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RESEARCH ARTICLE

Spectral Method for Fatigue Damage Assessment of Structures with Uncertain Parameters

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Abstract
This study presents a spectral method for fatigue damage evaluation of linear structures with uncertain-but-bounded parameters subjected to the stationary multi-correlated Gaussian random excitation. The first step of the proposed method is to model uncertain parameters by introducing interval theory. Within the framework of interval analysis, the approximate expressions of the bounds of spectral moments of generic response are obtained by the improved interval analysis via the Extra Unitary Interval and Interval Rational Series Expansion. Based on the cumulative damage theory and the Tovo-Benasciutti method, the lower and upper bounds of expected fatigue damage rate are accurately evaluated by properly combining the bounds of the spectral parameters of the power density spectral function of stress of critical points. Finally, a numerical example concerning a truss under random excitation is used to illustrate the accuracy and efficiency of the proposed method by comparing with the vertex method.

Keywords
Interval theory, Extra Unitary Interval, Interval Rational Series Expansion, structure with uncertain parameters, expected fatigue damage rate

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1 Introduction

Fatigue damage is one of the major factors in the failures of engineering structure and mechanical equipment. The traditional method on fatigue damage assessment usually can be divided into two groups, namely, time domain approach and spectral approach. For the former, the traditional method on fatigue damage assessment is the so-called nominal-stress approach [1], which uses numerical methods for cycle counting and damage accumulation stepwise. To compute the time histories of the stress or strain of critical points, time domain method is computationally very time-consuming, especially for large structures. For the spectral approach [2], the external loads are modeled as stationary Gaussian processes, and fatigue damage evaluation is derived from the power spectral density (PSD) function of the stress or strain process of critical points in frequency domain.

The above two approaches always consider structural parameters like geometry, material properties, constitutive laws and boundary conditions as fixed values. However, it has been recently recognized that the model parameters are poorly known too, giving arise to structures with uncertain parameters. In these cases, the uncertain parameters are described by the following two methods, known as probabilistic and non-probabilistic approaches. For the first group, structural uncertain parameters are modeled as random variables or stochastic field, and structural generic response under external load is given by *stochastic finite element method or Monte Carlo simulation method*, and then the fatigue damage of the structure will be obtained based on the *cumulative damage theory*. However, the reliable probabilistic model requires a large amount of accurate sample data, which are sometimes difficult to get particularly in the preliminary design stage. Meanwhile, the characteristics of the structural uncertainty are not random. Therefore, non-probabilistic model is more suitable than probabilistic model to define the structure with uncertain parameters.

In the past decade, the non-probabilistic theory, which has made a considerable progress as a new method to describe the uncertainty, generally comprises c*onvex set theory, fuzzy set theory and interval theory*. Ben-Haim [3] first introduced *convex set theory* into fatigue assessment, creating the estimation method of fatigue damage by a non-probabilistic approach. Qiu[4] evaluated fatigue damage of truss structures by comparing the convex set model with probabilistic model. Through combining the *fuzzy sets* and *convex set*, Sun [5] constrained structure life function by the *fuzzy sets* and obtained the life function based on the *second-order interval Taylor series expansion and Lagrange extremes condition*. In the above study, only one single uncertain factor such as structural parameters or external loads is taken into consideration according to non-probabilistic model. However, it is widely accepted that the uncertainties of structural parameters and external loads coexist simultaneously, and both of them have a considerable effect on fatigue damage [6–8].

Recently, Muscolino and Sofi [9–11] presented the *improved interval analysis via extra unitary interval* by introducing *Extra Unitary Interval (EUI)* and *Rational Series Expansion (RSE)* into interval analysis theory. The improved method could express explicitly the generic dynamic response and the bounds of *interval reliability function* of the structure with uncertain-but-bounded parameters under stationary Gaussian random excitation. Do [12] captured the range of dynamic reliability of the structure with interval parameters and safe bounds by the improved particle swarm optimization algorithm. Compared with the dynamic reliability founded on the *first-passage failure*, the fatigue damage evaluation based on the *damage accumulation theory* is expecting for further research.

Based on the above analysis, a new method is brought out to assess the fatigue damage of structures with uncertain parameters under stationary multi-correlated Gaussian random excitation, in which the uncertainty of structural parameters is modeled by non-probabilistic interval model, and meanwhile the uncertainty of external loads is defined by probabilistic model. In the framework of the *interval theory* and the *Tovo-Benasciutti method* [13, 14], the bounds of the interval expected fatigue damage rate $E(D^I)$ of linear structures with uncertain-but-bounded parameters are conveniently evaluated by properly combining the bounds of the interval variables, regardless of the number of uncertain parameters in the uncertain structural system. In terms of computational efficiency, the proposed method requires less computational consumption than the *vertex method* does and its application makes a lot of sense in case of complex structural systems with many interval input variables.

The paper is organized as follows. In section 2, interval model is briefly expressed. The fatigue failure problem of the structure with uncertain-but-bounded parameters under stationary multi-correlated Gaussian random excitation is introduced in section 3. In section 4, the interval *power spectral density (PSD)* function of generic response and interval spectral moments are obtained by the *improved interval analysis via extra unitary interval*. In section 5, the bounds of the interval expected fatigue damage rate are obtained by the proposed method derived from the *improved interval analysis and T-B method*. Finally, in section 6, a numerical example concerning a truss structural system under random excitation is used to illustrate the accuracy and efficiency of the proposed method by comparing with the *vertex method* [15]. The results also demonstrate usefulness of the proposed method in view of decision-making in engineering practice.

2 Interval model

The interval model, originally developed from interval analysis, is a non-probabilistic approach to address the uncertain-but-bounded parameter problems caused by observation error. However, the application of the interval analysis in real engineering area is quite difficult for overestimating the interval width. This phenomenon, so-called *dependency phenomenon* [15], is unacceptable from an engineering point of view. In order to eliminate the *dependency phenomenon in the classic interval analysis, affine arithmetic and generalized interval analysis* are respectively presented.

In the case of slight parameter fluctuations, the uncertain parameter vector $\boldsymbol{\alpha}$ is modeled as an uncertain-but-bounded vector. Then each element of the vector is defined in a closed-bounded real set. According to the interval theory, $\boldsymbol{\alpha}_0$ and Δ_{α} denote the nominal value and radius of the uncertain vector $\boldsymbol{\alpha}$, which are defined as:

$$\boldsymbol{\alpha}_0 = \frac{\underline{\boldsymbol{\alpha}} + \bar{\boldsymbol{\alpha}}}{2}; \quad \Delta\boldsymbol{\alpha} = \frac{\bar{\boldsymbol{\alpha}} - \underline{\boldsymbol{\alpha}}}{2} \tag{1}$$

where $\underline{\boldsymbol{\alpha}}$ and $\bar{\boldsymbol{\alpha}}$ are respectively the *LB* and *UB* of the uncertain vectors. Then the interval vector can also be expressed as:

$$\boldsymbol{\alpha} = [\underline{\boldsymbol{\alpha}}\,,\ \bar{\boldsymbol{\alpha}}] = \{\boldsymbol{\alpha} |\ \underline{\boldsymbol{\alpha}} \leq \bar{\boldsymbol{\alpha}}\} = \boldsymbol{\alpha}_0 + \Delta\boldsymbol{\alpha}\hat{\mathbf{e}}^I \tag{2}$$

where $\hat{e}_i^I$ is called the *Extra Unitary Interval (EUI)*, and its values are unknown but assumed to lie in the unitary interval. The algorithm of the *EUI* can be found in Refs.[9, 10].

Eq. can also be expressed as a component form, i.e.:

$$\alpha_i = \alpha_{i,0} + \Delta\alpha_i \hat{e}_i^I, \quad i = 1,\ 2,\ \cdots,\ n \tag{3}$$

For convenient application, the deviation amplitudes $\Delta\alpha i$ can be reasonably defined as dimensionless fluctuations of the uncertain-but-bounded parameters around their nominal values. Under this definition, Eq. can be expressed as:

$$\alpha_i = \alpha_{i,0}\left(1 + \Delta\alpha_i\, \hat{e}_i^I\right), \quad i = 1,2,\cdots,n \tag{4}$$

3 Problem statement

3.1 Equations of motion

The dynamic equations of n-DOF linear structure with uncertain parameters can be expressed as:

$$\mathbf{M}_0\ddot{\mathbf{u}}(\boldsymbol{\alpha},t) + \mathbf{C}(\boldsymbol{\alpha})\,\dot{\mathbf{u}}(\boldsymbol{\alpha},t) + \mathbf{K}(\boldsymbol{\alpha})\mathbf{u}(\boldsymbol{\alpha},t) = \mathbf{F}(t) \tag{5}$$

where $\mathbf{M}_0$ is the $n\times n$ mass matrix, and it is assumed deterministic; $\mathbf{C}(\boldsymbol{\alpha})$ and $\mathbf{K}(\boldsymbol{\alpha})$ are the $n\times n$ matrices, and they respectively represent damping and stiffness matrices which are the functions of the uncertain parameter vector $\boldsymbol{\alpha}$. Generally, the uncertain parameters affecting the damping and stiffness matrices are assumed to be fully disjoint. $\mathbf{F}(t)$ is an n-dimensional column vector representing the equivalent nodal loads of the structure; $\mathbf{u}(\boldsymbol{\alpha}, t)$, $\dot{\mathbf{u}}(\boldsymbol{\alpha}, \mathrm{t})$ and $\ddot{\mathbf{u}}(\boldsymbol{\alpha}, \mathrm{t})$ are all n-dimensional column vectors which respectively represent the node displacement, velocity and acceleration of structures and they are also the functions of the uncertain parameter $\boldsymbol{\alpha}$.

For convenience, in this paper, the Rayleigh damping model is adopted as follows:

$$\mathbf{C}(\boldsymbol{\alpha}) = a_0\mathbf{M} + a_1\mathbf{K}(\boldsymbol{\alpha}) \tag{6}$$

where a_0 and a_1 are the Rayleigh damping constants and units are s^{-1} and s respectively.

By applying the *interval theory* and the *EUI*, the uncertain vector $\boldsymbol{\alpha}$ can be defined as an interval vector. Therefore, the matrices of structure with uncertain-but-bounded parameters are respectively represented as:

$$\begin{aligned}\mathbf{K}(\boldsymbol{\alpha}) \equiv \mathbf{K}^I &= \mathbf{K}_0 + \sum_{i=1}^{r}\mathbf{K}_i\Delta\alpha_i\hat{\mathbf{e}}_i^I;\\ \mathbf{C}(\boldsymbol{\alpha}) \equiv \mathbf{C}^I &= \mathbf{C}_0 + a_1\sum_{i=1}^{r}\mathbf{K}_i\Delta\alpha_i\hat{\mathbf{e}}_i^I.\end{aligned} \tag{7}$$

where $\mathbf{K}_i$ are sensitivity matrices which are obtained by sensitivity analysis of the stiffness matrix about uncertain parameter α_i, and they can be represented as:

$$\mathbf{K}_i = \left.\frac{\partial\mathbf{K}(\boldsymbol{\alpha})}{\partial\alpha_i}\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0} \tag{8}$$

In some cases, it should be noticed that if the uncertain parameter ρ_i is a nonlinear function of the structure matrix, it needs to be equivalent to a linear function of the structure matrix α_i by a suitable transformation[16]. For instance, if the length l_i of beams or bars is an uncertain parameter, it can be modeled as a linear parameter $\alpha_i = 1/l_i$ for the stiffness matrix of the structure.

In engineering area, load vector $\mathbf{F}(t)$ commonly can be decomposed into the mean-value vector, $\boldsymbol{\mu}_\mathbf{F}$, and zero-mean vector, $\mathbf{f}(t)$. They respectively describe the static part and the fluctuation of the external load. Due to application of the *Goodman criterion*, the fatigue damage, caused by fluctuating load $\mathbf{f}(t)$, can be equivalent to that by dynamic load $\mathbf{F}(t)$. Therefore, the fatigue damage only caused by zero-mean fluctuating load is considered in this paper. Generally, the fluctuation of the external load, which is modeled as a zero-mean stationary multi-correlated Gaussian random process, is fully characterized by the correlation function matrix, $\mathbf{R}_{\mathbf{ff}}(\tau)$.

Based on the *random vibration*, the *PSD* function matrix of generic response, $\mathbf{G}_{\mathbf{yy}}(\boldsymbol{\alpha},t)$ (e.g. displacement, strain or stress), can be expressed as:

$$\begin{aligned}\mathbf{G}_{\mathbf{yy}}(\boldsymbol{\alpha},\omega) &= \mathbf{q}^T\mathbf{G}_{\mathbf{uu}}(\boldsymbol{\alpha},\omega)\mathbf{q}\\ &= \mathbf{q}^T\mathbf{H}^*(\boldsymbol{\alpha},\omega)\mathbf{G}_{\mathbf{ff}}(\omega)\mathbf{H}^T(\boldsymbol{\alpha},\omega)\mathbf{q}\end{aligned} \tag{9}$$

where $\mathbf{q}$ represents the relation between generic response and displacement response; $\mathbf{H}(\boldsymbol{\alpha}, \omega)$ is the *frequency response function* (*FRF*) matrix of the structure, expressed as:

$$\begin{aligned}\mathbf{H}(\boldsymbol{\alpha},\omega) &= \left[\mathbf{K}(\boldsymbol{\alpha}) - \omega^2\mathbf{M} + i\omega\mathbf{C}(\boldsymbol{\alpha})\right]^{-1}\\ &= \left[\mathbf{H}_0^{-1}(\omega) + \mathbf{R}(\boldsymbol{\alpha},\omega)\right]^{-1}\end{aligned} \tag{10}$$

where $\mathbf{R}(\boldsymbol{\alpha}, \omega)$ represents the impact of uncertain parameters on the *FRF* matrix of the structure; $\mathbf{H}_0(\omega)$ is the nominal value of the *FRF* matrix, calculated by the nominal values of the mass, damping and stiffness matrices of the structure; they can be expressed as:

$$\begin{aligned}\mathbf{H}_0(\omega) &= \left[\mathbf{K}_0 - \omega^2\mathbf{M}_0 + i\omega\mathbf{C}_0\right]^{-1};\\ \mathbf{R}(\boldsymbol{\alpha},\omega) &= (1 + i\omega a_1)\sum_{i=1}^{r}\mathbf{K}_i\Delta\alpha_i\hat{\mathbf{e}}_i^I.\end{aligned} \tag{11}$$

To state clearly, it is assumed that the vector $\mathbf{q}$ is independent of uncertain parameters. If the assumption is not satisfied, the vector $\mathbf{q}$ will turn out to be a function of uncertain parameters $\boldsymbol{\alpha}$ in Eq.(9).

3.2 Spectral method of fatigue damage evaluation

For structures with uncertain parameters, according to the relation between the *Palmgren-Miner rule and the marginal probabilistic density* of the stress amplitude, the expected fatigue damage rate E(D) of deterministic structural systems, neglecting the effect of any mean stress, can be expressed as [14, 17]:

$$E(D) = v_a C^{-1}\int_0^{+\infty} s^k \cdot p_a(s)\mathrm{d}s \tag{12}$$

where s is the stress amplitude; C and k are parameters of the *S-N* curve; v_a represents the number of counted cycles per unit time; $p_a(s)$ is the *marginal probabilistic density* of the stress amplitude s, and it depends on the adopted counting method.

Because the variance of fatigue damage rate $\sigma^2(D)$, is much less than the $E(D)$, the fatigue life of the structure with uncertain parameters under random loads is usually equal to the reciprocal of the $E(D)$ [18], namely:

$$T \approx E(T) = \frac{1}{E(D)} \tag{13}$$

Similarly with the interval reliability analysis, the evaluation of fatigue damage is transformed into assessment of interval expected fatigue damage rate $E(D^I)$ of structures with uncertain-but-bounded parameters. Then the interval expected fatigue damage rate $E(D^I)$ and interval fatigue life T^I can be expressed as:

$$E(D^I) \equiv E[D(\boldsymbol{\alpha})] = v_a(\boldsymbol{\alpha})C^{-1}\int_0^{+\infty} s^k \cdot p_a(\boldsymbol{\alpha}, s)\mathrm{d}s;$$
$$T^I \equiv T(\boldsymbol{\alpha}) \approx E[T(\boldsymbol{\alpha})] = \frac{1}{E[D(\boldsymbol{\alpha})]}. \tag{14}$$

Based on Eq.(12) and Eq.(14), fatigue damage assessment by a spectral method can only be evaluated after cycle identification by assuming a cycle counting method. It is widely accepted that the $E(D)$, obtained by *Rayleigh assumption*, has a great correlation with the direct damage calculations about simulated time histories by the *Rain-Flow counting method* and *Palmgren-Miner rule* for narrow-band frequency stress. However, if the random process cannot be assumed to be a narrow band process, the $E(D)$ will be overestimated by the *Rayleigh assumption*.

Many scholars have raised improved formulas for the sake of reducing the expected fatigue damage rate, such as *Wirsching's approximate method*, *Drilik's approximation method* and *T-B approximate method*. As showed in Ref.[14], the *Wirsching's approximate method* always gains over-conservative estimates. Whereas the best estimates are obtained by *Drilik's approximation method* and *T-B approximate method*. It needs to be stressed that the *Drilik's approximation method* is an empirical formula obtained by fitting large amounts of data and it has no clear theoretical framework but the *T-B method* does. Meanwhile, compared with the *Dirlik's approximation method*, the *T-B method* can get the complete distribution of the RFC cycles in terms of amplitudes and mean values. So the *T-B method* is accepted to estimate $E(D)$ in evaluation of the uncertain structure.

Similarly with Eq.(14), by applying the *T-B method*, the $E(D^I)$ of structures with uncertain parameters can be expressed as:

$$\begin{aligned} E(D^I) &= E[D(\boldsymbol{\alpha})] \\ &= \{b(\boldsymbol{\alpha}) + [1 - b(\boldsymbol{\alpha})] \cdot \alpha_2^{k-1}(\boldsymbol{\alpha})\} \cdot E[D_{LCC}(\boldsymbol{\alpha})] \end{aligned} \tag{15}$$

where $b(\boldsymbol{\alpha})$ is the interval weighting factor; $E[D_{LCC}(\boldsymbol{\alpha})]$ represents the interval expected damage rate, which is determined on the basis of amplitude distribution derived from the *Level-Crossing Counting method* (*LCC*). They can be expressed as[19]:

$$b = \frac{[\alpha_1(\boldsymbol{\alpha}) - \alpha_2(\boldsymbol{\alpha})][1 - \alpha_1(\boldsymbol{\alpha}) \cdot \alpha_2(\boldsymbol{\alpha})]}{[1 - \alpha_2(\boldsymbol{\alpha})]^2};$$
$$E[D_{LCC}(\boldsymbol{\alpha})] = \frac{1}{2\pi}\sqrt{2^k \lambda_0^{k-1}(\boldsymbol{\alpha}) \cdot \lambda_2(\boldsymbol{\alpha})} C^{-1}\Gamma(1 + \frac{k}{2}). \tag{16}$$

where $\alpha_1(\boldsymbol{\alpha})$ and $\alpha_2(\boldsymbol{\alpha})$ represent interval bandwidth parameters, i.e.:

$$\alpha_1(\boldsymbol{\alpha}) = \frac{\lambda_1(\boldsymbol{\alpha})}{\sqrt{\lambda_0(\boldsymbol{\alpha}) \cdot \lambda_2(\boldsymbol{\alpha})}}; \quad \alpha_2 = \frac{\lambda_2(\boldsymbol{\alpha})}{\sqrt{\lambda_0(\boldsymbol{\alpha}) \cdot \lambda_4(\boldsymbol{\alpha})}}. \tag{17}$$

where $\lambda_n(\boldsymbol{\alpha})$ is the nth interval spectral moment of stress process of the critical point.

Therefore, fatigue damage is not only related to the *S-N* curve, cycle distribution and expected rate of occurrence of cycle, but also related to structural uncertain parameters. So it turns out to be a function of structural uncertain parameters.

4 Improved interval analysis via extra unitary interval

4.1 Rational series expansion

To avoid the inversion of the parametric *FRF* matrix in Eq.(10), approximate expression of the *FRF* matrix can be effectively obtained by adopting the *Neumann Series Expansion* (NSE):

$$\begin{aligned} \mathbf{H}(\boldsymbol{\alpha}, \omega) &= \mathbf{H}_0(\omega) \\ &+ \sum_{s=1}^{\infty} (-1)^s \left[\mathbf{H}_0(\omega)\mathbf{R}(\boldsymbol{\alpha}, \omega)\right]^s \mathbf{H}_0(\omega) \end{aligned} \tag{18}$$

The convergence condition of Eq. is represented as:

$$\left\|\mathbf{H}_0^{-1}(\omega)\mathbf{R}(\boldsymbol{\alpha}, \omega)\right\|_2 < 1 \tag{19}$$

where $\|.\|_2$ represents the 2-norm of matrices. It should be noted that the convergence condition (see Eq.(19)) is usually satisfied for structural uncertain parameter $|\Delta\alpha_i|<1$.

Based on the *NSE*, Muscolino and Sofi decomposed the sensitivity matries $\mathbf{K}_i$ as a superposition of n rank-one matrices form, i.e.[9]:

$$\mathbf{K}_i = \sum_{l=1}^{p_i} \lambda_i^{(l)} \mathbf{v}_i^{(l)} \mathbf{v}_i^{(l)T} \tag{20}$$

where $\lambda_i^{(l)} = \mathbf{K}_0\boldsymbol{\Psi}_i^{(l)}$; $\boldsymbol{\Psi}_i^{(l)}$ and $\lambda_i^{(l)}$ are the lth eigenvector and the associated eigenvalue by solving the eigenproblem:

$$\mathbf{K}_i\boldsymbol{\Psi}_i^{(l)} = \lambda_i^{(l)}\mathbf{K}_0\boldsymbol{\Psi}_i^{(l)}, \quad (i=1,\cdots,r; l = 1,\cdots,p_i) \tag{21}$$

The eigenvectors $\boldsymbol{\Psi}_i^{(l)}$ satisfy the normalizing condition

$$\boldsymbol{\Psi}_i^T \mathbf{K}_0 \boldsymbol{\Psi}_i = \mathbf{I}_{p_i}; \quad \boldsymbol{\Psi}_i = [\boldsymbol{\Psi}_i^{(1)}\ \boldsymbol{\Psi}_i^{(2)} \cdots \boldsymbol{\Psi}_i^{(p_i)}] \tag{22}$$

So that the following relationship holds:

$$\boldsymbol{\Psi}_i^T \mathbf{K}_i \boldsymbol{\Psi}_i = \boldsymbol{\Lambda}_i; \quad \boldsymbol{\Lambda}_i = \mathrm{diag}[\lambda_i^{(1)},\ \lambda_i^{(2)}, \cdots, \lambda_i^{(p_i)}] \tag{23}$$

Then substituting Eq.(20) into Eq.(7), the interval stiffness matrix can be rewritten as:

$$\mathbf{K}(\boldsymbol{\alpha}) = \mathbf{K}_0 + \sum_{i=1}^{r}\sum_{l=1}^{p_i} \lambda_i^{(l)} \mathbf{v}_i^{(l)} \mathbf{v}_i^{(l)T} \Delta\alpha_i \hat{\mathbf{e}}_i^I \tag{24}$$

By substituting Eq.(24) into Eq.(18), an explicit alternative expression of the FRF matrix, so-called *Rational Series Expansion* (*RSE*), is proposed in Ref.[11]. If the uncertain parameters satisfy the condition $\|\Delta\alpha\| << 1$, an accurate approximation of the FRF matrix can be obtained by retaining only first-order terms of RSE and it can be expressed as a suitable affine form, i.e.:

$$\begin{aligned}&\mathbf{H}^I(\omega)=\mathbf{H}(\boldsymbol{\alpha},\omega)=\mathbf{H}_{\text{mid}}(\omega)+\mathbf{H}_{\text{dev}}(\boldsymbol{\alpha},\omega)\\&\boldsymbol{\alpha}=\boldsymbol{\alpha}_0+\Delta\boldsymbol{\alpha}\hat{\mathbf{e}}^I\in\boldsymbol{\alpha}^I=\left[\underline{\boldsymbol{\alpha}},\,\bar{\boldsymbol{\alpha}}\right]\end{aligned}\tag{25}$$

where $\mathbf{H}_{\text{mid}}(\omega)$ and $\mathbf{H}_{\text{dev}}(\boldsymbol{\alpha},\omega)$ are two complex matrices, representing the midpoint and the deviation of the FRF matrix, expressed as:

$$\begin{aligned}\mathbf{H}_{\text{mid}}(\omega)&=\text{mid}\left\{\mathbf{H}(\boldsymbol{\alpha},\omega)\right\}\\&=\mathbf{H}_0(\omega)+\sum_{i=1}^{r}\sum_{s=1}^{n}\Delta a_{0,is}(\omega)\mathbf{B}_{is}(\omega)\\\mathbf{H}_{\text{dev}}(\boldsymbol{\alpha},\omega)&=\text{dev}\left\{\mathbf{H}(\boldsymbol{\alpha},\omega)\right\}\\&=\sum_{i=1}^{r}\sum_{s=1}^{n}\Delta\alpha_{is}(\omega)\mathbf{B}_{is}(\omega)\,\hat{\mathbf{e}}_i^I\end{aligned}\tag{26}$$

with

$$\begin{aligned}&a_{0,is}(\omega)=\frac{[p(\omega)\lambda_i^{(s)}\Delta\alpha_i]^2b_{is}(\omega)}{1-\left[p(\omega)\lambda_i^{(s)}\Delta\alpha_ib_{is}(\omega)\right]^2};\\&\Delta\alpha_{is}(\omega)=\frac{p(\omega)\lambda_i^{(s)}\Delta\alpha_i}{1-\left[p(\omega)\lambda_i^{(s)}\Delta\alpha_ib_{is}(\omega)\right]^2};\\&b_{is}=\mathbf{v}_i^{(s)T}\mathbf{H}_0(\omega)\mathbf{v}_i^{(s)};\\&B_{is}(\omega)=\mathbf{H}_0(\omega)\mathbf{v}_i^{(s)}\mathbf{v}_i^{(s)T}\mathbf{H}_0(\omega).\end{aligned}\tag{27}$$

When the interval FRF matrix (see Eq.(25)) is substituted into Eq.(9), the PSD function matrix of the generic response $\mathbf{G}_{\mathbf{yy}}(\boldsymbol{\alpha},\omega)$ can be expressed as:

$$\begin{aligned}&\mathbf{G}_{\mathbf{yy}}(\boldsymbol{\alpha},\omega)=\text{mid}\left\{\mathbf{G}_{\mathbf{yy}}(\boldsymbol{\alpha},\omega)\right\}+\text{dev}\left\{\mathbf{G}_{\mathbf{yy}}(\boldsymbol{\alpha},\omega)\right\}\\&\boldsymbol{\alpha}=\boldsymbol{\alpha}_0+\Delta\boldsymbol{\alpha}\hat{\mathbf{e}}^I\in\boldsymbol{\alpha}^I=\left[\underline{\boldsymbol{\alpha}},\,\bar{\boldsymbol{\alpha}}\right]\end{aligned}\tag{28}$$

where $\text{mid}\{\mathbf{G}_{\mathbf{yy}}(\boldsymbol{\alpha},\omega)\}$ and $\text{dev}\{\mathbf{G}_{\mathbf{yy}}(\boldsymbol{\alpha},\omega)\}$ are the midpoint and deviation of the *PSD* function matrix of the generic response, $\mathbf{y}^I(t)$, respectively expressed as:

$$\begin{aligned}\text{mid}\left\{\mathbf{G}_{\mathbf{yy}}(\boldsymbol{\alpha},\omega)\right\}&=\mathbf{q}^T\mathbf{H}_{\text{mid}}^*(\omega)\mathbf{G}_{\mathbf{ff}}(\omega)\mathbf{H}_{\text{mid}}^T(\omega)\mathbf{q};\\\text{dev}\left\{\mathbf{G}_{\mathbf{yy}}(\boldsymbol{\alpha},\omega)\right\}&\approx\sum_{i=1}^{r}\mathbf{q}^T\begin{bmatrix}\mathbf{H}_{\text{mid}}^*(\omega)\mathbf{G}_{\mathbf{ff}}(\omega)\Delta\alpha_i\mathbf{B}_i^T(\omega)\\+\Delta\alpha_i\mathbf{B}_i^*(\omega)\mathbf{G}_{\mathbf{ff}}(\omega)\mathbf{H}_{\text{mid}}^T(\omega)\end{bmatrix}\mathbf{q}\hat{e}_i^I\\&=\sum_{i=1}^{r}\text{dev}\left\{\mathbf{G}_{\mathbf{yy},i}(\boldsymbol{\alpha},\omega)\right\}\hat{e}_i^I.\end{aligned}\tag{29}$$

For simplifying interval computations, higher-order terms appearing in the deviation of the interval *PSD* function matrix are neglected due to the powers of the generic fluctuation $\Delta\alpha_i$ greater than one.

After simple algebra, the *LB* and *UB* of the *PSD* function matrix of the generic response can be expressed as:

$$\begin{aligned}&\overline{\mathbf{G}}_{\mathbf{yy}}(\alpha,\omega)=\text{mid}\{\mathbf{G}_{\mathbf{yy}}(\omega)\}+\Delta\mathbf{G}_{\mathbf{yy}}(\omega);\\&\underline{\mathbf{G}}_{\mathbf{yy}}(\alpha,\omega)=\text{mid}\{\mathbf{G}_{\mathbf{yy}}(\omega)\}-\Delta\mathbf{G}_{\mathbf{yy}}(\omega).\end{aligned}\tag{30}$$

where $\Delta\mathbf{G}_{\mathbf{yy}}(\omega)$ represents the deviation of the interval generic response, i.e.:

$$\Delta\mathbf{G}_{\mathbf{yy}}(\omega)=\sum_{i=1}^{r}\left|\mathbf{q}^T\begin{bmatrix}\mathbf{H}_{\mathbf{mid}}^*(\omega)\mathbf{G}_{\mathbf{ff}}(\omega)\Delta\alpha_i\mathbf{B}_i^T(\omega)\\+\Delta\alpha_i\mathbf{B}_i^*(\omega)\mathbf{G}_{\mathbf{ff}}(\omega)\mathbf{H}_{\text{mid}}^T(\omega)\end{bmatrix}\mathbf{q}\right|\tag{31}$$

In the *random vibration* and interval analysis, the *l*th-order spectrum moment of interval generic response can be defined as:

$$\begin{aligned}\lambda_{l,y}^I&=\lambda_{l,y}(\boldsymbol{\alpha})=2\int_0^{+\infty}\omega^l\cdot G_{yy}^I(\boldsymbol{\alpha},\omega)d\omega\\&=\text{mid}\left\{\lambda_{l,y}(\boldsymbol{\alpha})\right\}+\text{dev}\left\{\lambda_{l,y}(\boldsymbol{\alpha})\right\};\;l=0,1,2,4\\&\boldsymbol{\alpha}=\boldsymbol{\alpha}_0+\Delta\boldsymbol{\alpha}\hat{\mathbf{e}}^I\in\boldsymbol{\alpha}^I=\left[\underline{\boldsymbol{\alpha}},\,\bar{\boldsymbol{\alpha}}\right]\end{aligned}\tag{32}$$

where $\text{mid}\{\lambda_{l,y}(\boldsymbol{\alpha})\}$ and $\text{dev}\{\lambda_{l,y}(\boldsymbol{\alpha})\}$ are respectively the midpoint and the deviation of the interval spectral moment, expressed as:

$$\begin{aligned}\text{mid}\left\{\lambda_{l,y}(\boldsymbol{\alpha})\right\}&=2\int_0^{+\infty}\omega^l\cdot\text{mid}\left\{G_y(\boldsymbol{\alpha},\omega)\right\}\mathrm{d}\omega\\\text{dev}\left\{\lambda_{l,y}(\boldsymbol{\alpha})\right\}&=\sum_{i=1}^{r}\left[2\int_0^{+\infty}\omega^l\cdot\text{dev}\left\{G_y(\boldsymbol{\alpha},\omega)\right\}\mathrm{d}\omega\cdot\hat{e}_i^I\right]\\&=\sum_{i=1}^{r}\text{dev}\{\lambda_{l,y,i}\}\cdot\hat{e}_i^I\\&l=0,1,2,4;\boldsymbol{\alpha}=\boldsymbol{\alpha}_0+\Delta\boldsymbol{\alpha}\hat{\mathbf{e}}^I\in\boldsymbol{\alpha}^I=\left[\underline{\boldsymbol{\alpha}},\,\bar{\boldsymbol{\alpha}}\right]\end{aligned}\tag{33}$$

Similarly, the *LB* and *UB* of interval spectral moment can be expressed as:

$$\begin{aligned}&\overline{\lambda}_{l,y}(\boldsymbol{\alpha},\omega)=\text{mid}\left\{\lambda_{l,y}\right\}+\Delta\lambda_{l,y}^I;\\&\underline{\lambda}_{l,y}(\boldsymbol{\alpha},\omega)=\text{mid}\left\{\lambda_{l,y}\right\}-\Delta\lambda_{l,y}^I.\end{aligned}\tag{34}$$

where $\Delta\lambda_{l,y}^I$ represents the deviation of the interval generic response, i.e.:

$$\Delta\lambda_{l,y}^I=\sum_{i=1}^{r}\left|2\int_0^{+\infty}\omega^l\cdot\text{dev}\left\{G_y(\boldsymbol{\alpha},\omega)\right\}\mathrm{d}\omega\right|,\;l=0,1,2,4\tag{35}$$

Since the generic response and the spectral moments are monotonic functions of uncertain parameters α_i^I, their exact bounds can be evaluated by applying the *vertex method*. The *vertex method* is a selecting extreme value process by combining all possible extreme values of uncertain parameters by requiring 2^r stochastic analysis procedures. However, for the *improved interval analysis via extra unitary interval*, the bounds of generic response and spectral moments can be assessed without resorting to any combinatorial procedure.

4.2 Sensitivity analysis of interval generic response

In the area of structural systems, sensitivity analysis is a valuable technique used to determine how the changes in the interval generic response, spectral moments and bandwidth parameters can be related to different sources of interval variable in its inputs.

By taking into account Eq (32)., the following explicit expression of the ith interval sensitivity of the lth-order interval spectral moment $\lambda^I_{l,y}$ through direct differentiation with respect to α^i can be expressed as [10]:

$$S^I_{\lambda_{l,y},i} = \left.\frac{\partial \lambda_{l,y}(\boldsymbol{\alpha})}{\partial \Delta\alpha_i}\right|_{\Delta\boldsymbol{\alpha}=0} = \left.\frac{\partial \mathbf{q}^T \boldsymbol{\lambda}_{l,\mathbf{uu}}(\boldsymbol{\alpha}^I)\mathbf{q}}{\partial \Delta\alpha_i}\right|_{\Delta\boldsymbol{\alpha}=0} \tag{36}$$

$$= \mathbf{q}^T \mathbf{S}^I_{\lambda_{l,\mathbf{uu}},i}\mathbf{q}\,\hat{e}^I_i = \Delta S_{\lambda_{l,y},i}\,\hat{e}^I_i,\ \ l=0,1,2,4.$$

where $S^I_{\lambda_{(l,uu)},i}$ represents the interval sensitivity of spectral moment matrix of displacement response $\lambda_{l,uu}(\boldsymbol{\alpha})$; $\Delta S_{\lambda_{l,uu},i}$ is the deviation amplitude of symmetric interval sensitivity $S^I_{\lambda_{l,uu},i}$; $\Delta S^I_{\lambda_{l,uu},i}$ and are defined as:

$$\mathbf{S}^I_{\lambda l,\mathbf{yy}} = \left.\frac{\partial \boldsymbol{\lambda}_{l,\mathbf{yy}}(\boldsymbol{\alpha})}{\partial \Delta\alpha_i}\right|_{\Delta\boldsymbol{\alpha}=0} = 2\int_0^{+\infty} \omega^l \Delta\mathbf{S}_{\mathbf{G}_{\mathbf{yy},i}}(\omega)\,\hat{e}^I_i\,d\omega$$

$$= \Delta\mathbf{S}_{\lambda l,\mathbf{yy}}\,\hat{e}^I_i \tag{37}$$

$$\Delta\mathbf{S}_{\mathbf{G}_{\mathbf{yy},i}}(\omega) = \mathbf{q}\left[\mathbf{H}^*_0\mathbf{G}_{\mathbf{ff}}(\omega)\mathbf{B}^T_i(\omega)p(\omega)\right]\mathbf{q}^T$$

$$+\mathbf{q}\left[p^*(\omega)\mathbf{B}^*_i(\omega)\mathbf{G}_{\mathbf{ff}}(\omega)\mathbf{H}^T_0(\omega)\right]\mathbf{q}^T$$

Notice that additional terms $\sum_i^r \mathbf{q}_i \Delta\alpha_i \hat{e}^I_i$ need to be added to Eq.(36), when $\mathbf{q}(\boldsymbol{\alpha})$ is a function of interval vector $\boldsymbol{\alpha}$.

Similarly, based on Eq. (36), the ith interval sensitivity $S^I_{\alpha_n,i}$ of the interval spectral bandwidth parameters of the generic response, $Y(\boldsymbol{\alpha},t)$, can be expressed as:

$$S^I_{\alpha_n,i} = \left.\frac{\partial \alpha_n(\boldsymbol{\alpha})}{\partial \Delta\alpha_i}\right|_{\Delta\boldsymbol{\alpha}=0}$$

$$= \frac{\left(2\lambda_0\lambda_{2n}\Delta S_{\lambda_{n,Y},i} - \lambda_n\lambda_{2n}\Delta S_{\lambda_{0,Y},i} + \lambda_0\lambda_n\Delta S_{\lambda_{2n,Y},i}\right)}{2\left(\lambda_0\lambda_{2n}\right)^{3/2}}\hat{e}^I_i \qquad n=1,2 \tag{38}$$

where $S^I_{\lambda_{0,y},i}$, $S^I_{\lambda_{n,y},i}$ and $S^I_{\lambda_{2n,y},i}$ are sensitivities of the spectral moment, which can be obtained by Eq.(36).

Under the assumption of slight deviation amplitudes of the uncertain parameters, the interval spectral parameter α^I_i, can be approximated by applying the *first-order interval Taylor series expansion*:

$$\alpha^I_n = \alpha_{n,0} + \sum_{i=1}^{r} S^I_{\alpha_n,i}\Delta\alpha_i\,\hat{e}^I_i,\ n=1,2. \tag{39}$$

Then the *LB* and *UB* of the interval spectral parameter can be expressed as:

$$\overline{\alpha}_n = \alpha_{n,0} + \sum_{i=1}^{r}\left|S_{\alpha_{n,\mathrm{mid}},i}\Delta\alpha_i\right|;$$

$$\underline{\alpha}_n = \alpha_{n,0} - \sum_{i=1}^{r}\left|S_{\alpha_{n,\mathrm{mid}},i}\Delta\alpha_i\right|,\ n=1,2. \tag{40}$$

where $S^I_{\alpha_n,i}$ is the ith interval sensitivity of interval spectral parameters, which is defined in Eq.(38).

5 Interval fatigue evaluation by a spectral method

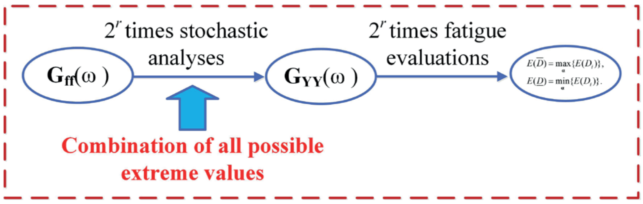

Fig.1 Flow chart of the vertex method

Within the framework of interval theory, the interval fatigue damage evaluation means that the *LB* and *UB* of interval expected fatigue damage rate $E(D^I)$ of critical points are derived when the structure is with uncertain-but-bounded parameters.

Since the $E(D^I)$ is a monotonic function of the uncertain parameter α_i, the exact bounds can be calculated by using the *vertex method*. As shown in Fig.1, the *vertex method* requires to evaluate the $E(D^I)$ of the selecting extreme value process, for all possible combinations of the bounds of the r uncertain parameters α_i, say 2^r, the *LB* and *UB* among all the possible $E(D)$s. Indeed, each evaluation of the $E(D)$ implies an onerous stochastic analysis of the structure for computing the spectral moments of the response pertaining to a specific combination of the bounds of the uncertain parameters. In this section, a simplified method is proposed to evaluate the bounds of $E(D^I)$ by properly combining interval variables. This method only needs one time stochastic analysis procedure based on the improved interval analysis.

The main idea is to view the $E(D^I)$, herein rewritten for the sake of clarity

$$E(D^I) = \left[b^I + (1-b^I)(\alpha^I_2)^{k-1}\right]E(D^I_{LCC})$$

$$= E[D^I(b^I, \alpha^I_2, E(D^I_{LCC}))] \tag{41}$$

as function of three interval variables b^I, α^I_2 and $E(D^I_{LCC})$, say the interval weight parameters, the interval spectral parameters and the $E(D^I)$ which is computed by the *level-crossing counting method*.

Eq.(41) decreases the amount of computing since the bounds of the $E(D^I)$ can be evaluated by

$$E(\overline{D}_{proposed}) = [\overline{b} + (1-\underline{b})(\overline{\alpha}_2)^{k-1}]E[\overline{D}_{LCC}],$$

$$E(\underline{D}_{proposed}) = [\underline{b} + (1-\overline{b})(\underline{\alpha}_2)^{k-1}]E[\underline{D}_{LCC}]. \tag{42}$$

where the *LB* and *UB* of the spectral parameter α_2 can be obtained by Eq.(40); the bounds of the b^I and the $E(D^I_{LCC})$ can be expressed as

$$\overline{b} = \max_{\substack{\alpha_1\in\alpha^I_1\\ \alpha_2\in\alpha^I_2}}\left\{b(\alpha^I_1,\alpha^I_2)\right\},\ \ \underline{b} = \min_{\substack{\alpha_1\in\alpha^I_1\\ \alpha_2\in\alpha^I_2}}\left\{b(\alpha^I_1,\alpha^I_2)\right\}. \tag{43}$$

$$E(\overline{D}_{LCC}) = \frac{1}{2\pi}\sqrt{2^k\,\overline{\lambda}_0^{k-1}\,\overline{\lambda}_2}\,C^{-1}\Gamma(1+\frac{k}{2}),$$

$$E(\underline{D}_{LCC}) = \frac{1}{2\pi}\sqrt{2^k\,\underline{\lambda}_0^{k-1}\,\underline{\lambda}_2}\,C^{-1}\Gamma(1+\frac{k}{2}). \tag{44}$$

where the bounds of interval spectral moments λ_i and spectral parameters α_2 can be obtained by Eq.(34) and Eq. (40).

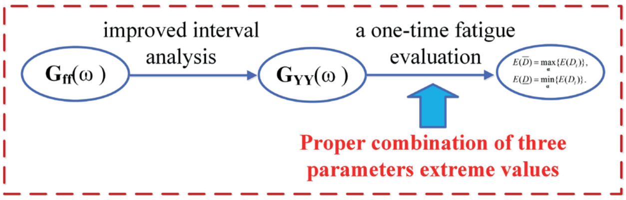

Fig. 2 Flow chart of the proposed method

According to Eq.(42), the *LB* and *UB* of $E(D^I)$ are often overestimated for ignoring the dependency of the coefficients b^I, α_2^I and the $E(D^I_{LCC})$. Meanwhile, the bounds of interval spectral parameters based on the *first-order interval Taylor series expansion* reduce the accuracy of the calculation of the $E(D^I)$. However, as shown in Fig.2, Eq.(42) can express the approximate formulas of the *LB* and *UB* of the $E(D^I)$, and it can avoid any selecting procedures of all possible values.

The main steps of the proposed procedure are summarized as follows: (i) to model the uncertain parameters as interval variables; (ii) to decompose the sensitivity matrices (see Eq.(20)); (iii) to express *FRF* matrix in an approximate explicit form by applying the *RSE* (see Eq.(25)); (iv) to obtain explicit expression of generic response and the bounds of zero-, second- and forth-order spectral moments of stress process of critical points; (v) to assess the bounds of the interval fatigue damage rate of structures with uncertain-but-bounded parameters (see Eq.(42)).

6 Numerical application

To verify the accuracy of the proposed method, the *LB* and *UB* of $E(D^I)$, namely $E(\underline{D})$ and $E(\overline{D})$, calculated by the proposed method are compared with the bounds obtained by the *vertex method*. For the latter, the bounds are calculated by combining the uncertain parameters according to the *vertex method*. It should be noted that if the number of uncertain parameter variables is N for the problem to be solved, the *vertex method* will require 2^N times stochastic analysis. Then computational cost exponentially increases with the amount of input uncertain parameters by the *vertex method*. While the proposed method in this paper requires only one time stochastic analysis, therefore, this method is particularly suitable for the estimation of structural fatigue damage with a large number of uncertain parameters.

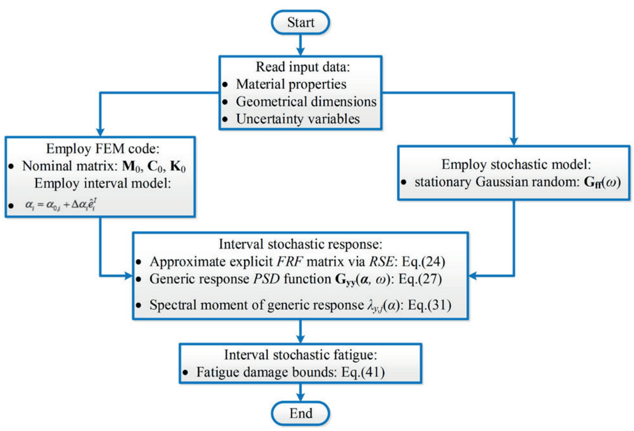

Fig. 3 Flow chart of the proposed method for interval fatigue evaluation

The numerical example in this paper is coded by Matlab R2014a. The computer used was an Intel Core i5-4200u with 1.60GHZ CPU and 8GB memory. The calculation procedure can be coded through the flow chart of the proposed method, reported in Fig.3.

For the numerical example, the deviation between the proposed method and the *vertex method* is calculated by:

$$\varepsilon = \frac{\left|E(D_{proposed}) - E(D_{exact})\right|}{E(D_{exact})} \times 100\% \qquad (45)$$

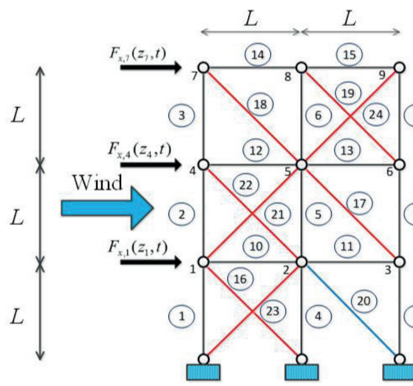

Fig. 4 24-bar truss structure under stochastic wind excitation

The scheme of example that is a 24-bar truss structure is shown in Fig.4. Lengths and elastic modulus of all the bars are deterministic values. Length L_i and elastic modulus E_i are selected as follows: $L_i = 3$m (i =1, 2, …, 9) and $E_i = 2.1\times10^{11}$N/m^2 (i =1, 2, …, 24). Furthermore, each node possesses a lumped mass $M = 500$kg. The cross-sectional areas of the diagonal bars are modeled as uncertain parameters. Their midpoint values are all $A_0 = 7\times10^{-4}$m^2 and deviation amplitudes $\Delta\alpha_i$ satisfy the condition $\Delta\alpha_i << 1$. The interval parameters are defined as $A_i^I = A_0(1 + \Delta\alpha_i\hat{e}_i^I)$ (i =16, 17, …, 24).

When the modal damping ratios for the first and third modes of the nominal structure are $\zeta_0 = 0.05$, the *Rayleigh damping* constants are taken as $c_0 = 3.517897$s^{-1} and $c_1 = 0.000547$s.

The truss is subjected to turbulent wind loads in the x-direction as shown in Fig.4. The nodal forces $F_{x,i}(z_i, t)$ are functions of the height z and time t, and it is often superimposed by the mean wind load $\overline{F}_{x,i}(z_i, t)$ and the turbulence wind load $f_{x,i}(z_i, t)$, i.e.[20]:

$$\begin{aligned} F_{x,i}(z_i,t) &= \overline{F}_{x,i}(z_i,t) + f_{x,i}(z_i,t); \\ \overline{F}_{x,i}(z_i,t) &\approx \frac{1}{2}\rho C_D A_i \overline{U}^2(z_i); \\ f_{x,i}(z_i,t) &\approx \rho C_D A_i u(z_i,t)\overline{U}(z_i). \end{aligned} \qquad (46)$$

where ρ represents the air density; C_D is the pressure coefficient. For this example, ρ and C_D are valued as follows: ρ=1.25kg/m^3 and C_D=1.2. A_i are the tributary areas of the nodal forces with values A_i=A_4=9m^2 and A_7=4.5m^2, respectively. $\overline{U}(z_i)$ and u(z_i, t) are the mean and turbulence wind speed. The mean wind speed $\overline{U}(z_i)$ follows the power law, which can be written as:

$$\overline{U}(z)=\overline{U}_{10}(\frac{z}{10})^{\alpha} \tag{47}$$

where $\overline{U}_{10}$ represents the mean wind speed at the reference height of 10m; α is the coefficient changing with the terrain roughness and the height range, set equal to $\alpha = 0.3$.

Generally, the turbulence wind $u(z, t)$ is always modeled as a zero-mean stationary multi-correlated Gaussian random process, and the statistical properties of turbulence wind velocity are completely determined by the *PSD* function matrix, which is represented as:

$$G_{u_iu_j}(\omega)=G_{u_iu_i}(\omega)\cdot f_{ij}(\omega) \tag{48}$$

where $G_{uu}(\omega)$ is the wind speed spectral; $f_{ij}(\omega)$ is the coherence function. $G_{uu}(\omega)$ and $f_{ij}(\omega)$ can be expressed as:

$$G_{u_iu_i}(\omega)=2\pi\cdot\frac{4kx^2}{(1+x^2)^{4/3}}\cdot\frac{\overline{U}_{10}^2}{\omega};$$
$$f_{ij}(\omega)=\exp\left\{-\frac{\omega\left[C_z^2(z_i-z_j)^2\right]^{1/2}}{\pi\left[\overline{U}(z_i)+\overline{u}(z_j)\right]}\right\}. \tag{49}$$

where k represents the non-dimensional roughness coefficient, herein set equal to $k = 0.03$, and $x = 600\omega/(\pi\overline{U}_{10})$; C_z is appropriate decay coefficient, herein set equal to $C_z = 10$.

According to Eq.(46) and Eq.(49), the *PSD* function matrix of turbulence wind load can be expressed as:

$$G_{f_if_j}(n)=(\rho C_D)^2A_iA_ju(z_i,t)u(z_j,t)G_{u_iu_j}(n) \tag{50}$$

The attention is concentrated on the $E(D^I)$ of the bar 20, say $E(D^I_{20})$, and the *LB* and *UB* of the $E(D^I)$, say $E(\overline{D}_{20})$ and , are evaluated according to two different kinds of S-N curves, namely $k=4$, $C=1940\times10^{12}$ and $k=3$, $C=3.26\times10^{12}$, for uncertainty $\Delta\alpha$, from $\Delta\alpha = 0$ to $\Delta\alpha = 0.1$. In Eq.(42), the estimates of the *LB* and *UB* of the $E(D^I)$ provided by the proposed method are in contrast with the exact bounds obtained by applying the *vertex method* requiring 2^9 stochastic analysis procedures. As shown in Fig.5, the region of the interval fatigue damage rate becomes wider as the uncertainty level increasing. The $E(D^I)$ in Fig.5 is normalized by $E(D)$, which is the nominal expected fatigue damage rate of structure.

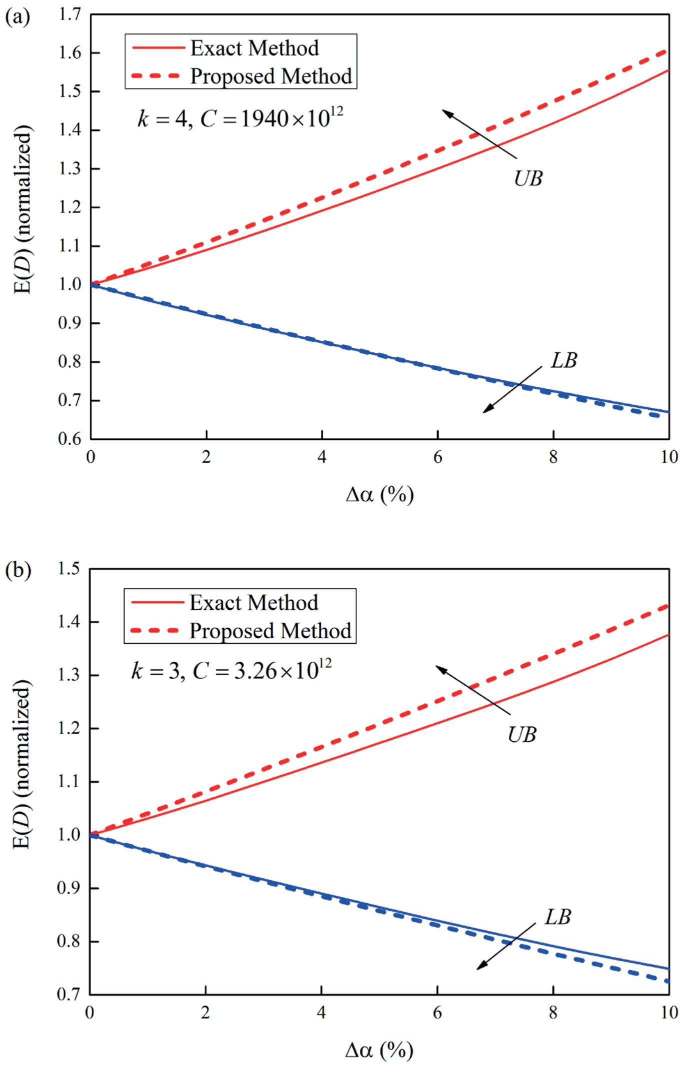

Fig. 5 Comparison between the exact and the proposed *LB* and *UB* of the $E(D^I)$ of the bar 20 of the truss structure for magnitudes of the interval cross-sectional areas $A_i^I = A_0\left(1+\Delta\alpha_i\hat{e}_i^I\right)$ $(i=16,17,...,24)$, fluctuation $\Delta\alpha$ from 0% to 10% by two different types of *S-N* curves: (a) $k = 4$, $C = 1940\times10^{12}$ and (b) $k = 3$, $C = 3.26\times10^{12}$.

In Fig.6, the absolute percentage errors versus $\Delta\alpha$ are plotted for fixed mean wind speed. As expected, the results of the proposed method are in good agreement with bounds $E(D^I)$ obtained by exact method.

With the wind speed of reference height increasing, the bounds of $E(D^I)$ which are calculated by the proposed method are compared with exact values as shown in Fig.7. For the sake of completeness, the nominal $E(D)$s are also plotted. The results also illustrate that the proposed method can get accurate estimation. The $E(D^I)$ in Fig.7 is normalized by $E(D)$, which is the nominal expected fatigue damage rate of structure with the mean wind speed of 15m/s. It should be emphasized that when evaluating the approximate bounds of the $E(D^I)$, the application of the proposed method requires only one time stochastic analysis procedure for each mean wind speed at the reference height.

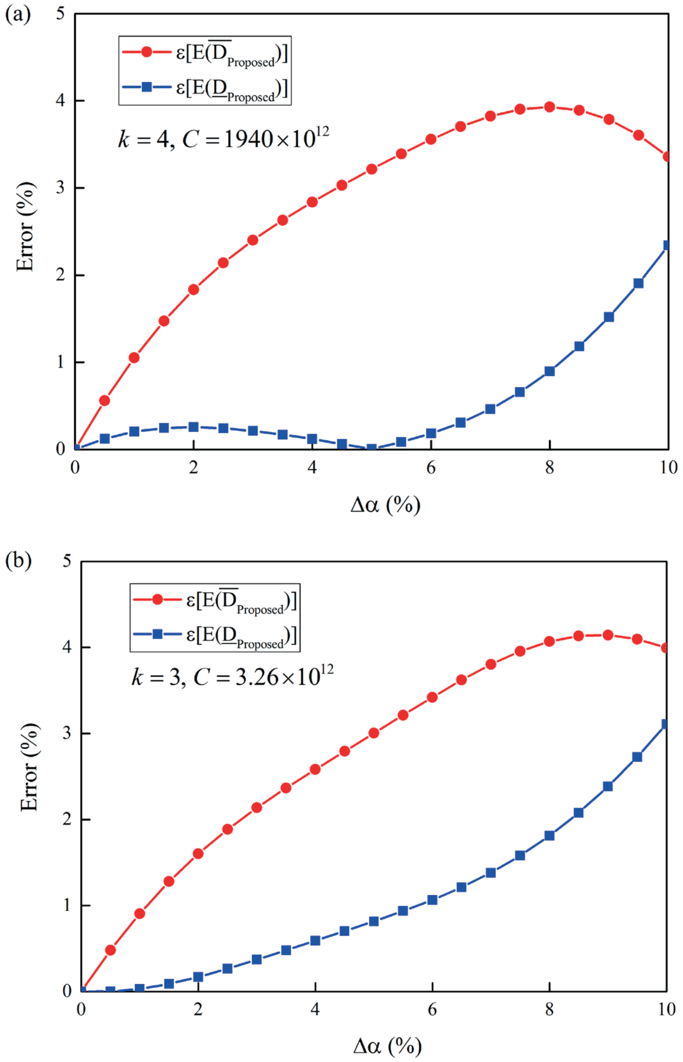

Fig. 6 Absolute percentage errors affecting the proposed estimates of the *LB* and *UB* of the $E(D^I)$ of the bar 20 of the truss structure for magnitudes of the interval cross-sectional areas $A_i^I = A_0\left(1+\Delta\alpha_i\hat{e}_i^I\right)$ $(i=16,17,...,24)$, fluctuation $\Delta\alpha$ from 0% to 10% by two different types of *S-N* curves: (a) $k = 4$, $C = 1940\times10^{12}$ and (b) $k = 3$, $C = 3.26\times10^{12}$.

It is worth emphasizing that the *vertex method* is a combinational approach which needs 2^9 times stochastic analyses and fatigue estimates. This procedure requires 1109s whilst the proposed method requires 29s to compute the bounds of $E(D^I)$ at each mean wind speed or with different deviation amplitudes for this numerical example. It is clear that the former is much more computationally expensive than the latter. Therefore, for the structure with a large number of DOFs, the computational time of the *vertex method* will increase rapidly.

Furthermore, comparison between Fig.5 and Fig.7, shows that fatigue damage assessment in terms of the $E(D)$ is much more affected by uncertainty of the cross-sectional area. Obviously, for a fixed mean wind speed at the reference height, only the *UB* of the $E(D^I)$ is useful for design purpose, since the *LB* may overestimate the actual performance of the structure in operating conditions. Therefore, for the fatigue life evaluation of structures with uncertain parameters, the *LB* of fatigue life is the inverse of the accumulation of the fatigue damage, which equals to integral with wind speed of reference height based on the product of the expected fatigue damage rate with different mean wind speeds and the probability density function of the mean wind speed.

7 Conclusions

In this paper, fatigue damage evaluation of linear structures with uncertain-but-bounded parameters has been obtained by a spectral method. In the framework of the interval theory, the interval model has been introduced into fatigue damage assessment by a spectral approach, and then interval spectral method for fatigue damage prediction has been proposed for estimating uncertain structure under random loads.

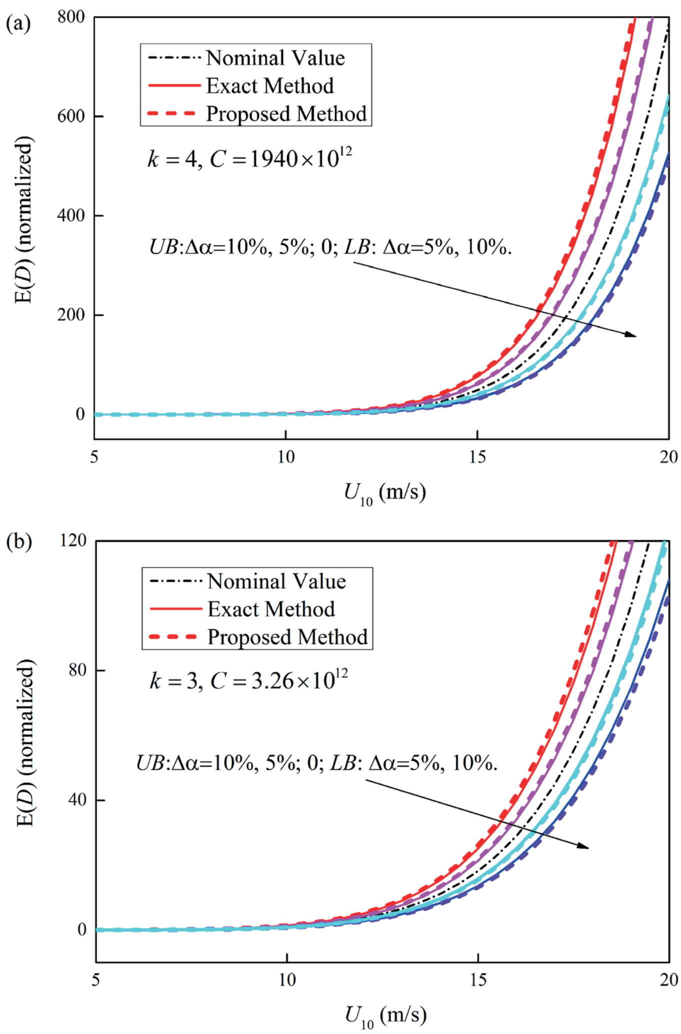

Fig. 7 Comparison between the exact and the proposed *LB* and *UB* of the $E(D^I)$ of the bar 20 of the truss structure with interval cross-sectional areas $A_i^I = A_0\left(1+\Delta\alpha_i\hat{e}_i^I\right)$ $(i=16,17,...,24)$, fluctuation $\Delta\alpha$ = 5% and $\Delta\alpha$ = 10% by two different types of *S-N* curves: (a) $k = 4$, $C = 1940\times10^{12}$ and (b) $k = 3$, $C = 3.26\times10^{12}$.

By applying the *Rational Series Expansion (RSE)* and *Extra Unitary Interval (EUI)*, the *FRF* matrix of structures with uncertain-but-bounded parameters and the *PSD* function matrix of generic response are explicitly expressed. And the *T-B method* is applied to assess fatigue damage through the *PSD* function of the stress of the critical point. Then the *LB* and *UB* of the $E(D^I)$ have been proposed by a reasonable combination of the maximum and minimum of the spectral moment and spectral parameters. Finally, the validity and accuracy of the proposed method are checked by comparing with the result of the *vertex method* through the numerical applications.

The proposed approach is much more efficient than crude combinatorial procedure, which requires a large amount of stochastic analyses determined by the combinations of the all possible extreme values of the interval parameters.

The interval fatigue damage prediction by a spectral method proposed in this paper has the following characteristics:

1. Based on the spectral method and the interval theory, structures with uncertain parameters are modeled as interval variables in this paper.

2. The *LB* and *UB* of the $E(D^I)$ calculated by the proposed method require only one time stochastic analysis procedure, whereas, for the *vertex method*, the computational cost increases exponentially with the number of input intervals increasing.

3. The $E(D^I)$ obtained by interval spectral method for fatigue damage prediction will have a high accuracy when generic response of structures with uncertain parameters is derived from the *improved interval analysis via Extra Unitary Interval* and the *Interval Rational Series Expansion*.

The analysis result shows that when the uncertainty of the structures is considered, the expected fatigue damage rate of structures will be greatly enhanced. Therefore for the critical structures, the uncertainty of structures is needed to be considered in the fatigue analysis.

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