New Practical Approach to Plastic Analysis of Steel Structures

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Abstract

The current practical methods for plastic analysis of steel structures are mainly based on plastic hinge or modified plastic hinge methods. These methods are simple and practical but they have some drawbacks. The main weakness of these methods is concentrating the nonlinear effects in one point and neglecting the gradual yielding of the material. This research focuses on the propagation effects of the plasticity in both section and length of the element. The proposed methodology employs a variable section in the plastic region of the element. The results of this method on selected practical cases are presented and compared with the exact solutions as well as the results of other methods. The comparison shows the proposed method to be more accurate, and also easier and more efficient to implement.

Keywords

Plastic Analysis · Steel Structures · Gradual Yielding · Semi Plastic Point

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1 Introduction

The nonlinear behavior of material is an important concept in analysis and particularly in design of steel structures. The nonlinear behavior of steel is linked to its ductile properties. The transition from elastic to plastic state, and consequently from plastic to strain hardening and necking state is not instantaneous. Plastic hinge theory is discussed in the pioneer work of Kazinczy [1]. These gradual transitions are path dependent. The path is generally a function of the shape and size of the cross-section and length of the member. Thus, the behavior of steel element is supposed to vary at every point along its length or throughout its section area. General methods to analyze these transition phases include plastic analysis, equilibrium, and kinematic methods. Simple plastic method refers to step-by-step analysis of member's elastic and plastic moments at every increment until plastic hinges are formed. The equilibrium method refers to determining plastic loads from moment diagrams, which are in equilibrium with externally applied loads. The kinematic method refers to observing the plastic collapse mechanism and obtaining the plastic load associated with that mechanism. Detail discussions about the determination of the deformations of elastoplastic and rigid-plastic structures, subjected to different loadings and great number of bounding theorems and methods, have been presented by Kaliszky [2].

Typical methods to consider the nonlinear behavior of structural elements are generally divided into two main categories: plastic area method and plastic hinge method. In the first method, the plastic area is assumed to propagate along the length and across the section of the member. This method can be incorporated in finite element analysis, in which finer mesh leads to better accuracy [3]. Although this method is accurate, it is time-consuming and needs computers with high computational capacities. Thus, it is not practical to use this type of analysis in every-day engineering practice. In the second method, plastic hinge, a hinge is activated in the section, where the applied moment reaches the plastic capacity of the section. Then, analysis is carried forward with the presence of the hinge. This method is very simple and popular, but at the same time it has some disadvantages. The plastic hinge is lumped at one point, while the

plasticity is propagated along the length of the member. Thus, semi-plastic regions are neglected in this method.

Different methods have been proposed in recent years to solve this problem, including the modified plastic hinge method. In this method, the gradual yielding of the material is addressed by incorporating adjustment coefficients. These coefficients modify the stiffness matrix throughout the transition from elastic to plastic behavior in the hinge area. This method is more accurate, but the coefficients cannot be obtained from analytical methods. Rather, these coefficients are determined by fitting analytical results over experimental results. For this reason, the method is not universally applicable to all cases with different sections and configurations.

Mamaghani, Usami, and Mizuno proposed a deflection analysis using finite element method. The inelastic large deflection analysis of structural steel members, such as pin-ended columns and beam-columns of strut type was examined. A multi-axial two-surface plasticity model (2SM) was developed in this study to determine the gradual plastification through the cross section and along the member length [4]. The Bauschinger effect, cyclic strain hardening, and residual stresses were produced during development of hysteretic plastic deformations. An elastic-to-plastic finite element formulation was used to find material and geometrical nonlinearities. The 2SM model incorporates the experimentally observed cyclic behavior of steel and describes the decrease and disappearance of the yield plateau, reduction of the elastic range and cyclic strain hardening. It also considers the degradation of post-buckling compressive resistance, deterioration of buckling load capacity in subsequent inelastic cycles, and progressive degradation of stiffness during cycles and plastic elongation in the column length. The predicted hysteretic behavior of structural steel members using this model was found to be in better agreement with the experimental results, as compared with other methods, i.e., the elastically perfect plastic (EPP), isotropic hardening (IP) and kinematic hardening (KH). Therefore, it was concluded that 2SM is quite promising to account for the material nonlinearity of steel members under cyclic loading. However, the 2SM model does not take into account the variability of material properties.

Another approach to analyze variability in behavior of steel under loading is the second-order plastic hinge analysis [5]. In this approach, the beam-column specimen was analyzed using a formulation based on stability interpolating functions for transverse displacements, as well as elastic coupling for axial, flexural, and torsional displacements. This method is particularly suitable for space frame structures in which the members are slender and subjected to high axial forces. Plastic action under cyclic loading is complex, as real materials initially experience strain hardening when the stress exceeds yield and may exhibit the Bauschinger effect when the loading is reversed [6, 7]. The proposed method was shown to be accurate in capturing the buckling load of columns with different end conditions using only one element per physical member. The plastic hinge

formulation allows plastic hinges to form at either the ends or within the length of the element. A study of a six-story space frame using a direct second-order plastic hinge analysis provided a better insight into the structural behavior up to the failure point. This insight relied on the load–displacement characteristic of the structure and the sequence of hinge formation in the frame. The method was shows to be particularly useful for flexible and non-symmetrical structures. It was noted that the accuracy of plastic hinge analysis was reasonable only for special cases, where the spread-of-plasticity is not significant and where the material stress–stain law is essentially elastic–plastic. Thus, this method was found to be suitable for slender space frame structures only.

A second-order spread-of-plasticity analyses was developed by Jiang, Chen, and Liew to analyze three-dimensional structures [8]. Spread-of-plasticity analyses refer to subdivision of the element cross-sections into grids to monitor the path-dependent nature of plasticity. This method is helpful in finding the current stress, the current yield stress at the current level of strain, and updates values at each increment of load. Further, this method can be used to study the more complex behaviors that involve torsional-flexural buckling, local buckling, and yielding under the combined action of compression and bi-axial bending. A 20-storey 3-D building was analyzed using this mixed element approach. The computational accuracy of results was found to be 10% more than the one obtained from plastic hinge analysis. It was concluded that the proposed method is effective in analyzing the semi-plastic region of steel members and could be applied to more complex steel-concrete composite structures. However, the applicability of this method is limited to the plastic points on grids, and does not apply to plastic points between grids. Kaliszky and Logo optimized the plastic design of bar structure considering the nonlinear behavior of the structure [9, 10].

Cocchetti and Maier proposed a plastic-hinge modeling using conventional finite element method for frames subjected to monotonic loading. This method assumed the possible plastic deformations in the member to be confined to some particular sections known as critical sections. The behavior of these critical sections could be demonstrated by elastic-plastic piece-wise-linear (PWL) models. These models relate the generalized stresses describing bending moment and axial force to the generalized strains describing rotation and axial elongation. Further, these models describe the transition from non-holonomic (path-dependent and irreversible) to holonomic region, which help in the formulation of a combination of limit and deformation analyses. These results can be further used to evaluate possible bifurcations and instability thresholds. The conclusion stated that despite these advantages, the PWL modeling has some disadvantages, such as tedious and time consuming computation process, and large number of variables involved in the modeling due to multiplicity of yield models. Further, the plastic deformations depend on variability of material, shape, size, and length of the

member at every point, thus, it can't be confined to some particular sections only. Therefore, the modeling assumption for these analyses needs to be revised [11]. Kaliszky presents solution methods for elastoplastic and shakedown analysis of linearly elastic, perfectly plastic bodies [12].

Thai and Kim considered an element with elastic segment in the middle and two plastic segments in two ends. The two end-segments of the element were consisted of series of fiber elements. The stiffness matrix was reduced to 12 degrees of freedom using static condensation. The results showed good accuracy in comparison with ABAQUS analyses [13]. Kim and Thai extended the numerical solution further to dynamic analysis [14]. Kaliszky and Logo studied through the choice of appropriate parameters of force-deformation relationships, the classical principles of linearly elastic and perfectly plastic structures can be obtained and the material and/or geometric nonlinearity, post buckling behavior [15].

In this research, a new practical method for plastic analysis of steel structures is proposed. This method considers the plasticity propagation along the length of the member. The analysis focuses on the gradual variations in geometry of steel members during semi-plastic to plastic state. This study presents experimental results of flexural testing on small-scale specimens and analytical investigations using the proposed method. Further, results have been compared with other analytical methods. The area of the study in this research is limited to two dimensional frames, but it has the capacity to be applied to three dimensional frames as well.

2 Experimental studies

A circular steel rod with 12.7 mm (0.5 in.) diameters was tested in a three-point-load configuration using universal testing machine. The rod was loaded until failure. Fig. 1 shows the load-displacement relationship and estimation of yield point based on experimental values. This figure indicates how simplifying assumptions are implemented on elastic-to-plastic transition to obtain an idealistic yield measure.
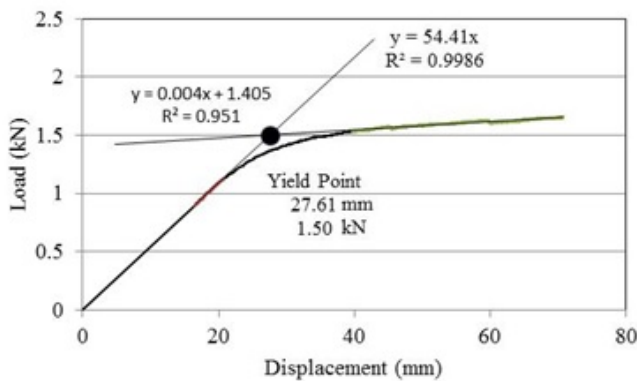

Fig. 1. Load-Displacement relationship of a solid rod subject to flexure

Fig. 2 provides comparisons between experimental and analytical results, in which stress and strain values are normalized

to yield stress and strain. This figure shows how conventional analytical methods differ from experimental results. Both simplified method present a sharp transition from elastic to plastic state, while, the experimental transition is gradual.
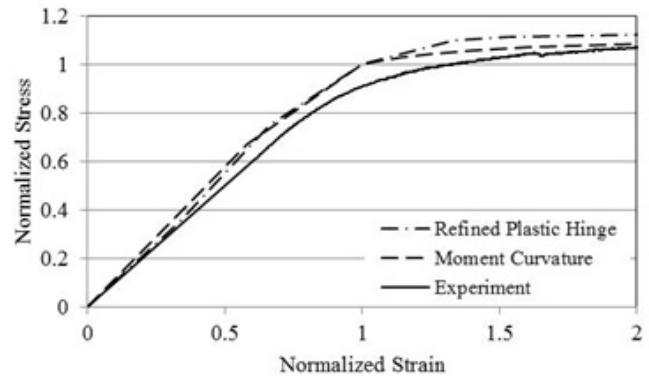

Fig. 2. Normalized Strain-Stress Relationship of the flexural solid rod

3 Effects of plasticity propagation

Consider a steel element under an increasing flexural loading, similar to the steel rod in experimental investigations. At the beginning, the whole section is elastic until the load reaches the yielding level. Then, the first point on the section along the furthest fiber from the centroid yields (yield moment). Increasing the load causes more points on the section and along the member to yield until the whole section reaches the plastic behavior (plastic moment). A plastic hinge is formed when the flexural resistance of the section gradually tends to zero. In simplified methods of analysis, the transition part between yielding moment and plastic moment is neglected. In other word, these methods consider a fully elastic section before reaching complete plastic behavior. This assumption is not correct as the reduction of the flexural stiffness of the section - from the first yielding to the development of the plastic hinge - may lead to redistribution of internal forces and change the response of the structure to the loading in respect to safety and serviceability. Thus, it is necessary to consider the propagation of the plasticity in analysis to have a better understanding of the behavior of the structure. In following sections, the effects of plasticity propagation along section and length of a member are studied.

3.1 Propagation of plasticity over the section

To define the distribution of plasticity over the section, the normal stresses in semi-plastic state must be determined in the section. In elastic phase, the distribution of stress over the section will be determined using section properties only, i.e., area, moment of inertia, and location of the neutral axis, which will remain constant and independent of the load. However, in the semi-plastic region, the distribution of nonlinear stress is completely dependent to the shape of the cross section and the state of the loading. Thus, it is not possible to propose a closed-form solution to determine the stress distribution over the entire sec-

tion. In this research, the section is modeled as a rigid plate on an elastic bed in order to consider the propagation of plasticity. This assumption facilitates determination of stress distribution and plastic area subject to different loadings. Using this method, the flexural stiffness of the section can be adjusted based on the spread of the plastic area in the section.

Fig. 3 shows the reduction of flexural stiffness of selected sections subject to different axial loads. These graphs are normalized to the elastic stiffness and the difference between plastic and yield moments. The graphs in this figure show the reduction in flexural stiffness as the applied moment on the section is increased. Further, the effect of normalized axial load, in respect to yield load, on the flexural stiffness is shown.

Fig. 4 emphasizes on the increase of the effects of propagation of plasticity when the ratio of plastic to yielding moment increases. According to this figure, the semi-plastic area is larger for solid section than hallow ones. This is well presented for the solid diamond section, in which the ratio between M_p and M_y reaches two, and the semi-plastic area is larger than elastic area in interaction graphs. In comparison, the plastic-to-yield moment ratio for the I-section is substantially smaller. Further, Fig. 4 reveals that the behavior of a solid rectangular section is similar to a diamond hallow section. This means a hollow diamond section can be a good substitute for a solid rectangular section based on similar elasto-plastic behavior.

3.2 Propagation of plasticity along the length of member

Although current simplified methods consider plasticity process to occur in one point, known as plastic hinge, the plasticity propagates along the length of member as well as the cross section. The length of the plastic area is correlated with the distance from the full plastic section to the elastic section with one yielding fiber at the furthest points from the neutral axis. Finding these two sections requires finding two axial forces and bending moments along the member which correspond to the above mentioned conditions. The length of plastic area in a member depends on section shape, moment distribution along the beam, and axial loading on the beam. The two later parameters are themselves related to external loading and stiffness distribution of structure.

By increase in loading, furthest point of a section reaches yielding and the plasticity propagates along the length of the member. This means that the stiffness and modulus of elasticity are reduced in this area and eventually tends to zero in perfect plastic state. Thus, the original cross-section will be changed to a variable cross-section over the semi-plastic part of the member which has variable stiffness from one end (elastic) to the other end (semi- or fully-plastic). Fig. 5 shows this process.

Calculation of the stiffness matrix of the element requires the stiffness of the end section, length of the plastic area, and the rate of stiffness reduction. The first two parameters can be obtained based on section properties and loading diagrams. The reduction rate is calculated using stiffness reduction diagrams, similar to
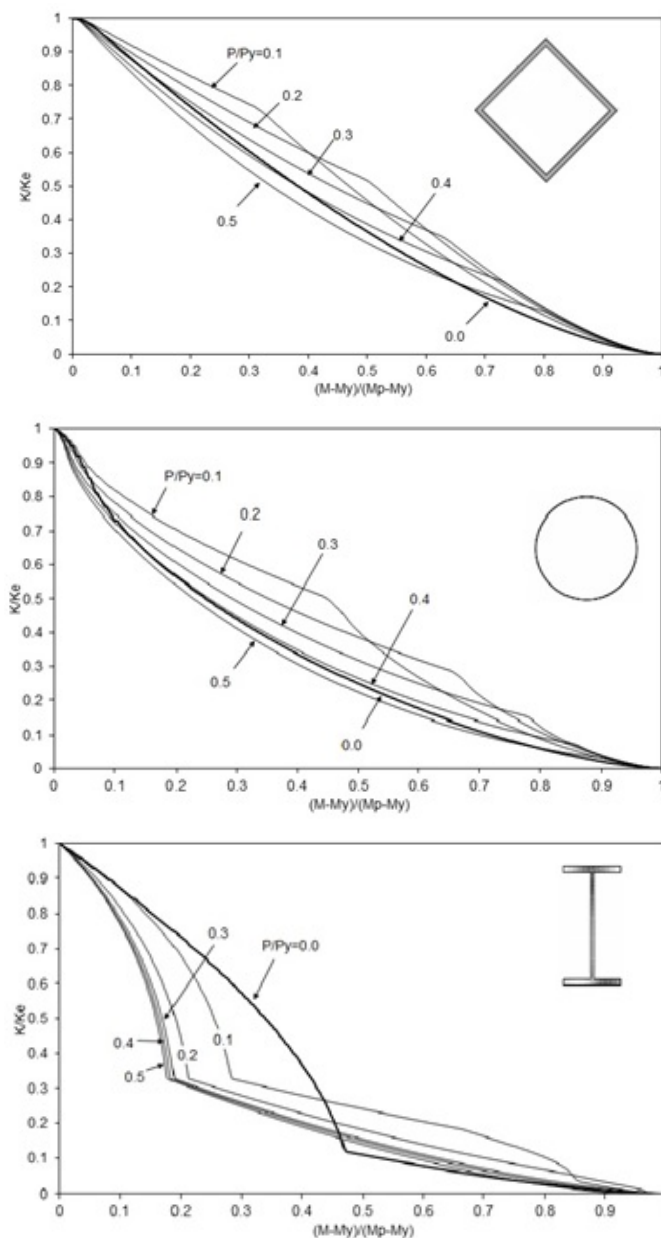

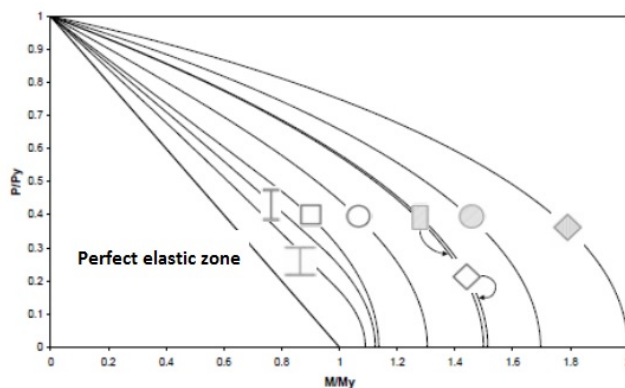**Fig. 3.** Stiffness reductions in some steel sections

Fig. 4. Normalized stiffness reduction graphs for different sections

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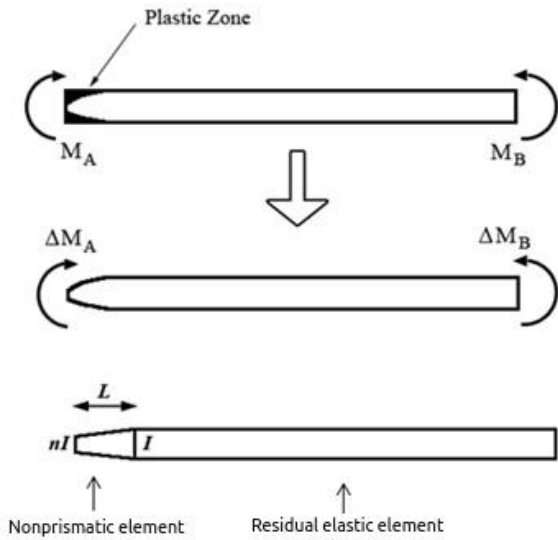
Fig. 5. Modeling of semi-plastic area with a variable cross-section along the element

those shown in Figs. 3 and 4, when available. So, the availability of stiffness reduction diagrams for various sections remains to be a problem. Further, if the algorithm of the solution is designed to have the exact stiffness reduction in each step, integration along the length of variable element becomes necessary. Due to complexity of the stiffness reduction functions, these integrations increase the time and cost of the analysis. In this research, it is assumed that the behavior of the semi-plastic element is mainly dependent on the stiffness of end section and length of plasticity rather than the rate of reduction. Based on this assumption, two functions (linear and second-order) are considered for reduction of stiffness in stiffness matrix of the element. In following sections it is shown these two different functions have no meaningful differences in results and the assumption is correct.

3.3 Algorithm of practical plastic analysis

Based on the previous discussions, an algorithm for analysis of the plastic area of a member is proposed. In this method, the plastic part of the member is substituted by an element with variable section. The stiffness matrix for a variable section in general can be written as:

$$K = (EI/L) \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \quad (1)$$

In this stiffness matrix, K, the node 1 represents the semi-plastic section, and the node 2 represents the fully plastic section in the element. L is the length of plasticity and EI is the flexural stiffness of elastic section. Part of the element here has constant stiffness and other part of the element has linearly varied stiffness. This means along the element there is not a unique stiffness condition. When we simplify the stiffness matrix the variations of parameters will be non-linear. If the reduction of stiffness is assumed to be a linear function, stiffness values K_{11}

to K_{22} can be obtained from following equations:

$$K_{11} = \frac{4n - 3n^2 - 1 + 2n^2 \ln n}{-2n + n\ln n + \ln n + 2} \quad (2)$$

$$K_{22} = \frac{-4n + n^2 + 3 + 2 \ln n}{-2n + n\ln n + \ln n + 2} \quad (3)$$

$$K_{12} = K_{21} = \frac{n^2 - 1 + 2n \ln n}{-2n + n\ln n + \ln n + 2} \quad (4)$$

In these equations, n is the ratio of the stiffness in semi-plastic section to the stiffness of a fully-plastic section. Using the second-order functions for stiffness reduction, following alternative equations can be proposed to calculate the elements of the stiffness matrix:

$$K_{11} = -2.099n^4 + 5.503n^3 - 4.852n^2 + 5.664n + 0.074 \quad (5)$$

$$K_{22} = -7.565n^4 + 18.151n^3 - 15.819n^2 + 7.755n + 1.434 \quad (6)$$

$$K_{12} = K_{21} = -3.855n^4 + 9.406n^3 - 8.488n^2 + 4.746n + 0.17 \quad (7)$$

It is important to notice the above equation with n equal to 0 and 1 are not precise. In these two conditions, the exact value must be introduced to the program. If n is 1, the stiffness matrix is calculated by elastic relations and there is no need to use the above equations. If n is zero, the value of K_{22} in linear and second-order formulations will be 2 and 1.02, respectively. Other values of stiffness matrix will be zero when n is zero. To avoid numerical problems in analysis of structure, these equations are substituted with above mentioned values, if n becomes smaller than a tolerated value, say 10^{-3}.

To use one element along each member, substructure technique is implemented for each semi-plastic element. This technique allows a member to be modeled with one element only. When the first section reaches yielding moment, the program automatically adds a node and considers two elements as substructures to replace the original element. For this purpose, the stiffness matrix for both variable and elastic elements in the member are calculated and assembled to form the member stiffness matrix. The stiffness matrix is statically condensed based on the end degrees of freedom. The algorithm of this technique is shown in Fig. 6. Further, the comparison between different practical inelastic analyses is shown in Fig. 7.

4 Results

In this section, selected practical examples are analyzed using different inelastic analysis methods. The results are compared to verify the accuracy of proposed method. The referenced analysis was performed using ANSYS, which is marked in following graphs as the exact solution. The modified plastic hinge method
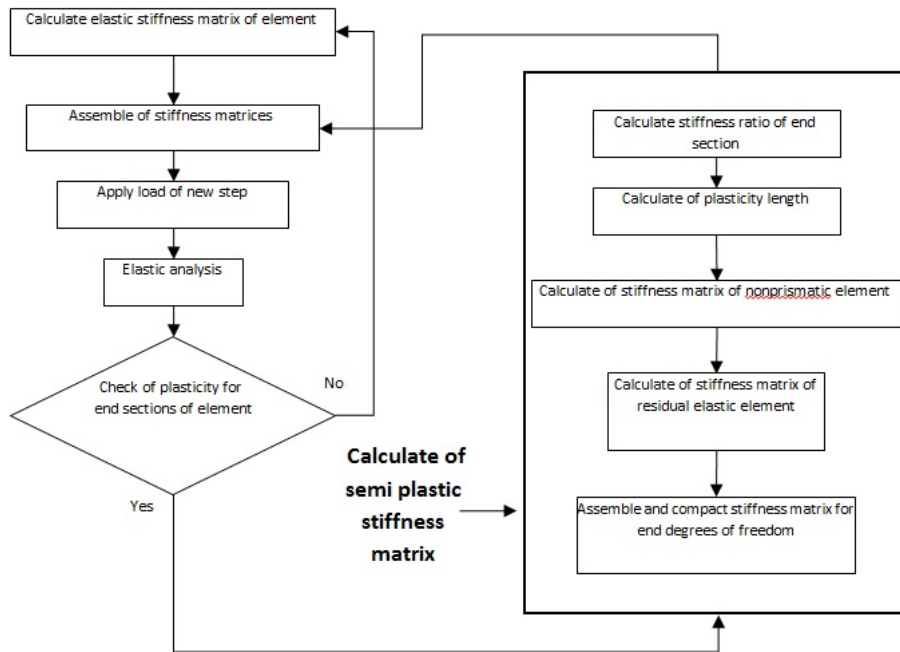
Fig. 6. Proposed algorithm for practical analysis of plastic area

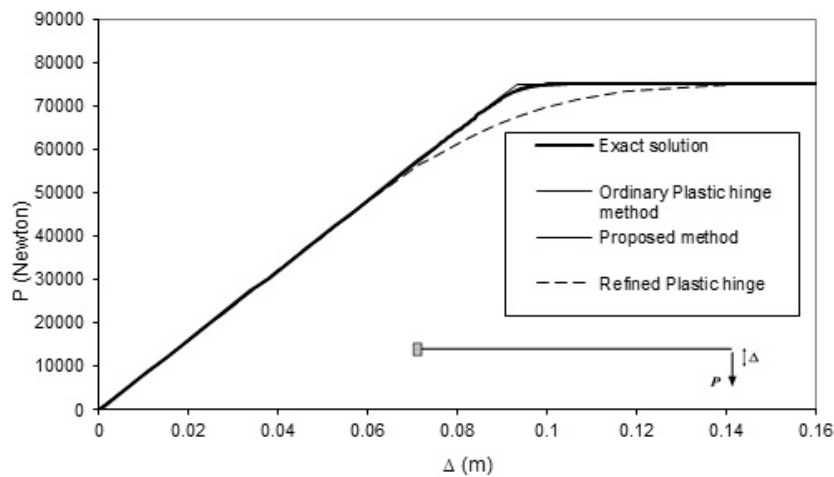**Fig. 7.** Comparison of proposed inelastic analysis with current methods

Fig. 8. Comparison of inelastic analysis for a cantilever beam with I-Shape section
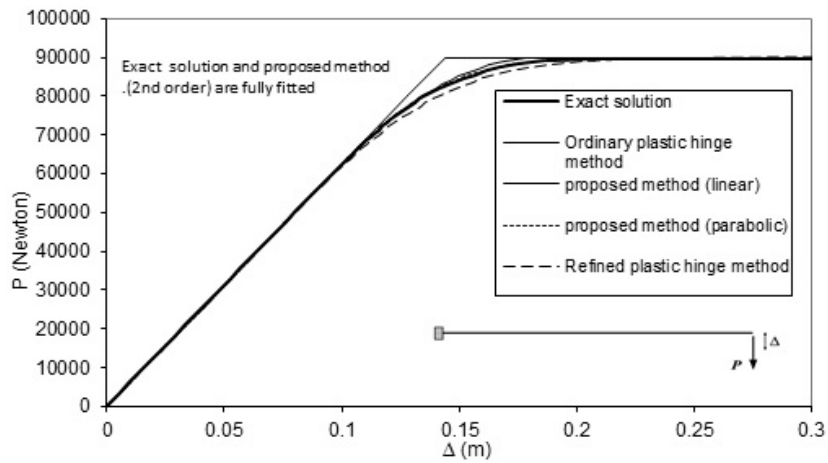
Fig. 9. Comparison of inelastic analysis for a cantilever beam with rectangular section
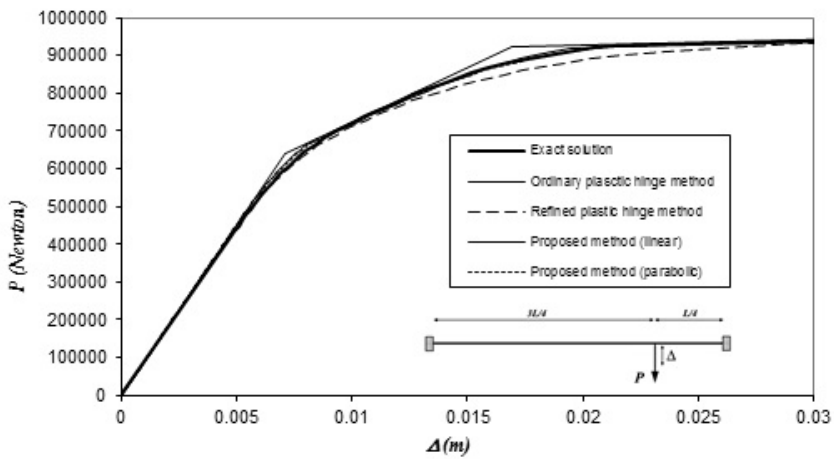

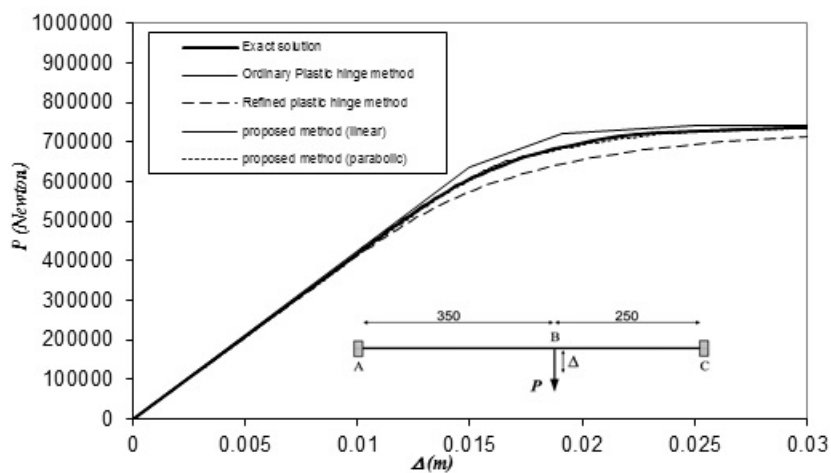**Fig. 10.** Comparison of inelastic analysis for a fix ended beam with rectangular section

Fig. 11. Comparison of inelastic analysis for a fix ended beam with rectangular section
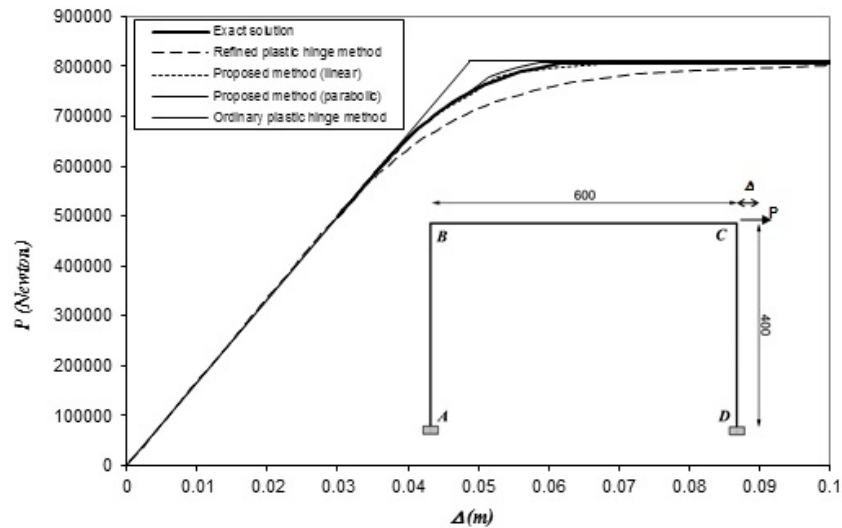
Fig. 12. Comparison of inelastic analysis for a frame with rectangular section

is carried based on the proposed method by Chen and Kim [16]. The result of simple plastic hinge method, which doesn't consider the propagation effects of plasticity, is also shown for comparison. The proposed method is considered with two different stiffness reduction functions (linear and second-order) to determine the sensitivity of the results to the type of implemented function. As discussed earlier, the effects of plasticity propagation are better manifested in behavior of solid sections rather than hollow sections. Thus, the comparative analysis is performed on rectangular beam (0.1 m by 0.3 m) and column (0.2 m by 0.3 m) sections. An I-shape section (HEA340) is also analyzed for comparison.

Figs. 8 and 9 provides comparison of results for a typical cantilever beam. Fig. 8 shows that proposed method fits the ANSYS solution, without the necessity of time-consuming finite element modeling and analysis. This figure also indicates that simple hinge method does not deviate substantially from accurate results for an I-shaped section. However, this is not necessarily correct for solid sections presented in Fig. 9. Comparison of Figs. 8 and 9 indicates that the proposed method, disregarding the reduction function type, is accurate for the analyzed beams as results are closer to the exact solution in comparison to simple and modified plastic hinge methods.

Figs. 10 and 11 show analysis results for a fixed-end beam subjected to a transverse load. In this model, plastic hinges are consecutively formed at A, B and C as shown in Fig. 11. As shown in these figures, development of one hinge might occur while the previously initiated hinge is still in progress, i.e. the section is semi-plastic. Regardless, presented results confirm the accuracy of proposed method using either linear or second-order functions.

The advantages of proposed method can be presented in analysis of a complete frame subjected to lateral loading. The progressive nature of failure in a simple frame, as shown in Fig. 12 justifies the application of proposed method instead of simplified methods. Fig. 12 clearly indicates that simplified methods

cannot accurately estimate the transition from elastic to plastic states. This level of inaccuracy has an eminent impact on the outcome of performance-based design approaches using weak-beam-strong-column, in which the ductility of the system relies on the appropriate prediction of progressive failure mechanism.

5 Conclusions

In this paper, the effects of plasticity propagation within the section and along the length of the element are investigated. The proposed methodology is based on formulation of a variable section in the plastic part of the member. Selected practical examples are provided to compare the proposed method with existing methods. Following results are attained:

- The propagation of plasticity is more important in sections with higher ratio of plastic moment (M_p) to yielding moment (M_y). This applies to many solid and hollow sections. Thus, incorporating semi-plastic formulations in analysis of structures containing such sections is essential.

- Simplified approximate methods might be appropriate for structures with single-hinge mechanisms. However, development of multiple hinges at the same time in a structure causes accumulation of errors due to these approximations and reduces the accuracty of analysis.

- The effect of plasticity propagation is less intense for I-shape sections than rectangular sections. Thus, the errors from simplified methods might be tolerable this type of sections.

- The proposed method in this research is more accurate than other practical inelastic analysis methods. Further, this method is easier and more practical than finite element methods.

- The proposed method is an explicit method, and thus, it is easier to be implemented in presence of other nonlinear effects caused by loadings or material characteristics.

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Further research is possible to extend the application of proposed methods to three-dimensional structures. Further, implementation of unloading and residual stresses can be incorporated in these analyses.

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