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ROLE OF THE GEOMETRY IN GPS POSITIONING

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Abstract

'Bad geometry' of GPS positioning degrades the precision; this is expressed in a multiplier called DOP. There are some erroneous statements concerning DOP (every DOP value is always greater than 1; vertical DOP is always greater than the horizontal one; $PDOP$ is inversely proportional to the volume of tetrahedron, etc.). One should treat the 'mathematical' and the 'geometrical' DOP differently: former is the square root of covariance matrix, latter is the reciprocal of the volume of a tetrahedron. For statistical analysis of DOP its values were calculated from computer generated satellite configurations. There are some illustrations (skyplots) demonstrating the simulation program of quasi GPS satellite configurations.

Keywords: accuracy of GPS positioning, dilution of precision (*DOP*), simulated satellite configurations, skyplot.

1. Geometry of Positioning

Any results of positioning are an interval surrounding the point of the error-free solution, because of the random errors. This interval is also called the *rms* (root mean square) error of the result and can be calculated from the *rms* error of the quantities measured using error propagation law.

When we say: the positioning 'has bad geometry', it means that the interval characterizing the results (and the *rms* error either) is wider than 'usual'. In GPS positioning the effect degrading the precision can be taken into consideration by a multiplier called DOP (Dilution of Precision). The same satellite geometry effects differently in horizontal and vertical positioning. That is why the value representing the precision of spatial positioning ($PDOP$) is reduced to horizontal ($HDOP$) and vertical ($VDOP$) components. The precision depends on the timing, represented by $TDOP$, because of the peculiarity of the positioning. $PDOP$ and $TDOP$ show the full precision ($GDOP$) of the GPS positioning.

The orbital elements of satellites are known, so their position relative to the observation point can be calculated at any time. So DOP values concerning the planned position and time can be forecasted. Although, since the full 'build-up' of GPS this question has lost its importance, misunderstandings concerning DOP values are still waiting for clearing up.

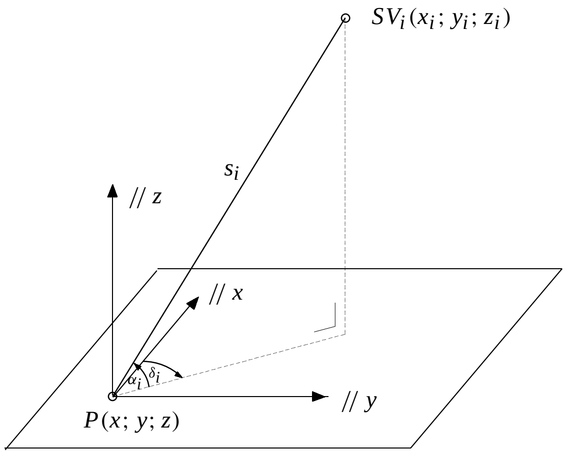

Fig. 1. Spatial positioning by distance measurements

2. Opinions about *DOP* Are Mistakable

Take a look at the three statements declared most frequently:

1. *Every DOP value is always greater than 1.* So if the *DOP* expresses the *dilution* of precision on one hand and it is a multiplier on the other hand, then it is expected 'by all means' to be greater than one.
2. Height determination is less accurate than horizontal positioning, so *VDOP is always greater than HDOP*. According to the most popular argumentation it is because the heighting is always 'one-sided' (non-symmetric) as only the satellites above the horizon can be observed.
3. *In case of four satellites PDOP value is inversely proportional to the volume of a tetrahedron having vertices on the sphere radii unit on the direction of satellites.*

DOP values are equally used both in 'mathematical' in 'geometrical' sense. At the separation of these two interpretations we suppose that

- only one position is measured,
- every measurement has the same weight of 1,
- measurements were carried out in time-synchronization so the distances measured are 'real'.

3. The 'Mathematical' *DOP*

The geometry of the positioning is shown in *Fig. 1.*

Given

$$\begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix};$$

the distance measured is s_i; the solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

The intermediate equation: $s_i = \sqrt{(x_i - x)^2 + (y_i - y)^2 + (z_i - z)^2}$. The i^{th} row in figure matrix **A**:

$$\frac{\partial s_i}{\partial x} = -\cos\delta_i \cos\alpha_i; \qquad \frac{\partial s_i}{\partial y} = -\sin\delta_i \cos\alpha_i; \qquad \frac{\partial s_i}{\partial z} = -\sin\alpha_i,$$

where δ_i is the azimuth, α_i is the elevation.

The matrix of coefficients in normal equations:

$$\mathbf{A}^*\mathbf{A} = \mathbf{N} = \begin{bmatrix} xx & xy & xz \\ byx & yy & yz \\ zx & zy & zz \end{bmatrix},$$

where

$$\begin{array}{ll} xx = \sum\cos^2\delta_i\cos^2\alpha_i; & xy = yx = \sum\sin\delta_i\cos\delta_i\cos^2\alpha_i; \\ yy = \sum\sin^2\delta_i\cos^2\alpha_i; & xz = zx = \sum\cos\delta_i\cos\alpha_i\sin\alpha_i; \\ zz = \sum\sin^2\alpha_i; & yz = zy = \sum\sin\delta_i\cos\alpha_i\sin\alpha_i. \end{array}$$

The covariance matrix:

$$\mathbf{M} = \left(\mathbf{A}^*\mathbf{A}\right)^{-1} = \mathbf{N}^{-1},$$

if the matrix **N** has an inverse.

The variance of the coordinates:

$$m_x^2 = \frac{(yy)(zz) - (yz)^2}{\det}; \quad m_y^2 = \frac{(xx)(zz) - (xz)^2}{\det}; \quad m_z^2 = \frac{(yy)(xx) - (yx)^2}{\det};$$

$$\det = (xx)(yy)(zz) - 2(xy)(xz)(yz) - (xx)(yz)^2 - (yy)(xz)^2 - (zz)(xy)^2.$$

The DOP values:

$$HDOP = \sqrt{m_x^2 + m_y^2}; \quad VDOP = \sqrt{m_z^2};$$

$$PDOP = \sqrt{m_x^2 + m_y^2 + m_z^2} = \sqrt{(HDOP)^2 + (VDOP)^2}.$$

So

$$PDOP = \sqrt{Sp\mathbf{M}} = \sqrt{Sp\,(\mathbf{A}^*\mathbf{A})^{-1}}$$

is invariant.

The ‘mathematical’ DOP is the square root of the trace of covariance matrix.

4. The ‘Geometrical’ DOP

The ‘geometrical’ DOP is the reciprocal of the volume of the tetrahedron described in statement no. 3. It is popular, because

– easy to calculate,
– invariant as the ‘mathematical’ DOP,
– based on four satellite observation.

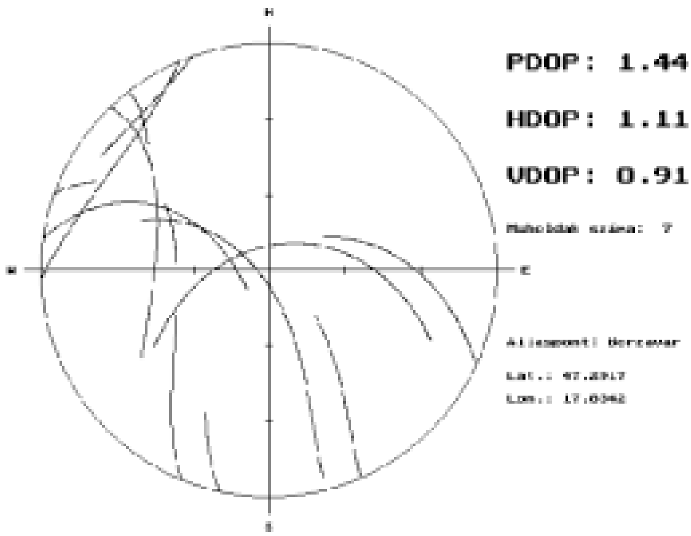

Fig. 2. Satellites over the horizon, Earth rotation ‘turned on’, time: ca. 5 hours after the ‘basic’ configuration (Műholdak száma=number of SVs visible; Álláspont=observation point)

The ‘geometrical’ $PDOP$ has at least the same serious disadvantages:

– cannot be used with more than four satellites,
– its components ($HDOP$ and $VDOP$) can hardly be interpreted geometrically,
– for zenith symmetric constellation the ‘mathematical’ $PDOP$ value is not the lowest when the volume of the tetrahedron is the highest,
– the volume of the tetrahedron becomes zero (and ‘geometrical’ $PDOP$ infinitely high) in every case, when the four satellites are on the same plane. The ‘mathematical’ $PDOP$ becomes infinitely high only if the point of observation is also in this plane.

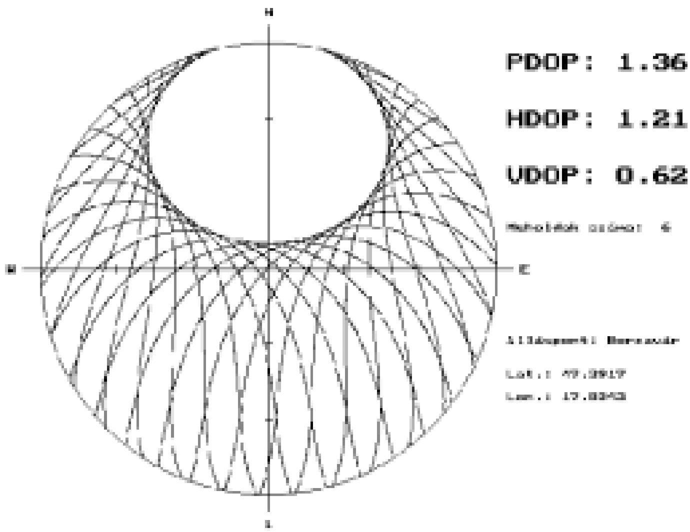

Fig. 3. Satellites over the horizon, Earth rotation 'turned on', time: random selected in the 12…24 h interval after the 'basic' configuration (Műholdak száma=number of SVs visible; Álláspont=observation point)

5. Is Statement No. 1 True?

Is there any satellite constellation when the $PDOP$ is smaller than unit? We have to examine both 'geometrical' and 'mathematical' $PDOP$ value, so only four satellites have to be taken into consideration in a configuration optimal for 'geometrical' $PDOP$: one satellite towards the zenith, the other three on the horizon of α elevation with azimuth difference of 120°. Results are shown in *Table 1*.

For more than four satellites (space vehicles, SVs) only the 'mathematical' $PDOP$ ($HDOP$, $VDOP$) was examined. The satellite configuration: 1 SV towards the zenith, the rest was on the horizon of $\alpha^* = \arcsin \frac{1}{n-1}$ elevation with the same azimuth difference. Results are shown in *Table 2*.

If the number of SVs increases, as a limit α becomes 0, $HDOP$ becomes 0, $VDOP$ becomes 1, so $PDOP$ becomes 1 either.

It seems that in this configuration *for limited n number of SVs the 'mathematical' $PDOP$ is always greater than 1.*

This configuration is, however, not the most favourable for the 'mathematical' $PDOP$. Since the $HDOP$ is calculated from the variance of two (x and y) coordinates and the $VDOP$ comes from only one (z), we can suppose the $PDOP$

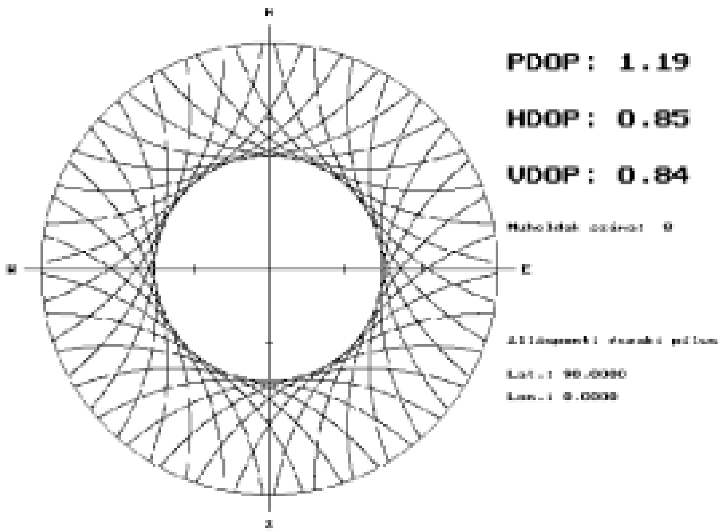

Fig. 4. North Pole station, satellites over the horizon, Earth rotation ‘turned on’, time: random selected in the 12…24 h interval after the ‘basic’ configuration (Műholdak száma=number of SVs visible; Álláspont=observation point)

Table 1.

$\alpha°$	‘geometrical’ $PDOP$	‘mathematical’ $PDOP$
0	0.3849	1.5275
5	0.4249	1.5236
10	0.4803	1.5139
15	0.5566	1.5039
20	0.6625	1.5001
25	0.8116	1.5081
30	1.0264	1.5327
35	1.3452	1.5781
40	1.8362	1.6488
45	2.6283	1.7512
60	11.4920	2.3751
75	168.6300	4.4908

Remarks: ‘geometrical’ $PDOP = 1$, if $\alpha = 29°28'44.4''$;
‘mathematical’ $PDOP$ has a minimum value of 1.5,
if $\alpha = 19°28'16.4''$.

value the lowest, if there is only double amount of satellites in the horizon as towards the zenith. Calculations concerning this configuration can be found in *Table 3*.

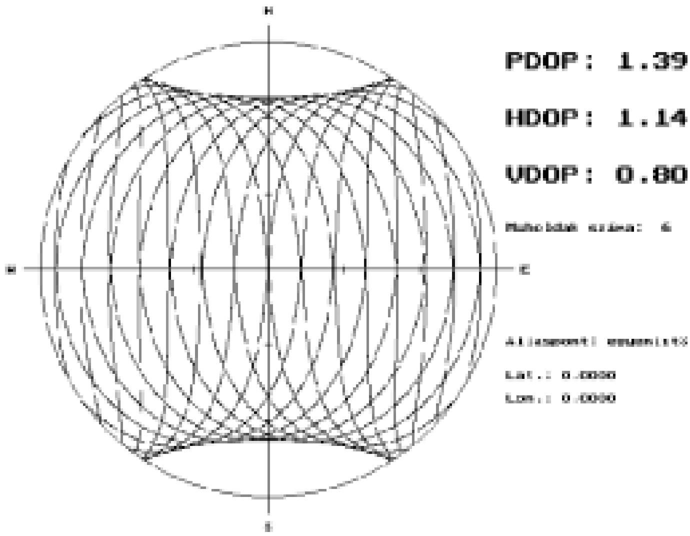

Fig. 5. Observation latitude 0°, longitude 0°, satellites over the horizon, Earth rotation 'turned on', time: random selected in the 12…24 h interval after the 'basic' configuration(Műholdak száma=number of SVs visible; Álláspont=observation point)

Table 2.

n	α	$HDOP$	$VDOP$	$PDOP$
4	α^*	1.22	0.87	1.5
	0	1.15	1	1.53
5	α^*	1.03	0.89	1.37
	0	1	1	1.41
6	α^*	0.91	0.91	1.29
	0	0.89	1	1.34
7	α^*	0.83	0.93	1.24
	0	0.82	1	1.29
8	α^*	0.76	0.94	1.21
	0	0.76	1	1.25
9	α^*	0.71	0.94	1.18
	0	0.71	1	1.22
10	α^*	0.67	0.95	1.16
x	0	0.67	1	1.18

It can be seen that *for nine SVs* $(PDOP)_{\min} = 1$ *for more it gets under 1.*

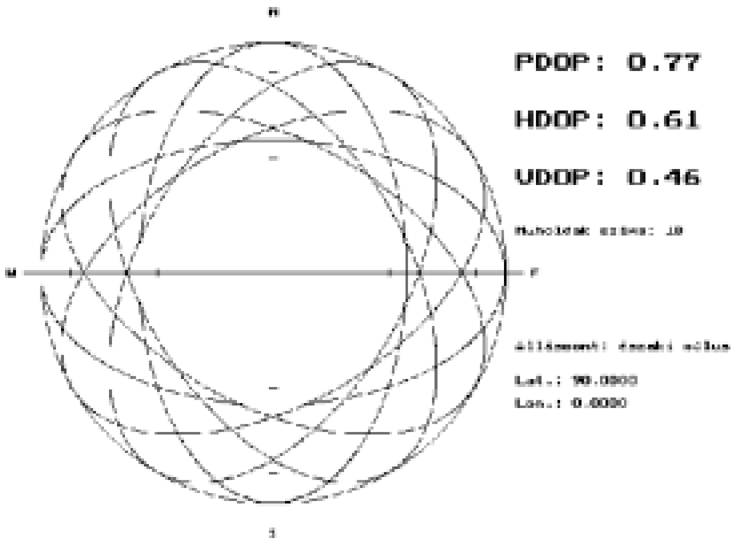

Fig. 6. ‘North Pole’ of geocentrum, all satellites, Earth rotation ‘turned off’, time: random selected in the 12…24 h interval after the ‘basic’ configuration (Műholdak száma=number of SVs visible; Álláspont=observation point)

Table 3.

n	xy	z	$HDOP$	$VDOP$	$PDOP$
5	3	2	1.15	0.71	1.35
6	4	2	1	0.71	1.23
7	5	2	0.89	0.71	1.14
8	5	3	0.89	0.58	1.06
9	6	3	0.82	0.58	1
10	7	3	0.76	0.58	0.95

6. Simulated Satellite Configurations

For statistical analysis of $PDOP$, the measure-planning forecast softwares cannot be used because they serve ‘geometrical’ $PDOP$ (in the case of more than four SVs the smallest of the different values). So, the ‘mathematical’ $PDOP$ is always less than forecast values: the difference can reach very high amount. It seemed to be practical to simulate satellite configurations by computer.

In the simpler case (hereinafter called the case of random appearing SVs)

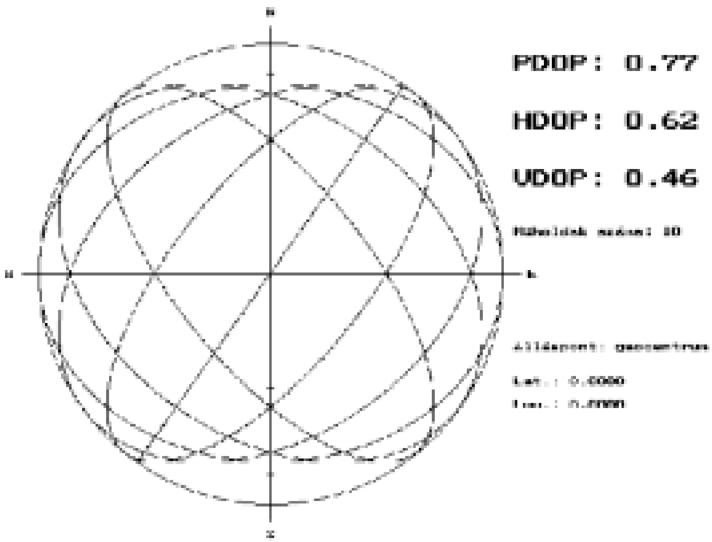

Fig. 7. Observation latitude 0°, longitude 0° of geocentrum, all satellites, Earth rotation ‘turned off’, time: random selected in the 12…24 h interval after the ‘basic’ configuration (Műholdak száma=number of SVs visible; Álláspont=observation point)

the azimuth and elevation were random generated. There were generated 10 series including 1000 elements each. After classification of $PDOP$ values the arithmetic means and standard deviations were calculated. Results are shown in *Table 4.*

Data show:

– the more SVs, the lower $PDOP$ values,
– $PDOP$ values smaller than 1 can occur only with more than 9 SVs.

A more intelligent simulation program (called the case of quasi GPS SVs) tries to get similar to ‘real’ GPS:

– SVs revolve around the spherical Earth on a circular orbit (with constant angular velocity),
– by default there are 18 SVs on 6 orbital planes.

By changing these parameters any kind of satellite system can be assembled because of the Earth radius, orbital radius, the number and the inclination of the orbital planes, the number of SVs and their position on the planes, the ratio of

Table 4.

n		0-1	1-1.5	1.5-2	2-2.5	2.5-3.5	3.5-5	5-8	8-13	13-20	20-∞
		Number of cases, when *PDOP* is between the limits									
3	k	0	0	64.6	147.5	202.8	166.3	152.3	99.1	57.3	110.1
	σ	0	0	7	12	10	14	9	9	6	7
4	k	0	0	325.5	226.7	215.7	115.8	71.3	26.7	10.4	7.9
	σ	0	0	20	14	13	10	8	4	3	2
5	k	0	1407	485.4	178.3	119.2	48.7	20.8	5.2	1.2	0.5
	σ	0	7	14	9	9	3	5	2	1	1
6	k	0	433.3	387.7	103.5	55.1	14.6	4.9	0.8	0.1	0
	σ	0	20	19	8	5	3	2	1	0	0
7	k	0	683.4	245.1	47.4	18.4	4.5	1.1	0.1	0	0
	σ	0	8	9	5	4	2	1	0	0	0
8	k	0	838.5	132.4	20.8	6.8	1.5	0	0	0	0
	σ	0	9	7	5	3	2	0	0	0	0
9	k	0	919.8	68.8	8.9	2.2	0.3	0	0	0	0
	σ	0	11	12	3	2	0	0	0	0	0
10	k	123.4	841.9	30.6	3.3	0.8	0	0	0	0	0
	σ	13	14	5	2	2	0	0	0	0	0
11	k	380.3	605.2	13.5	0.7	0.2	0.1	0	0	0	0
	σ	17	15	5	1	0	0	0	0	0	0

Table 5.

n		0-1	1-1.5	1.5-2	2-2.5	2.5-3	3.5-5	5-8	8-13	13-20	20-∞
		Number of cases, when PDOP is between the limits									
3	k	0	0	5.1	10.9	17.3	17.3	17.2	12.5	5.6	14.1
	σ	0	0	2	3	5	3	4	2	2	3
4	k	0	0	24.6	19.3	24.1	16.3	10.0	2.9	1.7	1.1
	σ	0	0	4	4	2	4	3	2	1	1
5	k	0	6.0	42.1	25.0	16.6	6.2	3.3	0.7	0.1	0
	σ	0	2	2	5	4	2	1	1	0	0
6	k	0	27.3	44.7	15.6	8.4	3.5	0.5	0	0	0
	σ	0	5	6	2	3	1	1	0	0	0
7	k	0	47.3	37.2	10.4	4.6	0.4	0.1	0	0	0
	σ	0	3	4	2	2	1	0	0	0	0
8	k	0	64.9	27.2	6.3	1.2	0.3	0.1	0	0	0
	σ	0	5	4	2	1	1	0	0	0	0

satellite revolution and Earth rotation angular velocities can also be changed. The observation circumstances can also be changed by ‘curtains’. By the simulation of continuous orbital movement moving skyplots with momentary DOP values can be displayed.

Configurations of quasi GPS satellites were also evaluated. In this case latitude and longitude of observation point was to be given. Computer-generated random value was the orbital position of the satellite No. 1 of the system. There were only eight ‘visible’ satellites because of the 15° elevation mask and the 18 SVs in the system. The evaluation of ten series, including 100 configurations each was the same as in the case of random appearing SVs. Results are shown in *Table 5*.

Fig. 2 shows a traditional skyplot. The same skyplot 12 hours later is in *Fig. 3*. *Fig. 4* is for North Pole, *Fig. 5* is for the point of Equator at longitude $0°$. In *Figs 6* and *7* there are two skyplots for the geocentrum (geocentrum means zero Earth radius). Every DOP value is time-independent in the geocentrum.

7. Why is Heighting Less Accurate?

In every examined case the $HDOP$ value was greater than $VDOP$. However, in practice, height can be determined less accurate than horizontal position.

The geometry of 'one-sided' (non-symmetric) positioning could support the practical experience – if there would be such an effect. But there isn't any and it is easy to demonstrate why. If we calculate the change in the trace of the covariance matrix as a consequence, on one hand in the change of azimuth δ and elevation α, on the other hand in the change of the opposite (azimuth $180° + \delta$ and elevation $-\alpha$) we get the same results. One-sided positioning does not exist in connection with random errors (including DOPs).

There is another reason for less accurate heighting. Suppose that the same systematic error exists in every distance measured. Such an error is caused by the inaccurate knowledge of tropospheric correction. In this case if the azimuth differences are more or less the same the systematic errors neutralize each other and have no effect in horizontal positioning. The systematic errors in vertical distances (components) cause errors in height determined, because there are no satellites 'visible' on the opposite side (at nadir). One-sided heighting does exists due to the systematic errors causing less accurate height determination. Systematic errors, however, are not expressed in DOP values designed for showing the role of random errors in the geometry of GPS positioning.

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