

Reliability Based Analysis and Optimum Design of Laterally Loaded Piles

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RESEARCH ARTICLE

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Abstract

In this study reliability based limit analysis is used to determine the ultimate capacity of laterally loaded piles. The aim of this study is to evaluate the lateral load capacity of free-head and fixed-head long pile when plastic limit analysis is considered. In addition to the plastic limit analysis to control the plastic behaviour of the structure, uncertain bound on the complementary strain energy of the residual forces is also applied. This bound has significant effect for the load parameter. The solution to reliability-based problems is based on a direct integration technique and the uncertainties are assumed to follow Gaussian distribution. The optimization procedure is governed by the reliability index calculation.

Keywords

reliability, laterally loaded pile, residual strain energy, probability, limit analysis, reliability index

1 Introduction

In engineering practice the uncertainties play a very important role [1-3] and need intensive calculations. There are several engineering problem where the designer should face to the problem of limited load carrying capacity of the connected elements of the structures [4, 5]. Evaluate of the lateral load capacity is an important component in the analysis and design of pile foundations subjected to lateral loadings and soil movements. Elastic-plastic solutions for free head and fixed head single laterally loaded piles were developed recently [6, 7]. They are subsequently extended to cater for response of pile groups by incorporating p-multipliers.

One of the most successful applications of the variational formulation in the incremental plasticity theory is the theory of limit analysis. The basic ideas of the principles of limit analysis were first recognized and applied to the steel beams by Kazinczy [8]. The fundamental problem of limit analysis is to determine the plastic limit load multiplier and the stresses, strain rates and velocities at the plastic limit state of the body. This can always be achieved by conducting an incremental analysis which is, however, usually very time-consuming. The main advantage of the extremum principles lies in the fact that, without study of the entire loading history, they directly provide the exact value of the upper and lower bounds of the plastic limit load multiplier. This is achieved merely by considering the sets of statically or kinematically admissible stress or strain rate fields of the body, Kaliszky [9]. At the plastic limit state the stresses can maintain a static equilibrium with the plastic limit load and, at the same time, satisfy the yield condition at every point in the body. Briefly, the plastic limit load is the largest load which can be balanced by the stresses satisfying the yield conditions, and the smallest load which can convert the body into a yield mechanism.

At the application of the plastic analysis and design methods the control of the plastic behaviour of the structures is an important requirement. Since the limit analysis provides no information about the magnitude of the plastic deformations and residual displacements accumulated before the adaptation of the structure, therefore for their determination several

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bounding theorems and approximate methods have been proposed. Among others Kaliszky and Lógó [10] suggested that the complementary strain energy of the residual forces could be considered an overall measure of the plastic performance of structures and the plastic deformations should be controlled by introducing a limit for magnitude of this energy. In engineering the problem parameters (geometrical, material, strength, manufacturing) are given or considered with uncertainties. The obtained analysis and/or design task is more complex and can lead to reliability analysis and design. Instead of variables influencing performance of the structure (manufacturing, strength, geometrical) only one bound modelling resistance scatter can be applied. The bound on the complementary strain energy of the residual forces controlling the plastic behaviour of the structure can be utilized. This bound has significant effect for the limit load multipliers [5]. The aim of this study is to evaluate the lateral load capacity of long pile with limited residual strain energy on the probabilistically given conditions. If the design uncertainties (manufacturing, strength, geometrical) are expressed by the calculation of the complementary strain energy of the residual forces the reliability based plastic limit analysis problems can be formed. In this study numerical procedures elaborated with a direct integration technique and the uncertainties are assumed to follow Gaussian distribution. The formulations of the problems yield to nonlinear mathematical programming. The optimization procedure is governed by the reliability index calculation. The parametric study is illustrated by the solution of examples.

2 Mechanical Modelling and the Analysis

Short and long piles fail under different mechanisms. A short rigid pile, unrestrained at the head, tends rotate or tilts as shown in (Fig. 1a) and passive resistance develops above and below the point of rotation on opposite sides of the pile. For long pile, the passive resistance is very large and pile cannot rotate or tilt. The lower portion remains almost vertical due to fixity while the upper part deflects in flexure. The pile fails when a plastic hinge is formed at the point of maximum bending moment, (Fig. 1b). Long pile fails when the moment capacity is exceeded (structural failure).

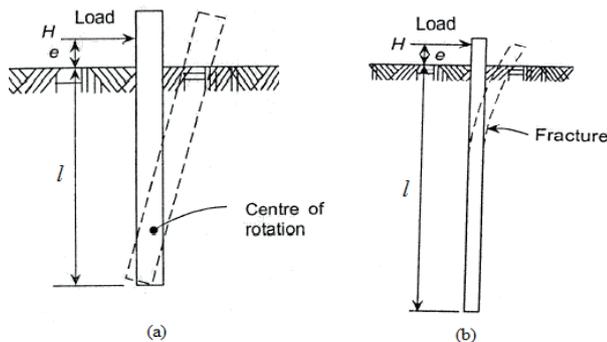


Fig. 1 Failure mechanisms of pile under horizontal load:
(a) short rigid pile, (b) long pile

Broms and Silberman [11] assumed simplified distribution of soil resistance for cohesionless soils and determined the load capacity of long piles in terms of the flexural rigidity of the pile. The design chart prepared by Broms is given (Fig. 2).

Assuming a uniform pile cross section, a plastic hinge with a moment of M_p will develop at the point of maximum bending moment that has no shear force, i.e. at point of failure in (Fig. 3). Pile under the lateral loading has a virtual lateral velocity V , V_0 at the pile head. The lateral velocity at any depth along the pile is assumed decreasing linearly from V_0 to 0 at point of failure and can be expressed as:

$$v = v_0 \left(1 - \frac{z}{l} \right) \quad (1)$$

where z is the depth measured from pile head, l is the depth where plastic hinge forms. This mechanism was originally proposed by Murff and Hamilton [12].

It is assumed that the lateral soil resistance is fully developed at the ultimate state. The ultimate soil resistance is described by the generic limiting force profile (LFP) proposed by Guo [13]

$$P_u = A_r (z + \alpha_0)^n \quad (2)$$

where P_u = ultimate soil resistance or limiting force per unit length; $A_r = s_u N_g d^{1-n}$ (cohesive soil) and $\gamma_s N_g d^{2-n}$ (cohesionless soil), gradient of the limiting force profile; d = the outer diameter of the pile; α_0 = an equivalent depth to consider the resistance at the ground surface, and $n (< 3)$ = the power governing the shape of the limiting force profile, the values of $n = 0.7$ and 1.7 are generally sufficient accurate for piles in clay and sand; z = depth below the ground level;

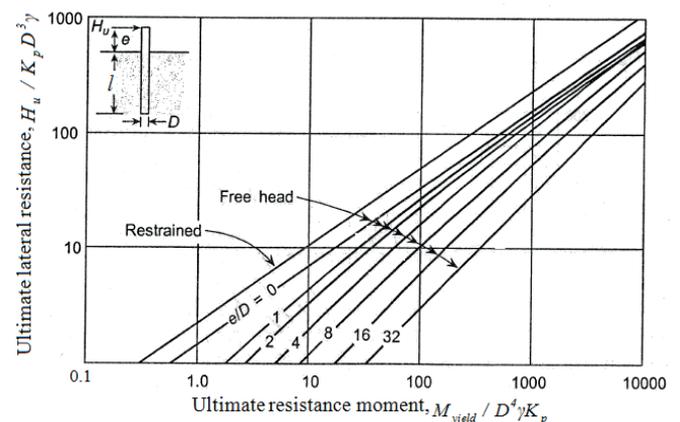


Fig. 2 Design chart for long piles in cohesionless soil

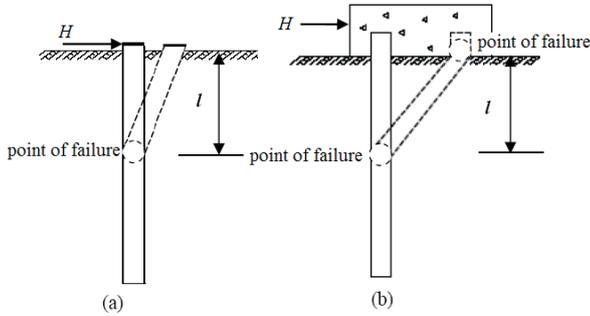


Fig. 3 Failure mechanism (a) free-head pile (b) fixed-head pile

S_u average undrained shear strength of cohesive soil; γ'_s effective unit weight of overburden soil (i.e. dry weight above water table and buoyant weight below); N_g gradient to correlate clay strength or sand weight with the ultimate resistance P_u . The magnitude of the three input parameters α_0 , N_g and n are independent of load levels over the entire loading regime. Guidelines for determining the values of the parameters are discussed by Guo [13-15]. The generic limiting force profile (LFP) becomes that suggested for sand by Broms [16], and that for clay by Matlock [17] and Reese et al. [18], by choosing an appropriate set of α_0 , N_g and n . For example, selecting $N_g = 3K_p$, $\alpha_0 = 0$ and $n = 1$, K_p = the coefficient of passive earth pressure, the limiting force profile becomes the Broms' [16] LFP for sand, while giving $\alpha_0 = 2d / N_g$, $N_g = \gamma'_s d / s_u + 0.5$, and $n = 1$, it reduces to Matlock's [17] LFP for soft clay. Here the virtual velocity v_0 will be cancelled. The best solution, i.e. the largest load, is found by maximizing the load H_u with respect to the optimization parameter l . The details of calculations are explained by Guo [13]. The solution for free head long piles are presented below:

$$l = \left[\alpha_0^{n+1} + (n+1) \frac{H_u}{A_r} \right]^{\frac{1}{n+1}} - \alpha_0 \quad (3)$$

The lateral load capacity can be calculated by:

$$\frac{M_p}{A_r} = \frac{1}{n+2} \left[\alpha_0^{n+1} + (n+1) \frac{H_u}{A_r} \right]^{\frac{n+2}{n+1}} - \left(\frac{\alpha_0^{n+2}}{n+2} + \alpha_0 \frac{H_u}{A_r} \right) \quad (4)$$

The influence of the loading eccentricity may be considered by replacing the plastic moment M_p with M_0 , where $M_0 = H_u e$, e is the eccentricity. Consequently:

$$\frac{M_p}{A_r} = \frac{1}{n+2} \left[\alpha_0^{n+1} + (n+1) \frac{H_u}{A_r} \right]^{\frac{n+2}{n+1}} - \left(\frac{\alpha_0^{n+2}}{n+2} + \alpha_0 \frac{H_u}{A_r} \right) + \frac{H_u e}{A_r} \quad (5)$$

For the case of a fixed-head pile, the energy dissipation due to the plastic moment M_p at the failure points is calculated. Following the same derivations as for the free-head piles, the ultimate lateral capacity for fixed-head piles can be easily determined.

3 Loadings

The structure is subjected to a dead load P_d and two independent, static working loads P_1 and P_2 with multipliers $m_1 \geq 0$, $m_2 \geq 0$ (Fig. 4). In the analysis five loading cases ($h = 1, 2, \dots, 5$) shown in (Table 1) are taken into consideration. For each loading case a plastic load multiplier m_{ph} can be calculated. Making use of these multipliers a limit curve can be constructed in the plane m_1, m_2 (Fig. 5). Structure does not shake down, under the action of the loads $m_1 P_1, m_2 P_2$, if the points corresponding to the multipliers m_1, m_2 lies inside or on the limit curve.

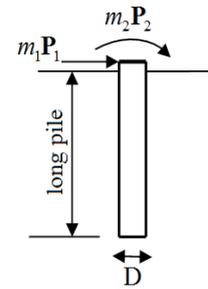


Fig. 4 Example of free head pile

At the application of the plastic analysis and design methods the control of the plastic behaviour of the structures is an important requirement. Following the suggestions of Kaliszky and Lógó [10] the complementary strain energy of the residual forces could be considered as an overall.

Table 1 Load combinations

h	Multipliers	Loads	Load multipliers
1	$m_2 = 0$	$Q_1 = P_1$	m_{s1}
2	$m_1 = 0$	$Q_2 = P_2$	m_{s2}
3	$m_1 = 0.5m_2$	$Q_3 = [0.5P_1, (0.5P_1 + P_2), P_2]$	m_{s3}
4	$m_1 = m_2$	$Q_4 = [P_1, (P_1 + P_2), P_2]$	m_{s4}
5	$m_1 = 2m_2$	$Q_5 = [2.0P_1, (2.0P_1 + P_2), P_2]$	m_{s5}

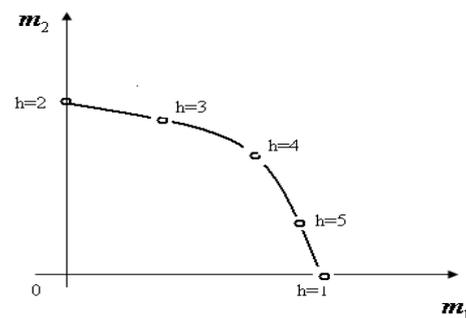


Fig. 5 Limit curve and safe domain

4 Reliability-Based Control of the Plastic Deformations

The measure of the plastic performance of structures and the plastic deformations should be controlled by introducing a bound for the magnitude of this energy:

$$\frac{1}{2} \sum_{i=1}^n \mathbf{Q}_i^r \mathbf{F}_i \mathbf{Q}_i^r \leq W_{p0} \quad (6)$$

Here W_{p0} is an assumed bound for the complementary strain energy of the residual forces and \mathbf{Q}^r residual internal forces. This constraint can be expressed in terms of the residual moments $M_{i,a}^r$ and $M_{i,b}^r$ acting at the ends (a and b) of the finite elements as:

$$\frac{1}{6E} \sum_{i=1}^n \frac{1}{I_i} \left[(M_{i,a}^r)^2 + (M_{i,a}^r)(M_{i,b}^r) + (M_{i,b}^r)^2 \right] \leq W_{p0} \quad (7)$$

By the use of (7) a limit state function can be constructed:

$$g(W_{p0}, \mathbf{M}^r) = W_{p0} - \frac{1}{6E} \sum_{i=1}^n \frac{1}{I_i} \left((M_{i,a}^r)^2 + (M_{i,a}^r)(M_{i,b}^r) + (M_{i,b}^r)^2 \right) \quad (8)$$

The plastic deformations are controlled while the bound for the magnitude of the complementary strain energy of the residual forces exceeds the calculated value of the complementary strain energy of the residual forces. Introducing the basic concepts of the reliability analysis and using the force method the failure of the structure can be defined as:

$$g(X_R, X_S) = X_R - X_S \leq 0; \quad (9)$$

where X_R indicates either the bound for the statically admissible forces X_S or a bound for the derived quantities from X_S . The probability of failure is given by

$$P_f = F_g(0) \quad (10.a)$$

and can be calculated as

$$P_f = \int_{g(X_R, X_S) \leq 0} f(X) dx. \quad (10.b)$$

Let assumed that due to the uncertainties the bound for the magnitude of the complementary strain energy of the residual forces is given randomly and for sake of simplicity it follows the Gaussian distribution with given mean value \bar{W}_{p0} and standard deviation $\hat{\sigma}_w$. Due to the number of the probabilistic variables (here only single) the probability of the failure event can be expressed in a closed integral form:

$$P_{f,calc} = \int_{g(W_{p0}, \mathbf{M}^r) \leq 0} f(\bar{W}_{p0}, \tilde{A}_w) dx \quad (10.c)$$

By the use of the strict reliability index a reliability condition can be formed:

$$\beta_{target} - \beta_{calc} \leq 0 \quad (10.d)$$

where \hat{a}_{target} and \hat{a}_{calc} are calculated as:

$$\beta_{target} = -\Phi^{-1}(P_{f,target}) \quad (10.e)$$

$$\beta_{calc} = -\Phi^{-1}(P_{f,calc}) \quad (10.f)$$

here Φ^{-1} : inverse cumulative distribution function (so called probit function) of the Gaussian distribution. (Due to the simplicity of the present case the integral formulation is not needed, since the probability of failure can be described easily with the distribution function of the normal distribution of the stochastic bound W_{p0}).

5 Plastic Limit Analysis

5.1 Basic design formulations

Determine the maximum load multiplier m_{ph} and cross-sectional dimensions under the conditions that (i) the structure with given layout is strong enough to carry the loads ($\mathbf{P}_d + m_{ph} \mathbf{Q}_h$), (ii) satisfies the constraints on the limited beam-to-column strength capacity, (iii) satisfies the constraints on plastic deformations and residual displacements, (iv) safe enough and the required amount of material does not exceed a given limit. The design solution method based on the static theorem of limit analysis [19] is formulated as below:

Maximize

$$m_{ph} \quad (11.a)$$

Subject to:

$$\mathbf{G}^* \mathbf{M}_d^p + \mathbf{P}_d = \mathbf{0} \quad (11.b)$$

$$\mathbf{G}^* \mathbf{M}_h^p + m_{ph} \mathbf{Q}_h = \mathbf{0} \quad (11.c)$$

$$\mathbf{M}_d^e = \mathbf{F}^{-1} \mathbf{G} \mathbf{K}^{-1} \mathbf{P}_d \quad (11.d)$$

$$\mathbf{M}_h^e = \mathbf{F}^{-1} \mathbf{G} \mathbf{K}^{-1} m_{ph} \mathbf{Q}_h \quad (11.e)$$

$$-2S_{0i} \sigma_y \leq (\mathbf{M}_{di}^p + \max \mathbf{M}_{hi}^p) \leq 2S_{0i} \sigma_y, \quad (i = 1, 2, \dots, n) \quad (11.f)$$

$$-\bar{\mathbf{M}}_j^p \leq (\mathbf{M}_{dj}^p + \max \mathbf{M}_{hj}^p) \leq \bar{\mathbf{M}}_j^p, \quad (j = 1, 2, \dots, n) \quad (11.g)$$

$$\mathbf{M}_i^r = \left[\left(\max \mathbf{M}_{hi}^e + \mathbf{M}_{di}^e \right) \right] - \left[\left(\max \mathbf{M}_{hi}^p + \mathbf{M}_{di}^p \right) \right] \quad (i=1,2,\dots,n) \quad (11.h)$$

$$\beta_{target} - \beta_{calc} \leq 0 \quad (11.i)$$

$$\sum_i A_i \ell_i - V_0 \leq 0. \quad (11.j)$$

\mathbf{P}_d is a vector of dead load. $\mathbf{M}_h^e, \mathbf{M}_d^e$ are vectors of fictitious elastic moments calculated from the live and dead loads assuming that the structure is purely elastic.

\mathbf{M}^r is vector of residual internal moment and $\mathbf{M}_d^p, \mathbf{M}_h^p$ are vectors of plastic moments. $\bar{\mathbf{M}}^p$ is a vector of limit moment. σ_y : yield stress, S_0 : statical moment of cross section. $\mathbf{F}, \mathbf{K}, \mathbf{G}, \mathbf{G}^*$: flexibility, stiffness, geometrical and equilibrium matrices, respectively; V_0 represents the total limit volume of the structure; \mathbf{Q}_h is a vector of load combinations, ($h = 1, 2, \dots, n$). Here (11.b) and (11.c) are equilibrium equations for the dead loads and for the live (pay) loads, respectively. Equations (11.d) and (11.e) express the calculations of the elastic fictitious internal forces (moments) from the dead loads and from the live (pay) loads, respectively. Equation (11.f) is the yield condition. Equation (11.g) is used as yield condition for lateral capacity of piles in term of plastic moment. Equations (11.h) is used to calculate the residual forces while (11.i) is the reliability condition which controls the plastic behaviour of the structure by use of the residual strain energy. Due to the mathematical nature of problem (11.a-j) an iterative procedure was elaborated which is governed by solving the equation (11.i). By selecting the diameter of long pile for each load combination \mathbf{Q}_h a plastic limit load multiplier m_{ph} can be determined and then the limit curve of the plastic limit state can be constructed. Due to the mathematical nature of problem (11.a)-(11.i) an iterative procedure was elaborated which is governed by solving (11.i). This is a nonlinear mathematical programming problem which can be solved by any appropriate solution method (e.g. NLP). Selecting one of the load combinations \mathbf{Q}_h a plastic limit load multiplier m_{ph} can be determined.

5.2 Alternative design formulation

Interchanging the objective function -Eq. (11.a)- and the last constraint -Eq. (11.j)- an alternative design formulation can be formulated:

Minimize

$$V = \sum_i A_i \ell_i \quad (12.a)$$

Subject to:

$$\mathbf{G}^* \mathbf{M}_d^p + \mathbf{P}_d = \mathbf{0} \quad (12.b)$$

$$\mathbf{G}^* \mathbf{M}_h^p + m_{ph} \mathbf{Q}_h = \mathbf{0} \quad (12.c)$$

$$\mathbf{M}_d^e = \mathbf{F}^{-1} \mathbf{G} \mathbf{K}^{-1} \mathbf{P}_d \quad (12.d)$$

$$\mathbf{M}_h^e = \mathbf{F}^{-1} \mathbf{G} \mathbf{K}^{-1} m_{ph} \mathbf{Q}_h \quad (12.e)$$

$$-2S_{0i} \sigma_y \leq (\mathbf{M}_{di}^p + \max \mathbf{M}_{hi}^p) \leq 2S_{0i} \sigma_y, \quad (i=1,2,\dots,n) \quad (12.f)$$

$$-\bar{\mathbf{M}}_j^p \leq (\mathbf{M}_{dj}^p + \max \mathbf{M}_{hj}^p) \leq \bar{\mathbf{M}}_j^p, \quad (j=1,2,\dots,k) \quad (12.g)$$

$$\mathbf{M}_i^r = \left[\left(\max \mathbf{M}_{hi}^e + \mathbf{M}_{di}^e \right) \right] - \left[\left(\max \mathbf{M}_{hi}^p + \mathbf{M}_{di}^p \right) \right], \quad (i=1,2,\dots,n) \quad (12.h)$$

$$\beta_{target} - \beta_{calc} \leq 0 \quad (12.i)$$

$$m_{ph} - m_0 \leq 0. \quad (12.j)$$

This is nonlinear mathematical programming problem (12) leads to same optimal solution as problem (11) which can be proved by the use of the optimality conditions.

6 Numerical Examples

To demonstrate the theories and solution strategy introduced above, a nonlinear mathematical programming procedure is elaborated where one has to determine the safe loading domain of a laterally loaded long pile with deterministic loading data and with probabilistic bound for the magnitude of the complementary strain energy of the residual forces.

The application of the method is illustrated by two examples. The first example shows a free-head steel pile subjected to a lateral load and bending moment at its top with diameter of D in cohesionless soil (Fig. 6). The working loads are $\mathbf{P}_1 = \mathbf{H} = 10kN$, $\mathbf{P}_2 = \mathbf{M} = 20kN$ and $\mathbf{P}_d = \mathbf{0}$. The yield stress and the Young's modulus are $\sigma_y = 21kN/cm^2$ and $E = 2.06 \cdot 10^4 kN/cm^2$.

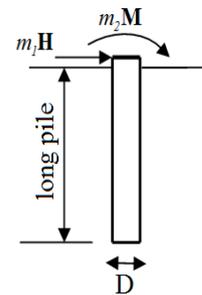


Fig. 6 Loads on the free-head pile

The results of the solution technique are presented in (Figs. 7 and 8) where deterministic loading is considered. The results are in very good agreement with the expectations. In (Fig. 7) one can see the safe loading domains in function of different expected probability. In (Fig. 8) the safe limit load domain is presented in case of different mean values of the complementary strain energy of the residual forces ($\bar{W}_{p0} = 30; 35; 40; 45$)

with standard deviation $\sigma_w = 3.5$ and target reliability index $\hat{a}_{target} = 3.2$. One can see that increasing the mean values results bigger safe loading domain.

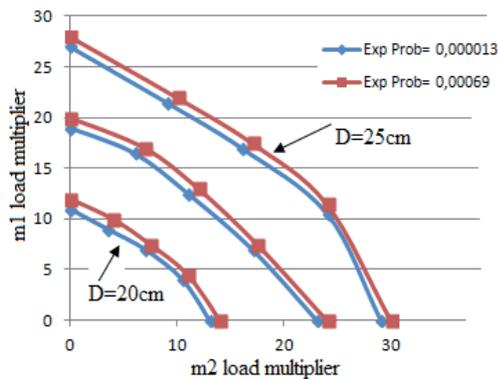


Fig. 7 Safe loading domain for plastic limit design

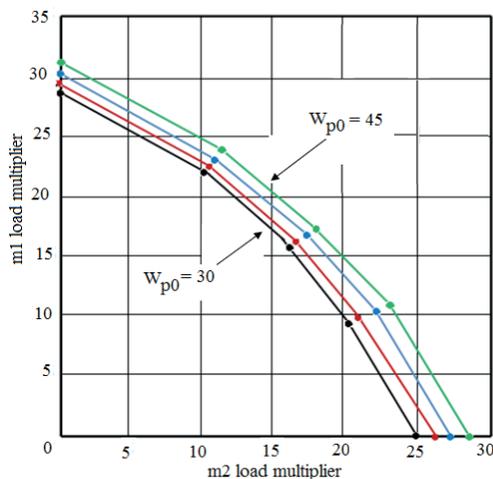


Fig. 8 Safe loading domain for plastic limit design

The second example shows a fixed-head steel pile subjected to a lateral load and bending moment at its top with diameter of D in cohesionless soil (Fig. 9). The working loads are $P_1 = H = 10kN$, $P_2 = M = 40kN$ and $P_d = 0$. The yield stress and the Young's modulus are $\sigma_y = 21kN/cm^2$ and $E = 2.06 \cdot 10^4 kN/cm^2$.

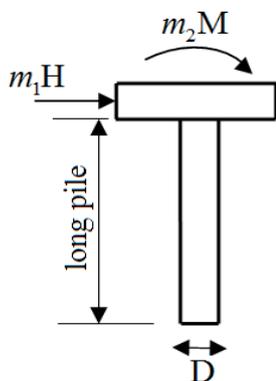


Fig. 9 Loads on the fixed-head pile

The results of the solution technique are presented in (Fig. 10) where deterministic loading is considered. In the figure one can see the safe loading domains in function of different expected probability.

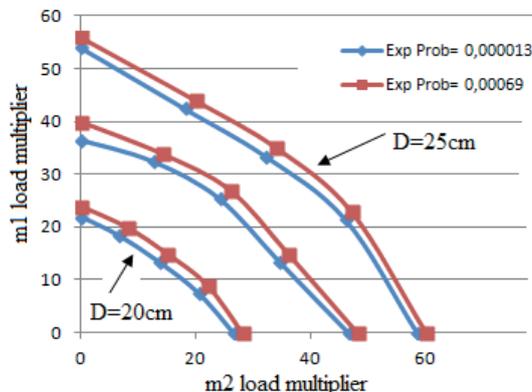


Fig. 10 Safe loading domain for plastic limit design

7 Conclusions

In this paper reliability based limit analysis is used to determine the ultimate capacity of laterally loaded long piles. To control the plastic behavior of the structure probabilistically given bound on the complementary strain energy of the residual forces is applied. Limit curves are presented for the plastic limit load multipliers. The numerical analysis shows that the given mean values and different expected probability on the bound of the complementary strain energy of the residual forces can influence significantly the magnitude of the plastic limit load. The presented investigation draws the attention to the importance of the problem but further investigations are necessary to make more general statements.

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