))\, \mathrm{d}\tau \right)\right)\right\},$$

$$-\mathbf{P}_e(t) - \mathbf{P}_d(t) + \mathbf{P}(t) = 0, \qquad \text{(decomposition of the external load)} \qquad (6.b)$$

$$\langle\rho\rangle^*[\mathbf{B}]^*\sigma_e(t) + \mathbf{P}_e(t) = 0, \qquad \text{(equilibrium equations for elastic part)} \qquad (6.c)$$

$$\langle\rho\rangle^*[\phi]^* \int_0^t \sigma_d(\tau)\, \mathrm{d}\tau + \int_0^t \mathbf{P}_d(\tau)\, \mathrm{d}\tau = 0, \qquad \text{(equilibrium equations for dynamic part)} \qquad (6.d)$$

$$\sigma_e(t) + \sigma_d(t) - \sigma_L \le 0, \qquad \text{(limit for stresses)} \qquad (6.e)$$

$$\begin{aligned} \overline{u}_i^j\big|_{S_{II_i}} - \hat{\hat{u}}_i^j\big|_{S_{II_i}} &= 0,\ i = 1, \dots, S_{II_n}, \\ j &= 1, \dots, s, \\ \overline{P}_i^j\big|_{S_{I_i}} - \hat{\hat{P}}_i^j\big|_{S_{I_i}} &= 0,\ i = 1, \dots, S_{I_n}, \\ j &= 1, \dots, z, \end{aligned} \qquad \text{(boundary conditions)} \qquad (6.f)$$

$$\sigma_e(t_0) + \sigma_d(t_0) = 0 \qquad \text{(initial conditions)} \qquad (6.g)$$

$$(t, \sigma_e(t), \sigma_d(t), P_e(t), P_d(t)) \in [\mathbf{t}_1, \mathbf{t}_2] \times \Re^{4mGz}, \quad \forall t, \quad t \in [t_1, t_2],$$

where σ_L is a given stress limit, the indexes $_e$ and $_d$ show the stresses originated from the elastic or dynamic parts of the external load, respectively, [**B**] and $[\phi]$ are the transfer matrices of the stresses originated from the elastic or dynamic parts. The inequalities cut the resonance state. G is the number of the Gaussian points, m denotes the number of elements and z is the stress-freedom of the Gaussian points.

The unknowns are: $\sigma_e(t), \sigma_d(t), \mathbf{P}_e(t), \mathbf{P}_d(t)$.

The dual problem can be formed by the help of Stieltjes derivatives [7]. The dual problem expresses the energy minimum theorem and it is proved by the equivalence of the results among the primal – dual problems and the classic solutions [6].

The dual problem is

$$\begin{aligned}
\min : \Bigg\{ \sum_{k=1}^{G} \rho \Bigg(& \frac{1}{2}\sigma_e^*(t)[\mathbf{F}]\sigma_e(t) + \\
& + \frac{1}{2}\left(\int_0^t (\sigma_d(\tau))\,\mathrm{d}\tau\right)^* \times \\
& \times [\mathbf{M}]^{-1}\left(\int_0^t (\sigma_d(\tau))\,\mathrm{d}\tau\right)\Bigg) \qquad \text{(energy function)} \qquad (7.\text{a}) \\
& - \left(\int_0^t \mathbf{P}_d(\tau)\,\mathrm{d}\tau\right)^* \mathbf{v}(t) - \\
& - (\mathbf{P}_e(t))^*\,\mathbf{u}(t)\Bigg\},
\end{aligned}$$

$$[\mathbf{F}]\sigma_e(t) + [\mathbf{B}]\mathbf{u}(t) + \mathbf{y}(t) = 0, \qquad \text{(compatibility equations for displacements)} \qquad (7.\text{b})$$

$$\begin{aligned}
& [\mathbf{M}]^{-1}\left(\int_0^t (\sigma_d(\tau))\,\mathrm{d}\tau\right) + \qquad \text{(compatibility equations for velocities of the displacements)} \qquad (7.\text{c}) \\
& + [\Phi]\mathbf{v}(t) + \mathbf{y}(t) = 0,
\end{aligned}$$

$$\mathbf{u}(t)^* = \mathbf{v}(t)^*\left[\frac{\mathbf{P}_d(t)}{\dot{P}_d^j(t)},\ j = 1, \dots, ns\right], \qquad \text{(the connection between the displacements and velocities of the displacements)} \qquad (7.\text{d})$$

$$(\sigma_e(t) + \sigma_d(t) - \sigma_L)^*\mathbf{y}(t) = 0, \qquad \text{(switch)} \qquad (7.\text{e})$$

$$\mathbf{y}(t) \geq 0, \qquad \text{(sign restriction)} \qquad (7.\text{f})$$

$$\begin{aligned}
& \overline{u}_i^j\big|_{S_{II_i}} - \hat{\hat{u}}_i^j\big|_{S_{II_i}} = 0,\ i = 1, \dots, S_{II_n}, \\
& \qquad j = 1, \dots, s, \qquad \text{(boundary conditions)} \qquad (7.\text{g}) \\
& \overline{P}_i^j\big|_{S_{I_i}} - \hat{\hat{P}}_i^j\big|_{S_{I_i}} = 0,\ i = 1, \dots, S_{I_n}, \\
& \qquad j = 1, \dots, z,
\end{aligned}$$

$$\sigma_e(t_0) + \sigma_d(t_0) = 0, \qquad \text{(initial conditions)} \qquad (7.\text{h})$$

$$(t, \sigma_e(t), \sigma_d(t), P_e(t), P_d(t), \mathbf{u}(t), \mathbf{v}(t), \mathbf{y}(t)) \in [\mathbf{t}_1, \mathbf{t}_2] \times \Re^{5mzG+3ns},$$
$$\forall t, \quad t \in [t_1, t_2],$$

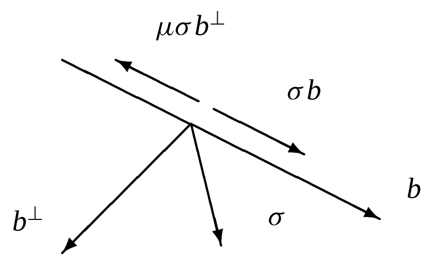

Fig. 1. The direction of the Burger's vectors

The unknowns are:

$$\sigma_e(t), \sigma_d(t), \mathbf{P}_e(t), \mathbf{P}_d(t), \mathbf{u}(t), \mathbf{v}(t), \mathbf{y}(t),$$

where $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are the dual functions ordered to the equilibrium equations for elastic part and equilibrium equations for dynamic part, respectively, $\mathbf{y}(t)$ are the dual functions of stress inequalities. Their values have to be zero if the resonance does not appear. The number of the nodes is denoted by n and s is the displacement-freedom of the nodes.

In the case of free motion the velocities are the derivative functions of the displacements functions. Because of the boundary conditions the structure has no free motion, the connections between the velocities and the displacements are presented by the third equation system.

4.2. Dynamical Problems with Damping

First the damping is taken into consideration as an internal friction according to the Coulomb law.

4.2.1. Application of the Coulomb-type friction model

In that computational model the damping is taken into consideration as an internal Coulomb-type friction at micro-level of the phenomena. The direction of the Burger's vector is determined by the microstructure of the phenomena [5]. Using this direction and knowing the internal friction angle it can be determined whether the internal friction exists or not and damping occurs or not.

$$\begin{aligned}
&\min : \left\{\left(\left(\frac{1}{2}\sigma_e^*(t)\langle\rho\rangle[\mathbf{F}]\sigma_e(t)+\right.\right.\right.\\
&+\frac{1}{2}\left(\int_0^t \sigma_d(\tau)\,\mathrm{d}\tau\right)^* \langle\rho\rangle[\mathbf{M}]^{-1}\left.\left.\left(\int_0^t \sigma_d(\tau)\,\mathrm{d}\tau\right)\right)\right)\\
&-\sum_{k=1}^{RR}\left(\frac{1}{2}\sigma_e^s(t)^*\langle\rho\rangle[\mathbf{F}]\sigma_e^s(t)\right)-\\
&-\sum_{k=1}^{RR}\frac{1}{2}\left(\left(\left(\int_0^t \sigma_d(\tau)\,\mathrm{d}\tau\right)^*\langle\rho\rangle[\mathbf{M}]^{-1}\times\right.\right.\\
&\left.\left.\left.\times\left(\int_0^t \sigma_d(\tau)\,\mathrm{d}\tau\right)\right)\right)\right\},
\end{aligned} \qquad \text{(coenergy function)} \qquad (8.a)$$

where

$$\begin{aligned}
&\sigma_e^s(t) = \sum_{kk=1}^{RR}\mu\left(\left((\sigma_e(t))\,\mathbf{b}_k^{\perp}\right)\otimes\mathbf{b}_k-\right.\\
&\left.-(\sigma_e(t))\mathbf{b}_k\right),\\
&\int_0^t \sigma_d^s(\tau)\,\mathrm{d}\tau = \sum_{kk=1}^{RR}\mu\left(\left(\int_0^t \sigma_d(\tau)\mathbf{b}_k^{\perp}\,\mathrm{d}\tau\right)\otimes\mathbf{b}_k-\right.\\
&\left.-\int_0^t \sigma_d(\tau)\mathbf{b}_k\,\mathrm{d}\tau\right),
\end{aligned}$$

$$-\mathbf{P}_e(t)-\mathbf{P}_d(t)+\mathbf{P}(t)=0, \qquad \text{(decomposition of the external load)} \qquad (8.b)$$

$$[\mathbf{B}]^*\langle\rho\rangle\sigma_e(t)-[\mathbf{B}]^*\langle\rho\rangle\sigma_e^s(t)+\mathbf{P}_e(t)=0, \qquad \text{(equilibrium equations elastic part)} \qquad (8.c)$$

$$\begin{aligned}
&[\Phi]^*\langle\rho\rangle\int_0^t \sigma_d(\tau)\,\mathrm{d}\tau-\\
&-[\Phi]^*\langle\rho\rangle\int_0^t \sigma_d(\tau)\,\mathrm{d}\tau+\int_0^t \mathbf{P}_d(\tau)\,\mathrm{d}\tau=0,
\end{aligned} \qquad \text{(equilibrium equations for dynamic part)} \qquad (8.d)$$

$$\begin{aligned}
&\overline{u}_i^j\big|_{S_{II_i}}-\hat{\hat{u}}_i^j\big|_{S_{II_i}}=0,\ i=1,\ldots,S_{II_n},\\
&\qquad j=1,\ldots,s,\\
&\overline{P}_i^j\big|_{S_{I_i}}-\hat{\hat{P}}_i^j\big|_{S_{I_i}}=0,\ i=1,\ldots,S_{I_n},\\
&\qquad j=1,\ldots,z,
\end{aligned} \qquad \text{(boundary conditions)} \qquad (8.e)$$

$\sigma_e(t) + \sigma_d(t) - \sigma_L \le 0$ (limit for stresses) (8.f)

$\sigma_e(t_0) + \sigma_d(t_0) = 0$ (initial condition) (8.g)

$$(t, \sigma_e(t), \sigma_d(t), P_e(t), P_d(t)) \in [\mathbf{t}_1, \mathbf{t}_2] \times G \times G \times F \times F, \quad \forall t, \quad t \in [t_1, t_2].$$

The dual problem is:

$$\begin{aligned}
&\min : \Bigg\{ \Bigg(\frac{1}{2}\sigma_e^*(t)\langle\rho\rangle[\mathbf{F}]\sigma_e(t) + \\
&+\frac{1}{2}\left(\int_0^t \sigma_d(\tau)\,\mathrm{d}\tau\right)^* \langle\rho\rangle[\mathbf{M}]^{-1}\left(\int_0^t \sigma_d(\tau)\,\mathrm{d}\tau\right)\Bigg) \\
&-\sum_{k=1}^{RR}\left(\frac{1}{2}\sigma_e^s(t)^*\langle\rho\rangle[\mathbf{F}]\sigma_e^s(t)\right) \\
&-\left(\int_0^t \mathbf{P}_d(\tau)\,\mathrm{d}\tau\right)^* \mathbf{v}(t) - (\mathbf{P}_e(t))^*\mathbf{u}(t) \\
&-\sum_{k=1}^{RR}\frac{1}{2}\left(\left(\int_0^t \sigma_d^s(\tau)\,\mathrm{d}\tau\right)^* \langle\rho\rangle[\mathbf{M}]^{-1}\times \right. \\
&\left.\times\left(\int_0^t \sigma_d^s(\tau)\,\mathrm{d}\tau\right)\right)\Bigg\},
\end{aligned}$$

(energy function) (9.a)

where

$$\begin{aligned}
&\sigma_e^s(t) = \sum_{kk=1}^{RR}\mu\left(\left((\sigma_e(t))\,\mathbf{b}_k^{\perp}\right)\otimes\mathbf{b}_k - \right. \\
&\left. -(\sigma_e(t))\mathbf{b}_k\right), \\
&\int_0^t \sigma_d^s(\tau)\,\mathrm{d}\tau = \sum_{kk=1}^{RR}\mu\left(\left(\left(\int_0^t \sigma_d(\tau)\mathbf{b}_k^{\perp}\,\mathrm{d}\tau\right)\otimes\mathbf{b}_k - \right.\right. \\
&\left.\left. -\int_0^t \sigma_d(\tau)\mathbf{b}_k\,\mathrm{d}\tau\right)\right),
\end{aligned}$$

$$\begin{aligned}
&\langle\rho\rangle[\mathbf{F}]\sigma_e(t) + [\mathbf{B}]\mathbf{u}(t) - \\
&-\langle\rho\rangle[\mathbf{F}]\sigma_e^s(t) - [\mathbf{B}]\mathbf{u}^s(t) + \mathbf{y}(t) = 0,
\end{aligned}$$

(compatibility equations for displacements) (9.b)

$$\langle\rho\rangle[\mathbf{M}]^{-1}\left(\int_0^t \sigma_d(\tau)\,\mathrm{d}\tau\right)+\langle\rho\rangle[\Phi]\mathbf{v}(t)-$$
$$\langle\rho\rangle[\mathbf{M}]^{-1}\left(\int_0^t \sigma_d^s(\tau)\,\mathrm{d}\tau\right)-[\Phi]\mathbf{v}^s(t)+$$
$$+\mathbf{y}(t)=0, \qquad \text{(compatibility equations for velocities of the displacements)} \qquad (9.c)$$

$$\mathbf{u}(t)=\mathbf{v}(t)^*\left[\frac{\mathbf{P}_d(t)}{\dot{P}_d^j(t)},\quad j=1,\dots,ns\right], \qquad \text{(connection between the displacements and velocities of the displacements)} \qquad (9.d)$$

$$(\sigma_e(t)+\sigma_d(t)-\sigma_L)^*\mathbf{y}(t)=0, \qquad \text{(switch)} \qquad (9.e)$$

$$\mathbf{y}(t)\geq 0, \qquad \text{(sign restriction)} \qquad (9.f)$$

$$\hat{\hat{P}}{}_i^j\big|_{S_{I_i}}-\overline{P}{}_i^j\big|_{S_{I_i}}=0,\ i=1,\dots,S_{I_n},\ j=1,\dots,s,$$
$$\overline{u}{}_i^j\big|_{S_{II_i}}-\hat{\hat{u}}{}_i^j\big|_{S_{II_i}}=0,\ i=1,\dots,S_{II_n},\ j=1,\dots,z, \qquad \text{(boundary conditions)} \qquad (9.g)$$

$$\sigma_e(t_0)+\sigma_d(t_0)=0, \qquad \text{(initial conditions)} \qquad (9.h)$$

$$(t,\sigma_e(t),\sigma_d(t),P_e(t),P_d(t),\mathbf{u}(t),\mathbf{v}(t),\mathbf{y}(t))\in[\mathbf{t}_1,\mathbf{t}_2]\times\Re^{2\times(mzG)+4\times(ns)},$$
$$\forall t,\quad t\in[t_1,t_2].$$

In the compatibility equations the effect of internal friction has appeared.

At the optimal function both the primal and the dual constraints are valid. Assuming the initial and boundary conditions are satisfied and the stress limit is neglected, the complementary problem is:

$$-\mathbf{P}_e(t)-\mathbf{P}_d(t)+\mathbf{P}(t)=0,$$

$$[\mathbf{B}]^*\langle\rho\rangle\sigma_e(t)-[\mathbf{B}]^*\langle\rho\rangle\sigma_e^s(t)+\mathbf{P}_e(t)=0,$$

$$[\Phi]^*\langle\rho\rangle\int_0^t\sigma_d(\tau)\,\mathrm{d}\tau-[\Phi]^*\langle\rho\rangle\int_0^t\sigma_d^s(\tau)\,\mathrm{d}\tau+\int_0^t\mathbf{P}_d(\tau)\,\mathrm{d}\tau=0,$$

$$[\mathbf{F}]\sigma_e(t)+[\mathbf{B}]\mathbf{u}(t)-\langle\rho\rangle[\mathbf{F}]\sigma_e^s(t)-[\mathbf{B}]\mathbf{u}^s(t)=0,$$

$$[\mathbf{M}]^{-1}\left(\int_0^t \sigma_d(\tau)\,\mathrm{d}\tau\right) + [\Phi]\mathbf{v}(t) - [\mathbf{M}]^{-1}\left(\int_0^t \sigma_d^s(\tau)\,\mathrm{d}\tau\right) - [\Phi]\mathbf{v}^s(t) = 0\,.$$

Expressing the stresses

$$\sigma_e(t) = -[\mathbf{F}]^{-1}[\mathbf{B}]\mathbf{u}(t) + [\mathbf{F}]^{-1}[\mathbf{F}]\sigma_e^s(t) + [\mathbf{F}]^{-1}[\mathbf{B}]\mathbf{u}^s(t),$$

$$\int_0^t \sigma_d(\tau)\,\mathrm{d}\tau = -[\mathbf{M}][\Phi]\mathbf{v}(t) + [\mathbf{E}]\left(\int_0^t \sigma_d^s(\tau)\,\mathrm{d}\tau\right) + [\mathbf{M}][\Phi]\mathbf{v}^s(t)$$

and substituting into the complementary problem:

$$-\langle\rho\rangle[\mathbf{B}]^*[\mathbf{F}]^{-1}[\mathbf{B}]\mathbf{u}(t) - \langle\rho\rangle[\mathbf{B}]^*[\mathbf{F}]^{-1}[\mathbf{B}]\left[\frac{\mathbf{P}_d(t)}{\dot{P}_d^j(t)},\ j=1,\dots,ns\right]\mathbf{v}(t) +$$

$$+\langle\rho\rangle[\Phi]^*[\mathbf{M}][\Phi]\mathbf{a}(t) - [\Phi]^*[\mathbf{M}][\Phi]\left\{\mathbf{v}(t)\left[\frac{\mathbf{P}_d(t)}{\dot{P}_d^j(t)},\ j=1,\dots,ns\right]^{-1} +\right.$$

$$\left. +\mathbf{u}^*(t)\left[\frac{\mathbf{P}_d(t)}{\dot{P}_d^j(t)},\ j=1,\dots,ns\right]^{-1}\right\} = \mathbf{p}(t).$$

Considering that $[M]\ddot{\mathbf{u}}^j(t)\mathbf{u}^j(t) = \frac{1}{2}[M]\dot{\mathbf{u}}^j(t)^2$, one can write $\ddot{\mathbf{u}}^j(t) = = \frac{1}{2}[M]\left[\frac{\dot{\mathbf{u}}^j(t)^2}{u^j(t)},\ j=1,\dots,ns\right]$, and $\mathbf{u}(t) = \dot{\mathbf{u}}(t)^*\left[\frac{\mathbf{P}_d(t)}{\dot{P}_d^j(t)},\ j=1,\dots,ns\right]$. By the use of these two expressions the damping matrix can be written in the following form:

$$[\mathbf{C}] = -\langle\rho\rangle[\mathbf{B}]^*[\mathbf{F}]^{-1}[\mathbf{B}]\left[\frac{\mathbf{P}_d(t)}{\dot{P}_d^j(t)},\ j=1,\dots,ns\right] -$$

$$- [\Phi]^*[\mathbf{M}][\Phi]\left[\frac{\mathbf{P}_d(t)}{\dot{P}_d^j(t)},\ j=1,\dots,ns\right]. \tag{10}$$

It can be seen that the damping matrix is the linear combination of the stiffness matrix and the mass matrix, as it is supposed in the literature [10]. The coefficients of that linear combination depend on the external dynamical forces and their derivatives. The external force can be decomposed into dynamical force and the ‘elastic’ force. The second one causes only elastic deformations. The proportion of these two forces changes in time and they are influenced by the geometry of the structure and the boundary conditions. This is the reason why the validity of the damping matrices determined by experience is rather narrow.

It is easy to create the well-known form of complementary problem of damping. The equation can be written in the following form:

$$\langle\rho\rangle[\Phi]^*[\mathbf{M}][\Phi]\ddot{\mathbf{u}}(t) + [\mathbf{C}]\dot{\mathbf{u}}(t) + \langle\rho\rangle[\mathbf{B}]^*[\mathbf{F}][B]\mathbf{u}(t) = \mathbf{p}(t), \tag{11}$$

where the stress limit and the boundary conditions are not taken into consideration, yet.

4.2.2. Application of internal variables

If the damping is not connected to Coulomb type internal friction but it is expressed by the internal properties of the phenomena, matrices [**A**] and [**L**] should be created by experience and they contain material coefficients belonging to the part of elastic and kinematic damping, respectively. A damping condition should be created what involves a stress limit. If the stresses reach this limit the damping is started. Furthermore an additional equation is needed able to control the existence of the damping. For the sake of simplicity the temperature T is taken into consideration with a given value which means that the effect of the temperature is neglected. The internal variables are signed by $\xi(t)$. The primal problem can be written in the following form:

$$\begin{aligned}\min :& \Bigg\{\Bigg(\frac{1}{2}\sigma_e^*(t)\langle\rho\rangle[\mathbf{F}]\sigma_e(t)+ \\ &+\frac{1}{2}\left(\int_0^t \sigma_d(\tau)\,\mathrm{d}\tau\right)^*\langle\rho\rangle[\mathbf{M}]^{-1}\left(\int_0^t \sigma_d(\tau)\,\mathrm{d}\tau\right) \\ &+T\xi_e^*(t)\langle\rho\rangle[\mathbf{A}]\xi_e(t)+ \\ &+T\left(\int_0^t \xi_d(\tau)\,\mathrm{d}\tau\right)^*\langle\rho\rangle[\mathbf{L}]\left(\int_0^t \xi_d(\tau)\,\mathrm{d}\tau\right)\Bigg)\Bigg\},\end{aligned} \qquad \text{(coenergy function)} \tag{12.a}$$

$$-\mathbf{P}_e(t) - \mathbf{P}_d(t) + \mathbf{P}(t) = 0, \qquad \text{(decomposition of the external load)} \tag{12.b}$$

$$\mathbf{f}(\sigma_e, \sigma_d, \text{material coefficients}) \le 0, \qquad \text{(damping conditions)} \tag{12.c}$$

$$\mathbf{f}(\sigma_e, \sigma_d, \text{material coefficients})\,\xi(t) = 0, \qquad \text{(control equations)} \tag{12.d}$$

$$\begin{aligned}&\langle\rho\rangle[\mathbf{B}]^*\sigma_e(t) + \langle\rho\rangle T[\mathbf{B}]^*\xi_e(t)+ \\ &+\mathbf{P}_e(t) = 0,\end{aligned} \qquad \text{(equilibrium equations for elastic part)} \tag{12.e}$$

$$\langle\rho\rangle[\Phi]^*\int_0^t \sigma_d(\tau)\,\mathrm{d}\tau + \langle\rho\rangle T[\Phi]^*\int_0^t \xi_d(\tau)\,\mathrm{d}\tau + \int_0^t \mathbf{P}_d(\tau)\,\mathrm{d}\tau = 0, \quad \text{(equilibrium equations for dynamic part)} \tag{12.f}$$

$$\sigma_e(t) + \sigma_d(t) - \sigma_L \leq 0, \quad \text{(stress limit)} \tag{12.g}$$

$$\begin{aligned} &\hat{\hat{P}}{}_i^j\big|_{S_{I_i}} - \overline{P}{}_i^j\big|_{S_{I_i}} = 0,\ i = 1,\ldots,S_{I_n},\ j = 1,\ldots,s,\\ &\overline{u}{}_i^j\big|_{S_{II_i}} - \hat{\hat{u}}{}_i^j\big|_{S_{II_i}} = 0,\ i = 1,\ldots,S_{II_n},\ j = 1,\ldots,z, \end{aligned} \quad \text{(boundary conditions)} \tag{12.h}$$

$$\begin{aligned} &\overline{\mathbf{u}}(t)\big|_{S_{II}} = 0,\ \overline{\mathbf{P}}(t)\big|_{S_I} = 0,\\ &\sigma_e(t_0) + \sigma_d(t_0) = 0, \end{aligned} \quad \text{(initial conditions)} \tag{12.i}$$

$$(t, \sigma_e(t), \sigma_d(t), \xi_e(t), \xi_d(t), \mathbf{P}_e(t), \mathbf{P}_d(t)) \in [t_1, t_2] \times \Re^{6mzG},$$

$$\forall t, \quad t \in [t_1, t_2].$$

According to the Wolf's dual theory the dual problem can be written in the following way:

$$\begin{aligned} \min: \Bigg\{ &\frac{1}{2}\Bigg(\sigma_e^*(t)\langle\rho\rangle[\mathbf{F}]\sigma_e(t) + \\ &+ \left(\int_0^t \sigma_d(\tau)\,\mathrm{d}\tau\right)^* \langle\rho\rangle[\mathbf{M}]^{-1} \times \\ &\times \left(\int_0^t \sigma_d(\tau)\,\mathrm{d}\tau\right) + T\xi_e^*(t)\langle\rho\rangle[\mathbf{A}]\xi_e(t) + \\ &+ T\left(\int_0^t \xi_d(\tau)\,\mathrm{d}\tau\right)^* \langle\rho\rangle[\mathbf{L}]\left(\int_0^t \xi_d(\tau)\,\mathrm{d}\tau\right)\Bigg) - \\ &- \left(\int_0^t \mathbf{P}_d(\tau)\,\mathrm{d}\tau\right)^* \mathbf{v}(t) - (\mathbf{P}_e(t))^*\mathbf{u}(t)\Bigg\}, \end{aligned} \quad \text{(energy function)} \tag{13.a}$$

$$\langle\rho\rangle[\mathbf{F}]\sigma_e(t) + [\mathbf{B}]\mathbf{u}(t) + \mathbf{y}(t) = 0, \quad \text{(compatibility equations of the displacements)} \tag{13.b}$$

$$\langle\rho\rangle[\mathbf{M}]^{-1}\left(\int_0^t (\sigma_d(\tau))\,\mathrm{d}\tau\right) + [\Phi]\mathbf{v}(t) + \mathbf{y}(t) = 0, \quad \text{(compatibility equations of the displacement velocities)} \tag{13.c}$$

$$T\langle\rho\rangle[\mathbf{A}]\xi_e(t) + T\langle\rho\rangle[\mathbf{L}]\frac{\xi_d(t)}{\dot{\xi}_d(t)} + +[\mathbf{B}]\mathbf{u}(t) + T\langle\rho\rangle[\Phi]^*\frac{\xi_d(t)}{\dot{\xi}_d(t)}\mathbf{v}(t) = 0, \quad \text{(compatibility equations of the internal variables)} \tag{13.d}$$

$$\mathbf{u}(t) = \mathbf{v}(t)^*\left[\frac{\mathbf{P}_d(t)}{\dot{P}_d^j(t)},\ j = 1,\ldots,ns\right], \quad \text{(connection between the displacements and the displacement velocities)} \tag{13.e}$$

$$\mathbf{f}(\sigma_e, \sigma_d,\ \text{material coefficients})\,\lambda(t) = 0, \quad \text{(control equations)} \tag{13.f}$$

$$(\sigma_e(t) + \sigma_d(t) - \sigma_L)^*\mathbf{y}(t) = 0,\quad \lambda(t) \geq 0,\ \mathbf{y}(t) \geq 0, \quad \text{(sign constraints)} \tag{13.g}$$

$$\begin{aligned} &\hat{\hat{P}}{}_i^j\big|_{S_{I_i}} - \overline{P}{}_i^j\big|_{S_{I_i}} = 0,\ i = 1,\ldots,S_{I_n},\\ &\qquad j = 1,\ldots,s,\\ &\overline{u}_i^j\big|_{S_{II_i}} - \hat{\hat{u}}{}_i^j\big|_{S_{II_i}} = 0,\ i = 1,\ldots,S_{II_n},\\ &\qquad j = 1,\ldots,z, \end{aligned} \quad \text{(boundary conditions)} \tag{13.h}$$

$$\sigma_e(t_0) + \sigma_d(t_0) = 0, \quad \text{(initial conditions)} \tag{13.i}$$

$$(t, \sigma_e(t), \sigma_d(t), \xi_e(t), \xi_d(t), \mathbf{P}_e(t), \mathbf{P}_d(t), \mathbf{u}(t), \mathbf{v}(t), \mathbf{y}(t), \lambda(t)) \in$$

$$\in [\mathbf{t}_1, \mathbf{t}_2] \times \Re^{6mzG+4ns}, \quad \forall t, \quad t \in [t_1, t_2],$$

where $\lambda(t)$ is vector of the dual variables connected to the damping.

5. Conclusion

Comparing both models created by the use of Coulomb type internal friction and the use of internal variables we can write the following:

Coulomb type internal friction	internal variables
• before the application of the model one can decide whether damping exists or not,	• the friction condition controls when damping exists
• the displacements due to the internal friction are determined by the Coulomb type model,	• the direction of displacements due to the damping is determined by the direction of the gradient of the friction condition,
• Coulomb type model is used in the expressions of energies,	• in the expression of energies the formula for damping is given in general form,
• using this model the dissipative energy cannot be determined,	• the dissipative energy can be determined by the use of internal variables,
• Coulomb type model can be derived as a special case of the model created by the use of internal variables,	• the different types of damping models can be created as special cases,
• the number of the unknowns are identical with the unknowns of model without damping.	• the number of the unknowns is significantly higher.

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