

Non-Equilibrium Thermodynamical Framework for Rate- and State-Dependent Friction

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Received 16-05-2015, revised 30-06-2015, accepted 30-06-2015

Abstract

Rate- and state-dependent friction laws for velocity-step and healing are analysed from a thermodynamic point of view. Assuming a logarithmic deviation from steady-state, a unification of the classical Dieterich and Ruina models of rock friction is proposed.

Keywords

rock friction · Dieterich law · Ruina law · steady state

1 Introduction

The rock experiments of sliding friction are understood by the so-called rate- and state-dependent friction laws. The equations of these laws unify the results obtained from two types of rock experiments; the first one is the time dependence of the coefficient of static friction [1] and the second one is slip velocity dependence of the coefficient of kinetic or sliding friction [2].

The properties of dynamic friction are the following [3]:

- 1 frictional coefficient in stable sliding conditions with a constant load-point velocity depends on the logarithm of the load-point velocity. ¹;
- 2 the magnitude of the instantaneous jump of the frictional coefficient depends on the change of the logarithm of the quotient of the corresponding load-point velocities;
- 3 the following evolution of the frictional coefficient to new value in stable sliding also depends on the instantaneous change of the load-point velocity;
- 4 oscillation occurs in some cases (e.g., large load-point velocity, polished surfaces, thin sand interface layer between the samples) (see e.g., [4]).

In healing experiments² the properties of static friction are:

- 1 recovery magnitude is proportional to the logarithm of healing time;
- 2 larger velocity or larger elasticity increases the recovery magnitude of the static friction.

These properties can be reproduced by using two classical equations. The first one is the *constitutive law* Eq. (1), expressing the relation between frictional coefficient μ and slip velocity V with an additional variable, called state variable θ . The second

¹ Load-point velocity is the velocity of the point of the sample, where the load is applied. This can be measured more exactly, than the velocity of the sliding surfaces.

² Stationary sliding is followed by a particular period when the load point velocity is zero and then sliding with the initial velocity.

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one is the *evolution law* Eq. (2) expressing the time evolution of the state variable depending on the slip velocity [5]:

$$\mu = \frac{\tau}{\sigma} = \mu_* + a \ln\left(\frac{V}{V_0}\right) + b \ln\left(\frac{V_0\theta}{L}\right), \quad (1)$$

$$\frac{d\theta}{dt} = 1 - \frac{V\theta}{L}, \quad (2)$$

where τ is the shear stress, σ is the normal stress, μ_* is the constant frictional coefficient for steady-state slip at reference slip velocity V_0 , a and b are material parameters, L is a material parameter called the critical slip distance, and t is time.

An important improvement is the evolution equation of Ruina [6]:

$$\frac{d\theta}{dt} = -\frac{V\theta}{L} \ln\left(\frac{V\theta}{L}\right). \quad (3)$$

Experimental data of static friction is better reproduced by the equations of the Dieterich law [5] (Eqs. (1) and (2)), and of dynamic friction by the equations of Ruina law [6] (Eqs. (1) and (3)). A comparison with experimental data is given in the works [3, 7]. In particular, the Dieterich law assumes time-relaxation and therefore it is asymmetric for upward and downward jumps in displacement, contrary to the experiments. On the other hand, the experimentally observed time dependent healing is properly reproduced by the Dieterich law and is not reproduced by the Ruina law. Thus other versions have been proposed (e.g., [7–9]) in order to reproduce the experimental data better. However, none of them are completely satisfactory.

Nakatani reformulated the Dieterich-Ruina law introducing a new variable [10]:

$$\theta = \frac{L}{V_0} \exp\left(\frac{\Theta}{b}\right). \quad (4)$$

Then the constitutive law is linearized with this variable:

$$\mu = \frac{\tau}{\sigma} = \mu_* + a \ln\left(\frac{V}{V_0}\right) + \Theta, \quad (5)$$

and the evolution laws of Dieterich and Ruina become

$$\frac{d\Theta}{dt} = \frac{b}{L} \left(V_0 e^{-\frac{\Theta}{b}} - V \right), \quad (6)$$

$$\frac{d\Theta}{dt} = -V \left(\frac{b}{L} \ln\left(\frac{V}{V_0}\right) + \frac{\Theta}{L} \right), \quad (7)$$

respectively. Nakatani suggested this modification together with a particular interpretation of Θ as strength and interpreted the modified evolution equation of Dieterich in the framework of thermal activation theory. He did not investigate the modified form of Ruina law Eq. (7).

2 Thermodynamics of the frictional layer

A thermodynamic approach of Mitsui and Ván introduced a minimal model of rock friction by using permanent and recoverable parts of the total observed displacement as state variables in the spirit of continuum plasticity [11].

They interpreted the state variable of the rate- and state-dependent friction law as the elastic part of the displacement. It was argued that the constitutive and the evolution laws are originated in a thermodynamic framework. Their calculation of the entropy production was the following.

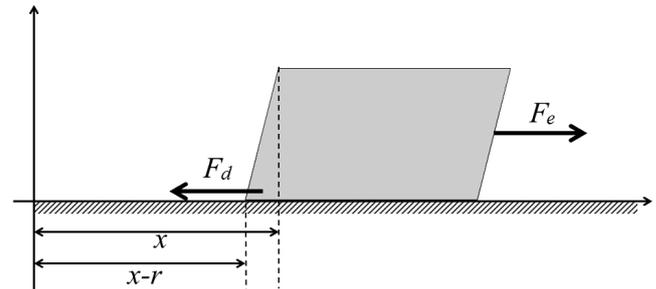


Fig. 1. Sliding thermomechanical body. Experimental realisations try to restrict the deformation to the neighbourhood of the sliding surfaces. Our figure is simplified in this respect.

The general setting is a sliding body on a horizontal surface with mass m . There are two forces that determine the motion of the body: the external force F_e , and the damping force F_d , due to friction (Fig. 1). The position of the body is denoted by x . The body is not considered completely rigid, however one assumes that one particular material point of the body characterize its instantaneous position. The equation of motion is

$$m\ddot{x} = F_e - F_d. \quad (8)$$

Moreover, the work of the external force changes the energy of the body, E . Therefore

$$\dot{E} = F_e \dot{x}. \quad (9)$$

In this case thermodynamics requires that the damping force contributes only to the internal energy of the body. It is assumed that the external force accelerates the body and also that the body is deformable. In this homogeneous model, the deformation is expressed by the recoverable displacement, r . Accordingly, the kinetic and elastic energies of the body are distinguished. This particular interpretation from [11] is not necessary, r may denote a general internal variable.

The internal energy, U , is the difference of the total energy, E , the kinetic energy and a quadratic contribution of the internal variable. This form follows from the condition of thermodynamic stability in the state space [12]. That is

$$U = E - m \frac{\dot{x}^2}{2} - k \frac{r^2}{2}, \quad (10)$$

where k is a parameter like Young's modulus in elasticity. One assumes a particular kinematic condition in order to introduce an interpretation of r . When the instantaneous position of the body is the sum of a permanent and a recoverable displacement then a convenient method of their distinction is an additive separation of the displacement rates:

$$V = \dot{x} = \dot{r} + z, \quad (11)$$

where V is the rate of the position x , and z is the rate of the permanent displacement. This rate type kinematical condition is convenient when distinguishing permanent and recoverable changes. Eq. (11) is analogous to the condition used in plasticity for the distinction of plastic and elastic strains (see e.g., [13, 14]). However, in friction the internal variable is not necessarily identical to the recoverable strain. The background physical mechanisms may contribute to the displacement, but do not represent a completely recoverable, elastic change.

Entropy production, including the consequent dissipation, can be calculated by the entropy balance, assuming that the entropy is the function of the internal energy only:

$$\begin{aligned}\dot{S}(U) &= \frac{1}{T} (F_e \dot{x} - mV\dot{V} - kr\dot{r}) \geq 0 \\ \Rightarrow T\dot{S} &= F_e V - kr\dot{r} \geq 0.\end{aligned}\quad (12)$$

The damping force and the rate of the internal variable \dot{r} are the constitutive quantities to be determined in accordance with the requirement of nonnegative entropy production. We connect the first term to friction interaction and the second term to frictional healing. In the vicinity of thermodynamic equilibrium, where both forces and fluxes are zero, the usual linear relationship is a consequence of Lagrange's mean value theorem [15, 16]. V and r are thermodynamic state variables, consequently they can be considered as thermodynamic forces, while μ and kr are to be determined constitutively, as being thermodynamic fluxes. The standard linear approximation results in equations that reflect well the thermodynamic admissibility of both velocity weakening and strengthening, but do not incorporate the observed direct effect (logarithmic relaxation) [11].

However, in velocity-step experiments dynamic friction is a steady state phenomenon so we are looking for a deviation of the frictional coefficient from a fixed value, μ_0 , at a given V_0 reference velocity. On the other hand, the internal variable is expected to be zero, when its time derivative is zero that is, in thermodynamic equilibrium. Hence, we face at a mixed, partial steady state and partial equilibrium situation. Unfortunately, the dissipation inequality Eq. (12) may not characterize the deviation from steady state. What we need is an estimation of the deviation of the entropy production from its steady state value.

In our framework, the steady state means a constant displacement rate, denoted by V_0 . Therefore the internal energy should be determined accordingly

$$U = E_0 - m \frac{(V - V_0)^2}{2} - k \frac{r^2}{2}, \quad (13)$$

where E_0 is the energy of the body moving with the velocity

V_0 ³. Then the calculation of the entropy production results in

$$\begin{aligned}T\dot{S}(U) &= F_e V - m(V - V_0)\dot{V} - kr\dot{r} = \\ &= F_e V_0 + F_d(V - V_0) - kr\dot{r} \geq 0.\end{aligned}\quad (14)$$

We can introduce a friction specific entropy production with Amontons' law:

$$\frac{T}{F_n A} \dot{S} = \Sigma = \mu_0 V_0 + \Delta\mu(V - V_0) - \kappa r\dot{r} \geq 0, \quad (15)$$

where $\mu_0 = F_e / (AF_n)$ is the external shear stress divided by the normal force, $\Delta\mu = F_d / (AF_n)$ is the frictional stress divided by the normal force, and $\kappa = \frac{k}{AF_n}$.

Now, it is straightforward to introduce a linear approximation for the increment of entropy production, as for the near-equilibrium situation. However, in the following we apply a different starting point. Our fundamental assumption is that the leading term of the deviation from the steady state is *logarithmic*, while it is linear around equilibrium. We can formulate this hypothesis analogously to the classical exploitation of the entropy principle introducing the concept of *incremental entropy production*. We require that it is minimal at the steady state

$$\Delta\Sigma = \Sigma - \Sigma_{steady} = \Sigma - \mu_0 V_0 = \Delta\mu \ln\left(\frac{V}{V_0}\right) - \kappa r\dot{r} \geq 0 \quad (16)$$

We will call this hypothesis *the principle of minimal incremental entropy production*.

The required minimality ensures the asymptotic stability of the steady state if the evolution equations are constructed accordingly. In this respect, the principle is similar to the role of the second law near to the equilibrium. On the other hand, this hypothesis is a modification of the requirement of non-negative excess entropy production of Glansdorff and Prigogine [18]. The difference from the Prigogine-Glansdorff requirement is the logarithmic deviation instead of a linear one. (The possibility of partial steady state, a specific property of friction, is important, too.) Along with these we want to emphasize that the inequality here is not a general law of nature, it is to be regarded as a convenient stability assumption during seeking for an adequate model [19, 20].

An example of similar logarithmic deviation is the thermodynamics of chemical kinetics, where the entropy production is a product of the chemical affinity and the reaction rate, but the Guldberg-Waage kinetic equations introduce an exponential relation [21]. It is remarkable that chemical equilibrium is considered as a steady state from a thermodynamic point of view, when forward and backward reactions are properly distinguished [22, 23]. Our formula corresponds to a simple

³The proper relation of kinetic and internal energies in a Galilei relativistic framework is a delicate question from a thermodynamic point of view. A detailed treatment of objective, frame independent thermodynamic modeling in case of single component fluids is given in [17]. A similar approach is required in our case, too.

monomolecular reaction, which is actually equivalent to a single internal degree of freedom [21]. In friction the logarithmic deviation may be further motivated according to this chemical analogy [10].

Finally, we remark that the logarithmic form can be directly derived assuming that the internal energy is modified by a logarithmic velocity dependent term instead of a quadratic one: $U = E_0 - mV(\ln(V/V_0) - 1)$.

Eq. (16) is similar to the usual entropy production in many respects. First of all, the two terms are zero in the reference steady state and we assume that $\Delta\mu$ is a constitutive function of the logarithmic deviation, a function of the thermodynamic state variable. Therefore, in case of smooth functions, the Lagrange mean value theorem ensures a linear homogeneous relationship between these constitutive quantities and the related thermodynamic forces also in this mixed steady-state equilibrium case.

The linear solution of inequality Eq. (16) results in the following expressions:

$$\Delta\mu = l_1 \ln\left(\frac{V}{V_0}\right) - l_{12}kr, \quad (17)$$

$$\dot{r} = l_{21} \ln\left(\frac{V}{V_0}\right) - l_2kr. \quad (18)$$

We will call Eqs. (17)-(18) as *thermodynamical aging law*. Eq. (17) is identical to the Nakatani form of the constitutive law Eqs. (5) and (18) is similar to Eq. (7).

The coefficient matrix may depend on the thermodynamic forces, in particular, l_1 , l_2 , l_{12} and l_{21} may depend on the state variables V and r [24]. In the following we assume a strict linear relationship, when the coefficient matrix is constant. It is remarkable that there are no reasons to assume symmetricity or antisymmetricity of the matrix. The conditions of Onsagerian statistical background cannot be introduced without a particular interpretation of the internal variable (more detailed arguments are given in [25,26]).

3 Different mechanisms of different relaxations

According to the experimental observations in case of velocity step and healing experiments, sometimes the internal variable changes when the surfaces slip, and sometimes its evolution is seemingly independent of the relative motion of the surfaces. This distinction is connected to the detailed mechanism of friction, and requires an extension of the modeling framework. In the following we will investigate the question of slip related internal variable evolution.

Ruina [6] applied this assumption directly to the relaxation. In our case that requires the modification of Eq. (18), assuming that the slip is what makes the internal variable change, and the rate of the variable is a consequence. In this case the time derivative in Eq. (18) is substituted by the space derivative and the rate is obtained as a consequence:

$$\frac{dr}{dt} \rightarrow \frac{dr}{dx} = \frac{\dot{r}}{V}. \quad (19)$$

Performing this substitution in Eq. (18) leads to

$$\dot{r} = l_{21}V \ln\left(\frac{V}{V_0}\right) - V l_2kr. \quad (20)$$

Eq. (20), together with Eq. (17) will be called *thermodynamical slip law*. One can see, that the Nakatani transformed Ruina law Eqs. (5),(7) can be obtained if $l_1 = a$, $l_{12} = -1/\kappa$, $l_{21} = -b/L$ and $l_2 = 1/(\kappa L)$. The number of phenomenological parameters is increased by one, from three to four, compared to the original Ruina theory. It is because κ appears only as a multiplier of the cross-coefficients l_{12} and l_{21} . The slip governed modification represents a particular quasilinear form of the strictly linear relations of the thermodynamic aging law Eqs. (17)-(18).⁴

The slip condition of Ruina, expressed by Eq. (19) assumes that V is the relative surface velocity, related to the permanent parts of the displacement. When introducing a distinction between permanent and recoverable parts of the apparent displacement beyond the difference of the load point and relative surface velocities, one may expect that only the permanent part contributes to the entropy production, to the evolution of the internal variable. For example, an interpretation of the internal variable as recoverable displacement with the condition Eq. (11) leads to the following permanent displacement:

$$x_{per} = \int_{t_0}^t z(s)ds = x - r.$$

Particular mechanisms require that the internal variable influences the slip. For example, when the internal variable is connected to deformation of surface irregularities then this conclusion is straightforward. Therefore, we assume in general that the internal variable directly influences the displacement, and the reduced part is what influences the (incremental) entropy production. Hence, we introduce $x_{red} = x - \alpha r$, where $0 \leq \alpha \leq 1$ is the *factor of slip reduction*. If $\alpha = 1$ then the internal variable can be interpreted as the recoverable part of the displacement [11].

The evolution equation of the internal variable is obtained by substituting the time derivative with the slip related change of the internal variable, as follows:

⁴In our thermodynamical framework, a slip related change may lead to further consequences. In order to keep the integrity of the thermodynamic considerations, the calculation of the entropy rate may be substituted by the calculation of the slip related entropy change.

For example, with the internal energy Eq. (13), the derivative of the entropy by the displacement will be the following:

$$\frac{dS}{dx} = \frac{1}{T} \left(F_e \frac{V_0}{V} + F_d \frac{V - V_0}{V} - kr \frac{dr}{dx} \right). \quad (21)$$

Then we proceed assuming logarithmic increment and obtain Eq. (17) and (20) as a consequence. However, slip and displacement are not the same, the calculation of slip related changes should distinguish between the permanent and recoverable parts. Here we do not analyse this possibility further, we accept the approach of Ruina at this point.

$$\frac{dr}{dt} \rightarrow \frac{dr}{d(x - \alpha r)} = \frac{\dot{r}}{V - \alpha \dot{r}}.$$

For the sake of simplicity, α is constant. Then a rearrangement leads to the following evolution equation of the internal variable:

$$\dot{r} = V \frac{l_{21} \ln\left(\frac{V}{V_0}\right) - l_2 \kappa r}{1 + \alpha \left(l_{21} \ln\left(\frac{V}{V_0}\right) - l_2 \kappa r\right)}. \quad (22)$$

Eq. (22) together with Eq. (17) is called *thermodynamical friction law* in the following. This is another particular quasi-linear form of the thermodynamic aging law Eqs. (17) - (18).

In the above thermodynamical models, there is a direct effect and the conventional step test parameters are $a = l_1$ and $b = \frac{l_2 l_{21}}{l_2}$. In the following we compare the performance of the obtained thermodynamic models with the classical models and experiments.

4 Velocity step tests

A comparison of the different rate- and state-dependent friction laws is shown in Fig. 2. The velocity weakening experiment is modeled with the following parameters: $\mu_0 = 0.6$, $V_0 = 1 \mu\text{m/s}$, $V_1 = 10 \mu\text{m/s}$, $L = 20 \mu\text{m}$, $a = 0.015$, and $b = 0.02$.

Dieterich and Ruina models are shown by solid thin lines. The parameters of the thermodynamic friction model are calibrated to give the proper step conditions and relaxation speed: $l_1 = a$, $l_2 = 1/L$, $l_{12} = b$, $l_{21} = l_2$. There are two additional parameters, the static recovery strength κ , analogous to a spring constant, and the factor of slip reduction α . The dotted curve runs exactly over the line obtained by the Ruina model because thermodynamic slip model recovers the Ruina case when $\kappa = 1$ and $\alpha = 0$. One obtains highly asymmetric relaxation curves choosing higher values of the slip reduction parameter α , and with a proper choice one calculates curves that are close to the Dieterich model. E.g., the thick dashed curve was calculated with $\kappa = 0.85$ and $\alpha = 8$. Moreover, one can obtain symmetric relaxation curves close to either the down or up relaxation curves of the Dieterich model with an appropriate choice of the parameters. For example, the thick dotdashed line was calculated using $\kappa = 0.7$ and $\alpha = 2$.

The thermodynamic aging law produces asymmetric curves, similar to the Dieterich law.

5 Healing

The interpretation of healing is contradictory, here we have chosen the experiments and strategy of Beeler and Tullis [3, 27] for demonstration. They performed healing experiments with various machine rigidities. One of them, $k = 0.002$, was the natural elasticity of the experimental device, and in this case the load point velocity was $V_0 = 1 \mu\text{m/s}$. With the $k = 0.074$ servocontrolled value the load point velocity was $V_0 = 0.316 \mu\text{m/s}$. In both cases the rock parameters were $L = 3 \mu\text{m}$, $a = 0.009$, and $b = 0.008$. The data points in the first and second cases

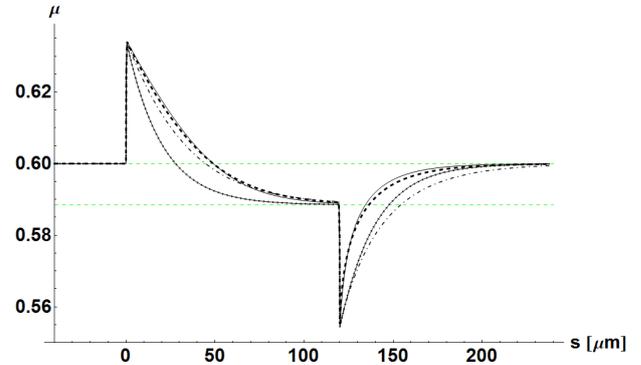


Fig. 2. Simulation of a velocity step test with various friction laws. The Dieterich and Ruina laws are drawn by solid thin lines. The thermodynamic friction law leads to the dotted line with $\kappa = 1$ and $\alpha = 0$ parameters running over the line of Ruina law. The dashed line with parameters $\kappa = 0.85$ and $\alpha = 8$ runs close to the Dieterich law. For the dotdashed line the parameters are $\kappa = 0.7$ and $\alpha = 2$.

are shown on Fig. 3 by circles and rectangles, respectively. The simulation used the experimental rock parameters of Beeler et al. [27], and the additional parameters were chosen as $\alpha = 0.75$ and $\kappa = 0.9$. Then we have obtained the dashed curve for the $k = 0.074$ case and the dotted one for $k = 0.002$.

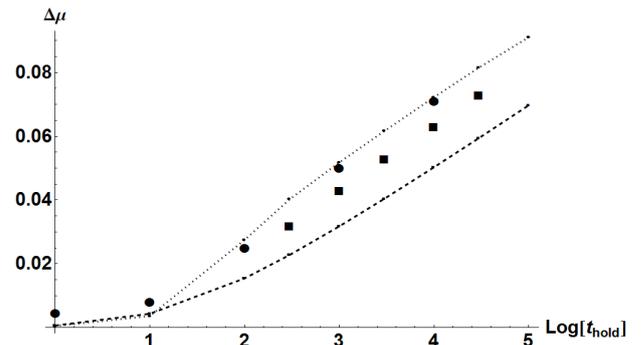


Fig. 3. Simulation of the healing experiments of Beeler et al. [27]. The experimental data for the $k = 0.002$ case is shown by the big circles and for the $k = 0.074$ case by the rectangles. The dotted curve is calculated by the thermodynamic friction model for $k = 0.002$. The $k = 0.074$ parameter resulted in the dashed curve.

Figures 4 and 5 show the effect of changing the initial velocity and the machine rigidity. In Fig. 4 the simulation of the healing experiment with the more rigid machine is shown, where the parameter values are $k = 0.002$, $V_0 = 1 \mu\text{m/s}$, $L = 3 \mu\text{m}$, $a = 0.009$, $b = 0.008$, $\alpha = 0.75$ and $\kappa = 0.9$ (dotted curve in Fig. 3). Increasing the velocity to $V_0 = 2 \mu\text{m/s}$ pushes the curve upward and parallel to the original one, shown by the dashed curve. In Fig. 5 the effect of softening is demonstrated, the dashed curve is calculated with $k = 0.02$. The dotted curve is identical to the one in Fig. 4. The increase of the healing effect qualitatively corresponds to the experimental observations.

6 Conclusions

A non-equilibrium thermodynamic model with a single internal variable of rate- and state-dependent friction was proposed. The model introduced the following basic assumptions:

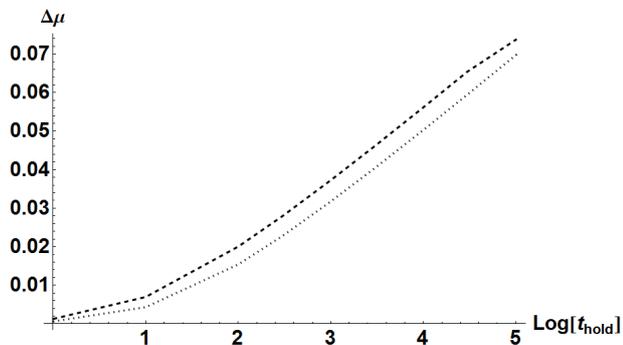


Fig. 4. The effect of larger steady state velocity for healing. $V_0 = 1 \mu\text{m/s}$, dotted curve, $V_0 = 2 \mu\text{m/s}$, dashed one.

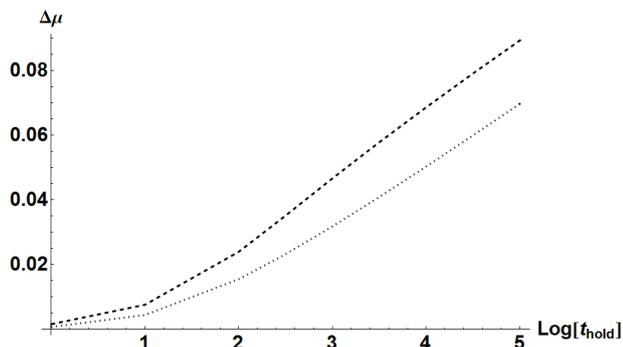


Fig. 5. The effect of softened machine for healing. $k = 0.002$, dotted curve, $k = 0.02$, dashed one.

- The deviation from the steady state is logarithmic.
- The change of internal variable is due to a reduced slip, and the reduction is proportional to the value of the internal variable.

The obtained thermodynamic friction model generalizes the well-known Dieterich and Ruina laws with two additional parameters.

- For velocity step tests, it interpolates between the Dieterich and Ruina laws. The form of the relaxation depends on the additional parameters. One can obtain symmetric up and down relaxation curves that are either close to the curves of Ruina relaxation or are close to the Dieterich type relaxation, but with less apparent linear part.
- Simulations show promising results for healing experiments, both quantitatively and qualitatively.

Acknowledgement

The work was supported by the grants OTKA K81161 and K104260.

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