periodica polytechnica

*Civil Engineering*
*58/1 (2014) 63–69*
*doi: 10.3311/PPci.7433*
*http://periodicapolytechnica.org/ci*


RESEARCH ARTICLE

# Hyperbolic transition curve

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Received 2013-04-14, accepted 2013-12-23

***Abstract***

*This paper describes a transistion curve geometry based on sine hyperbolic function. This geometry combines the advantages of the widely used chlotoide and cosine transition curves. The hyperbolic transition curve is parameterized, the characteristics of the transition curve depends on the parameter, creating a continuous transition between the chlotoide and the cosine geometry.*

***Keywords***
*transition curve · clothoide*

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## 1 Introduction

In case of clothoide transition curves the change of lateral acceleration is even, the values of the third order motion parameters are low. However at the ends of the transition curves the function is discontinuous which in case of high speeds may result unfavorable stresses according to the practical experiences. At other transition curve geometries (cosines, etc) the change of the third order parameters is continuous, but it results significantly higher values for them.

As the speed of railway vehicle increased the geometry of transition curves became more nad more important. New methods are developed to analyze the optimal geometry for connecting the sections of different curvature [1–3]. This paper describes the bases of a new, efficient transition curve geometry based on hyperbolic functions. At this hyperbolic transition curve the third order **h** vector – which is particularly important at the geometry design of high-speed railway tracks [4] – is continuous and its value is very close to those values of the clothoide transition curve. The exact characteristics of the hyperbolic transition curve depends on a $p$ parameter.

## 2 The curvature function of the hyperbolic transition curve

The hyperbolic transition curve is based on the well-known sine hyperbolic function, $sh(x)$ (Figure 1.a) [5]:

$$\mathrm{sh}\,(x) = \frac{e^x - e^{-x}}{2} \qquad (1)$$

If we want to use the sine hyperbolic function as a transition curve then we have to modify it to get a horizontal tangent at least in two points. It means that the derivate of the modified function must have at least two points with the value of 0 (Figure 1.b). The easiest way to do this is to shift the derivate function of the $\mathrm{sh}(x)$ (3) (Figure 2.b). From this the wanted function could be created by integration (4) (Figure 2.a):

$$\mathrm{sh}\,(x)' = \mathrm{ch}\,(x) \qquad (2)$$

$$f_1' = \mathrm{ch}\,(x) - z\,(p) \qquad (3)$$

$$f_1 = \int (\mathrm{ch}(x) - z(p))\, dx = \mathrm{sh}(x) - z(p)\, x \quad (4)$$

The rate of the shift is a parameter of $f_1$ function, so we could apply a $p$ parameter instead of $z(p)$ parameter. But for practical purposes it is subservient to use the $z(p) = \mathrm{ch}(p)$ form, because this way the local maximum and minimum places will be at $x = -p$ and $x = p$ values. So the $f_1$ function after this first step is:

$$f_1 = \mathrm{sh}(x) - \mathrm{ch}(p)\, x \quad (5)$$

At the local maximum and minimum places of $f_1$ function the tangent is horizontal. To have the start point of curvature function the function $f_1$ (Figure 2.a) is multiplied by -1 and its local minimum is moved into the pole (6) (7) (Figure 3):

$$f_2 = -f_1(x - p) + f_1(-p), \quad (6)$$

$$f_2 = -\mathrm{sh}(x - p) + \mathrm{sh}(-p) + x \cdot \mathrm{ch}(p). \quad (7)$$

To have the end point of the curvature function the local maximum value of function $f_2$ should be $1/R$ at point $X$. So we have to transform function $f_2$ accordingly (8) (9) (Figure 4):

$$f_3 = \frac{f_2(x)}{f_2(2p) \cdot R}, \quad (8)$$

$$f_3 = \frac{1}{2 \cdot R} \cdot \frac{\mathrm{sh}(p - x) - \mathrm{sh}(p) + x \cdot \mathrm{ch}(p)}{p \cdot \mathrm{ch}(p) - \mathrm{sh}(p)}. \quad (9)$$

Finally to have the curvature function of the transition curve in the desired form the function $f_3$ should be written with arc length parameterization, where $L$ denotes the total length of the transition curve:

$$G(l) = f_3\left(\frac{2p}{L} \cdot l\right), \quad (10)$$

$$G(l) = \frac{1}{2R} \cdot \frac{\mathrm{sh}\left(p - \frac{2p}{L} \cdot l\right) - \mathrm{sh}(p) + \frac{2p}{L} \cdot l \cdot \mathrm{ch}(p)}{p \cdot \mathrm{ch}(p) - \mathrm{sh}(p)}. \quad (11)$$

Figure 5 shows the curvature function of the hyperbolic transition curve for different $p$ parameter values. For comparison the figure also depicts the curvature functions of clothoide and cosines transition curves.

## 3 The function describing the transition curve

The curvature function of transition curves between straight section and curve is described by (11). From this the angle of the tangent ($\tau_l$) is:

$$\tau_l = \int_0^\tau d\tau = \int_0^l G_l dl$$
$$= \frac{1}{2LR} \frac{l^2 p^2 \cosh p - Llp \sinh p + L^2 \sinh \frac{lp}{L} \sinh\left(p - \frac{lp}{L}\right)}{p^2 \cosh p - p \sinh p}. \quad (12)$$

Based on $\tau_1 = f(l)$ function the equations of the perpendicular coordinates in the function of the length are [4]:

$$x = \int_0^x dx = \int_0^l \cos \tau_l dl \quad (13)$$

and

$$y = \int_0^y dy = \int_0^l \sin \tau_l dl. \quad (14)$$

These formulas can not be calculated directly by basic integrals. Instead the power series of $\sin\tau_l$and $\cos\tau_l$ functions are used to describe the perpendicular coordinates of the transition curve:

$$x = \int_0^l \cos \tau_l dl = \int_0^l \left(1 - \frac{\tau_l^2}{2!} + \frac{\tau_l^4}{4!} - \ldots\right) dl \quad (15)$$

$$y = \int_0^l \sin \tau_l dl = \int_0^l \left(\tau - \frac{\tau_l^3}{3!} + \frac{\tau_l^5}{5!} - \ldots\right) dl. \quad (16)$$

We have created the approximate formulas of $x$ and $y$ by taking the first two terms of the power series, but these are not included due to space constraints.

## 4 Determination of the most important parameters of the transition curve

### 4.1 The first derivate of the curvature function

The analysis of the first derivate of the curvature function is practically substituting the analysis of the third order motion characteristics (change of acceleration: $h$, m/s$^3$) [6], because the value of the third order motion parameter is primarily determined by the change of the curvature [7]. The first derivate of the curvature function of the hyperbolic transition curve (11):

$$\frac{dG}{dt} = \frac{p}{RL} \frac{\mathrm{ch}(p) - \mathrm{ch}\left(p - \frac{2pl}{L}\right)}{p \cdot \mathrm{ch}(p) - \mathrm{sh}(p)}. \quad (17)$$

### 4.2 The second derivate of the curvature function

Similarly to the relationship between the first derivate and the third order motion characteristics the fourth order motion characteristics ($m$, m/s$^4$) is primarily determined by the second derivate of the curvature function [6]. Derived from formula (17):

$$\frac{d^2G}{dt^2} = \frac{2p^2}{RL^2} \frac{\mathrm{sh}\left(p - \frac{2pl}{L}\right)}{p \cdot \mathrm{ch}(p) - \mathrm{sh}(p)}. \quad (18)$$

### 4.3 Geometrical parameters

The geometric parameters of the hyperbolic transition curve are described according to Figure 6.

The $X$ and $Y$ coordinates of the end of the transition curve could be derived from (17, 18) in case of $l = L$, which results:

| | $p = 1$ | $p = 5$ | $p = 20$ |
|---|---|---|---|
| $X$ | $L - 0,02287 \frac{L^3}{R^2}$ | $L - 0,02351 \frac{L^3}{R^2}$ | $L - 0,02445 \frac{L^3}{R^2}$ |
| $Y$ | $0,15046 \frac{L^2}{R} - 0,00276 \frac{L^4}{R^3}$ | $0,15584 \frac{L^2}{R} - 0,00282 \frac{L^4}{R^3}$ | $0,16291 \frac{L^2}{R} - 0,00291 \frac{L^4}{R^3}$ |

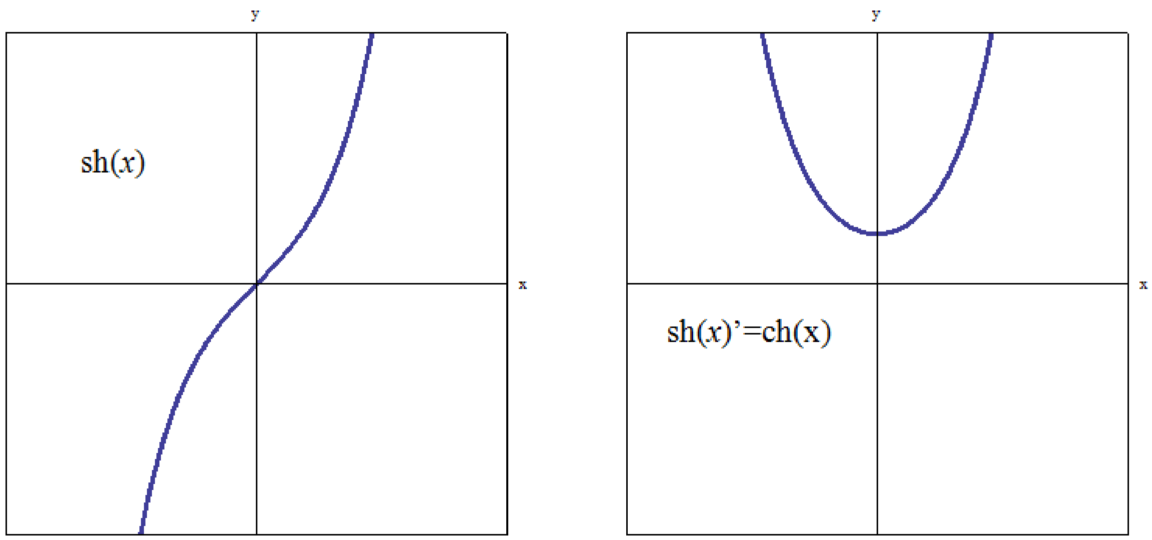


**Fig. 1.** The sine hyperbolic function and its derivate, the cosines hyperbolic function

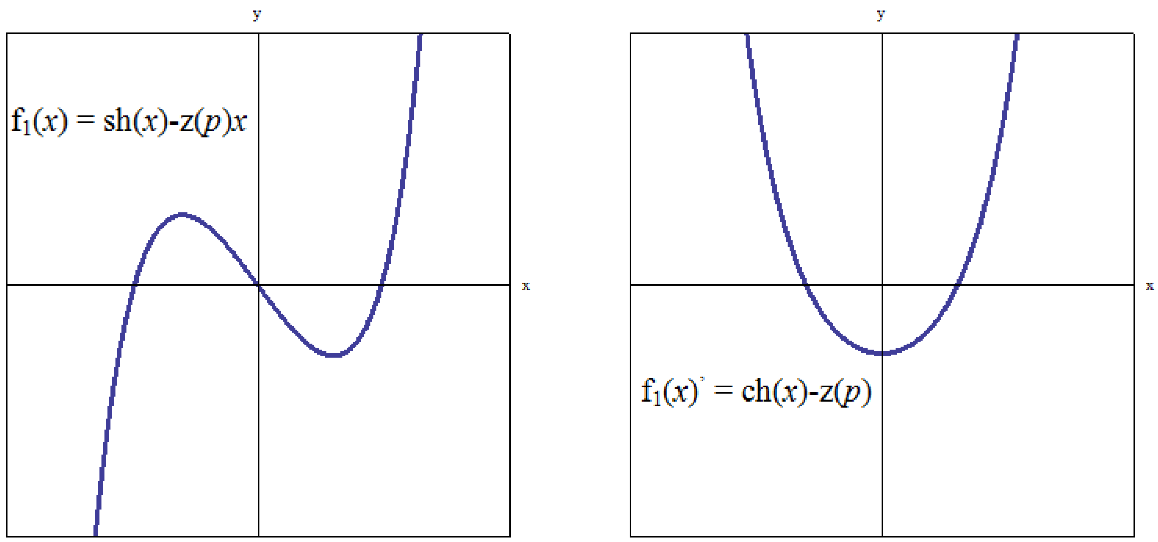


**Fig. 2.** Function $f_1$, after the shift of the original derivate function

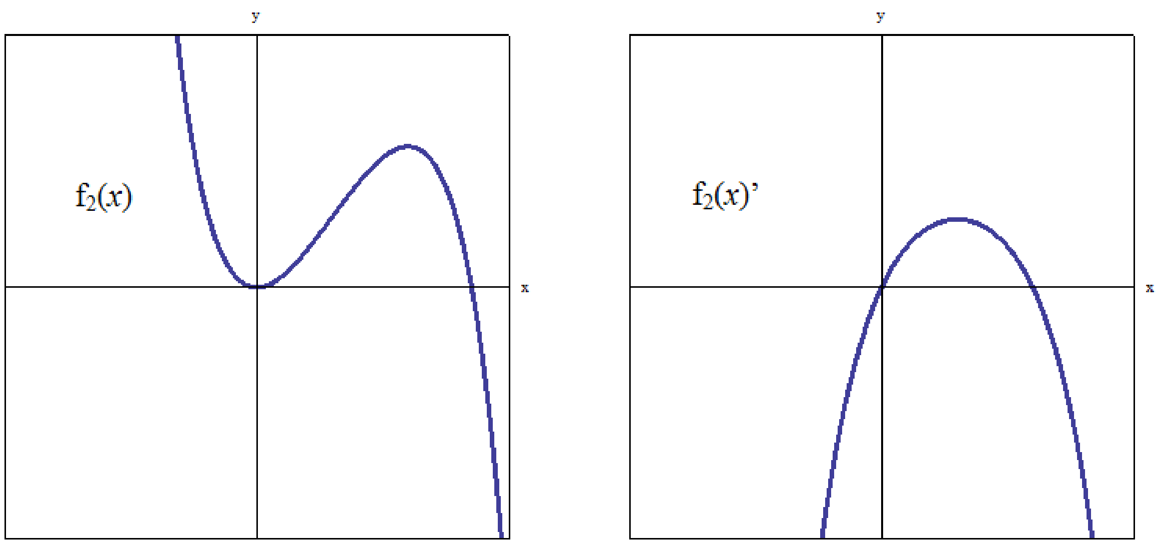


**Fig. 3.** Function $f_2$, after moving the local minimum of $f_1$ to the pole

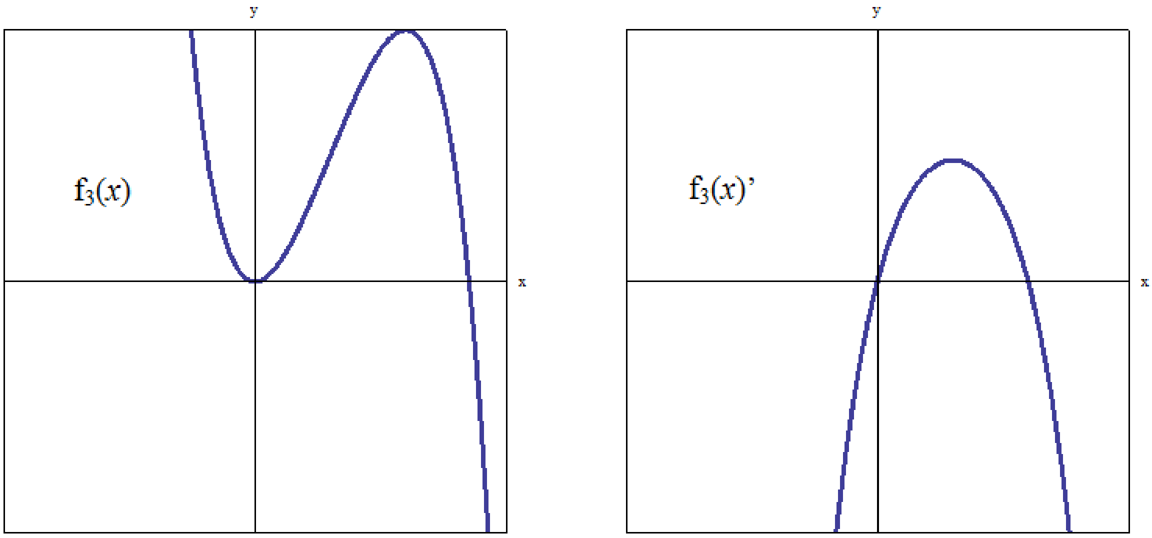


**Fig. 4.** Function $f_3$, after fitting $f_2$ to the curvature of the connecting curve

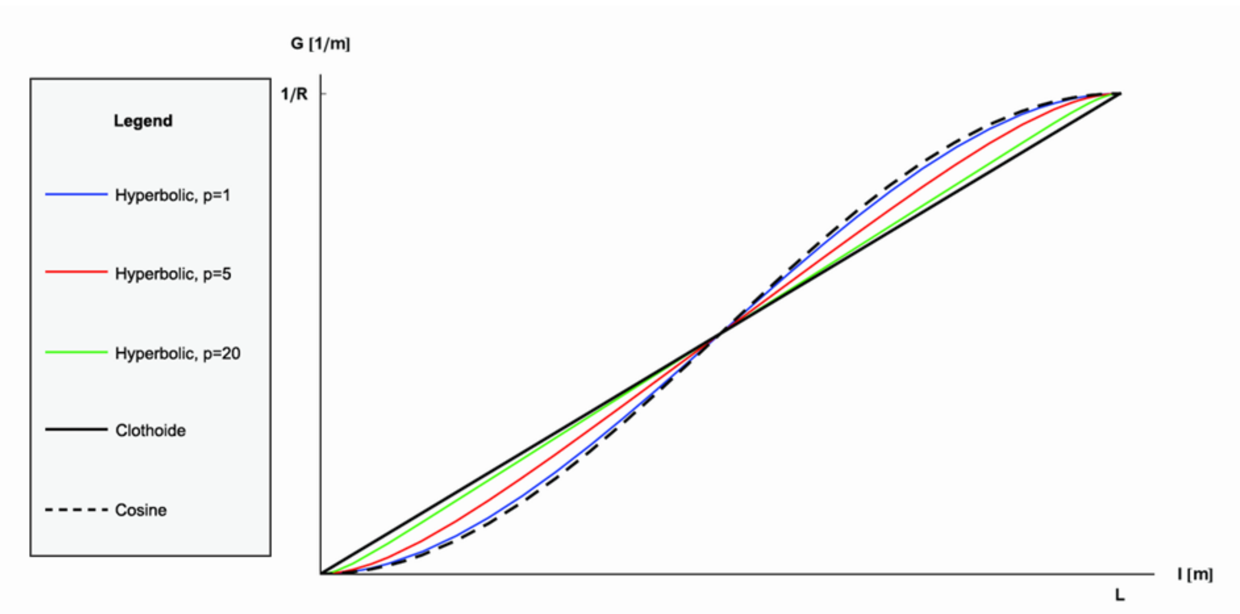


**Fig. 5.** Curvature function of the hyperbolic transition curve for different p parameters, and the curvature function of the clothoide and cosines transition curves

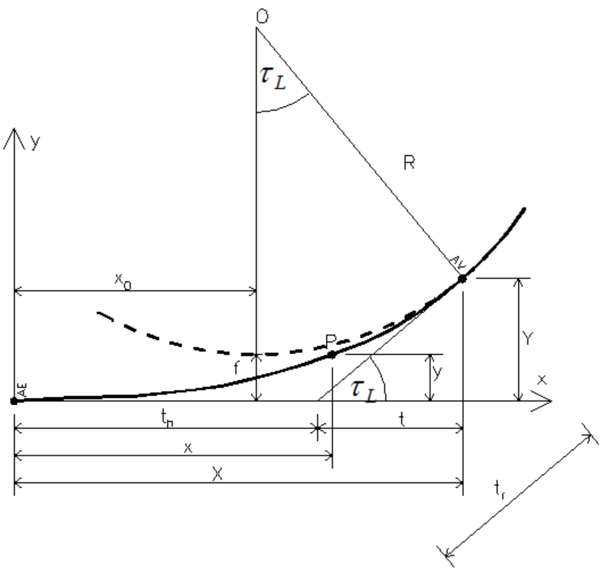


**Fig. 6.** Geometric parameters of transition curves

The angle of the tangent at the end point of the transition curve is the same as the ones of the other transition curves:

$$\tau_L = \frac{L}{2R}. \tag{19}$$

The offset of the curve:

$$f = Y - (R - R \cdot \cos(\tau_L)) \approx Y - \left(R - R \cdot \left(1 - \frac{\tau_l^2}{2!}\right)\right), \tag{20}$$

| | $p = 1$ | $p = 5$ | $p = 20$ |
|---|---|---|---|
| $f$ | $0,02546\frac{L^2}{R} - 0,00276\frac{L^4}{R^3}$ | $0,03084\frac{L^2}{R} - 0,00282\frac{L^4}{R^3}$ | $0,03791\frac{L^2}{R} - 0,00291\frac{L^4}{R^3}$ |

## 5 Evaluation of the hyperbolic transition curve in the function of p parameter

The evaluation of a new transition curve should be done by comparison to the currently applied transition curves. In this paper we compared the hyperbolic transition curve to two widely used geometries, the clothoide and the cosines transition curves:

- Clothoide transition curve:

$$G(l) = \frac{l}{R \cdot L}. \tag{21}$$

- Cosines transition curve (in the literature it is also called Japanese or sine transition curve):

$$G(l) = \frac{1}{2R}\left(1 - \cos\frac{\pi}{L}l\right) = \frac{1}{R}\sin^2\left(\frac{\pi}{2}\frac{l}{L}\right). \tag{22}$$

For transparency on Figure 5, 7 and 8 all three analyzed types of transition curves are shown.

The comparison and evaluation could be done only in the function of parameter $p$. To determine the effect of $p$ parameter on the characteristics in the first step we analyzed the two extremes:

1 $p \to 0$,

2 $p \to \infty$.

The limits for each geometric parameters were determined by (12–26), and their values are summarized in the first two columns of Table 1. Because every function could be decomposed into an $m(L, R) \cdot n(p)$ product, the results could be written in a form easily comparable to the other transition curves.

Besides the calculated geometric parameters it is subservient to analyze the length of the transition curve. One possible way to do this is to compare the lengths of transition curves which have the same curve offset. The approximate values of the curve offset are also shown in Table 1. The length extension which are shown in the last row of Table 1 could be calculated based on these values. .

Based on the results summarized in Table 1 the following consequences could be drawn:

- If the $p$ parameter of the transition curve is $p \to \infty$, then the hyperbolic transition curve infinitely approximates the clothoide transition curve. (This could be seen well in Figures 5, 7 and 8.)
- If the $p$ parameter of the transition curve is $p \to 0$, then the hyperbolic transition curve becomes very similar to the cosine transition curve, but differs from it slightly (as it is shown in Figures 5, 7 and 8.)
- The first derivate of the curvature function and consequently the third order motion characteristics are more favorable than the ones of the cosine transition curve.
- On the other hand the second derivate of the curvature function and consequently the fourth order motion characteristics are less favorable than the ones of the cosine transition curve.
- Taking into account the aspects important in case of speed-increasing reconstructions and of new geometry construction (curve offset, ordinate of the end point, length extension) the hyperbolic transition curve is more favorable than the cosines one.

Besed on the above mentioned results it could be stated that the hyperbolic transition curve has very favorable characteristics. However to state it definitely that the hyperbolic transition curve is more favorable than the cosines one further questions should be answered as it is concluded in the next chapter.

## 6 Conclusions and future development possibilities

By modifying the $p$ parameter of the hyperbolic transition curve the maximum of the third order motion characteristics could be set to the desired value, but the smaller this maximum is the higher the value of the fourth order motion characteristics. The question to be answered is: what is the optimal $p$ value, which assures the most favorable motion characteristics from the point of view of the railway transportation and of the stresses of the track and the vehicle. To answer the question the role and the effect of the fourth order motion characteristics from the point of view of stresses of the track and the vehicle should be analyzed.

**Tab. 1.** The comparison of the hyperbolic transition curve to the clothoide and the cosine transition curves

| Analyzed geometric parameters | Hyperbolic | | Cosine | Chlotoide |
|---|---|---|---|---|
| | $p \to 0$ | $p \to \infty$ | | |
| $\left(\frac{dG}{dl}\right)_{max}$ at the middle of the transition curve | $1,5\frac{1}{RL}$ | $\frac{1}{RL}$ | $\frac{\pi}{2RL} = 1,57\frac{1}{RL}$ | $\frac{1}{RL}$ (discontinuous) |
| $\left(\frac{d^2G}{dl^2}\right)_{max}$ at the end of the transition curve | $6,0\frac{1}{RL^2}$ | $\infty$ | $4,93\frac{1}{RL^2}$ | $\infty$ |
| Approximate value of curve offset, $f$ | $\frac{L^2}{40R}$ | $\frac{L^2}{24R}$ | $\frac{L^2}{42,23R}$ | $\frac{L^2}{24R}$ |
| Ordinate of the end point*, $Y$ | $0,150\frac{L^2}{R}$ | $0,167\frac{L^2}{R}$ | $0,149\frac{L^2}{R}$ | $0,167\frac{L^2}{R}$ |
| Angle of the end tangent, $\tau_L$ | $\frac{L}{2R}$ | $\frac{L}{2R}$ | $\frac{L}{2R}$ | $\frac{L}{2R}$ |
| Length extension compared to the chlotoide, in case of the same $f$ | 29,1 % | 0 % | 33 % | - |

* Approximate value

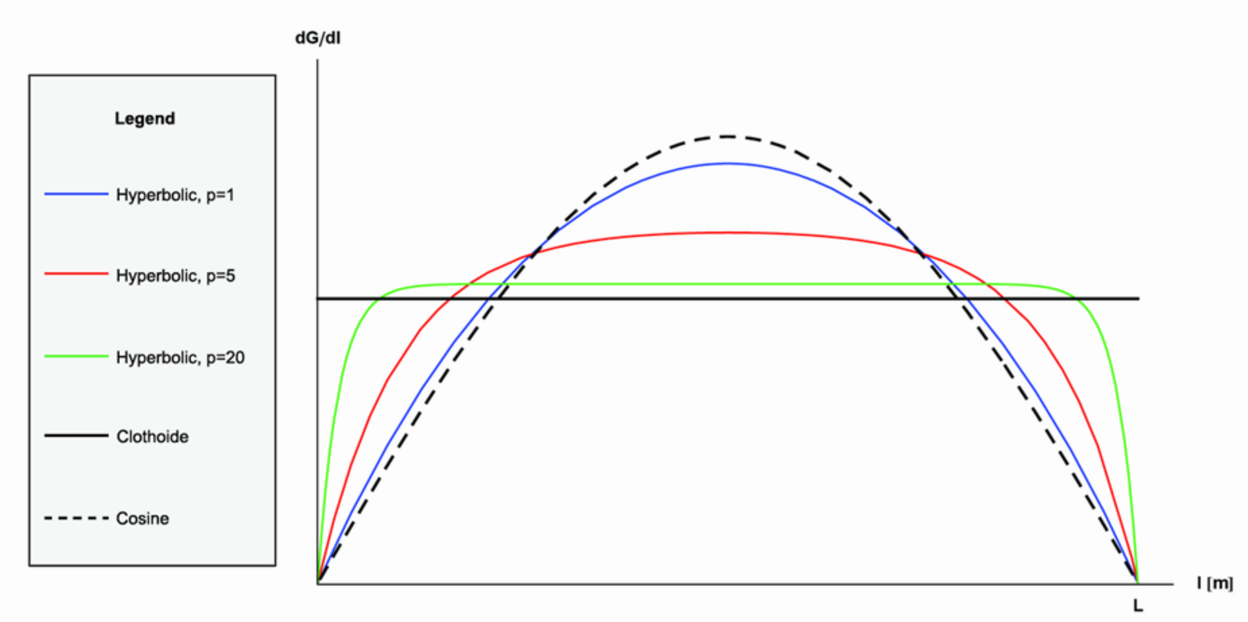


**Fig. 7.** Comparison of first derivates of curvature functions – hyperbolic transition curve with $p = 1, 5, 20$ parameter values; clothoide and cosines transition curves

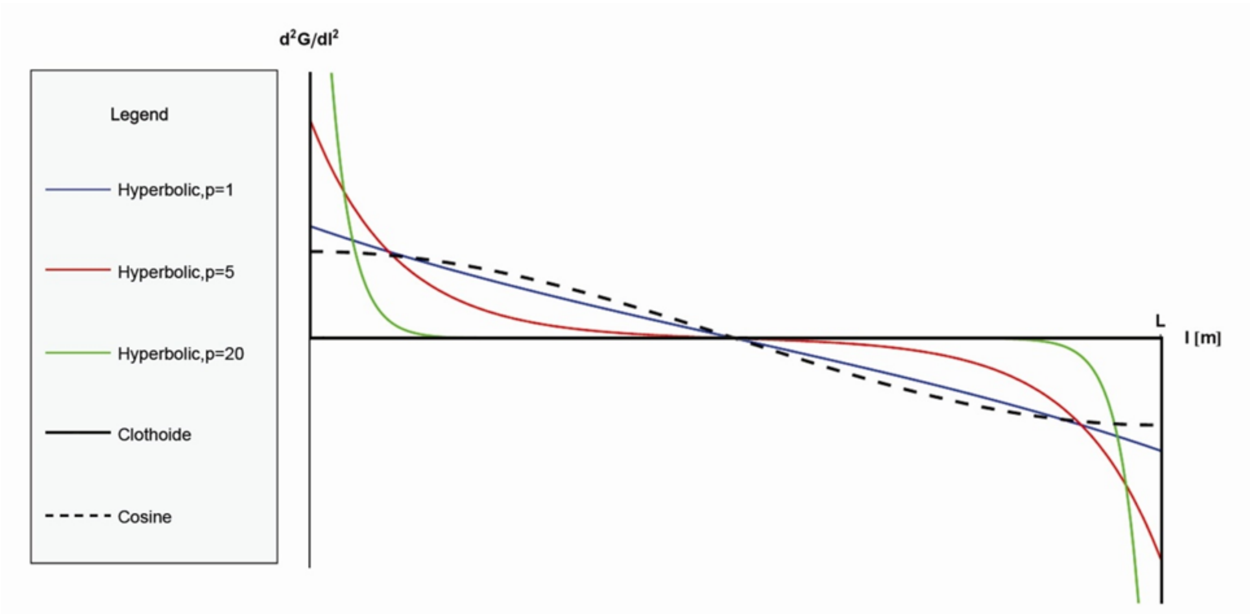


**Fig. 8.** Comparison of second derivates of curvature functions – hyperbolic transition curve with $p = 1, 5, 20$ parameter values; clothoide and cosines transition curves

The next step will be to find the answer to the optimization of parameter $p$. We will examine how and in what extent the fourth order motion characteristics influences the stresses. We will also research how to calculate this effect.

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