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APPROXIMATE CALCULATION OF EQUIVALENT EXCITATION LENGTH IN INDUCTION HEATING WITH FLUX CONDUCTOR CONTAINING FERROMAGNETIC PLATE[1]

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Abstract

In the first part of the article the electromagnetic field is determined between the ferromagnetic plate and the flux conductor – inductor system with approximating analytical calculation. Then the power is calculated, penetrated into the ferromagnetic plate and from this the equivalent excitation length can be obtained with the condition that a uniform excitation of equivalent length gives the same power. The ferromagnetic skin effect is handled by the MacLean's model, which considers the magnetisation curve as rectangular lines.

Keywords: induction heating, heating of ferromagnetic plates.

1. Introduction

During induction heating, it is very important to concentrate the heat at a given place, to reduce the size, to increase the efficiency of the energy transformation and to shield the outer magnetic field. These requirements can be fulfilled mostly by inductors with flux conductor, therefore in the practice of the induction heating this kind of inductor is frequently used, for example to heat plate load made of steel (may be by continuous feeding). In some processes the heating stops below the Curie temperature, so the plate remains ferromagnetic, while in other cases the plate has to be heated up to a higher temperature, where its magnetic property will be lost and its relative permeability will be reduced to unity. Even in the latter case, the heating of the plate crosses the period of ferromagnetic state. That is why the examination of this state is important. Therefore our examination is connected to the heating of a ferromagnetic plate by inductor with flux conductor; or to be more precise, it gives an important geometrical datum for the design of the inductor–load system. This date is the equivalent excitation length, which is computed based on

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the distribution of the field quantities in the air gap that can be seen in *Fig. 1*, with a size of Δ. (The Figure shows the upper half of an arrangement.)

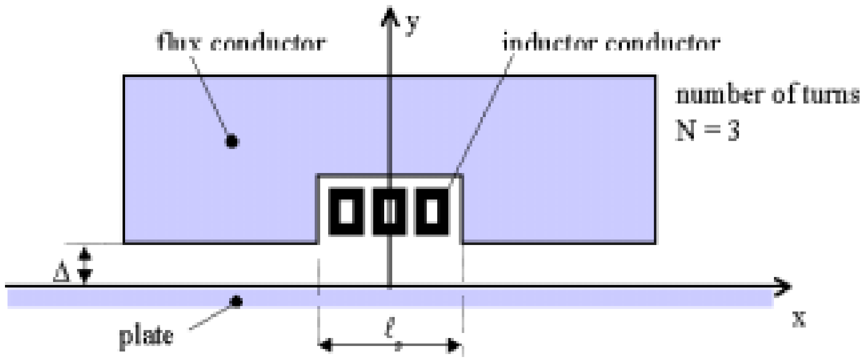

Fig. 1. Intersection model of the arrangement

2. Approximation Hypothesis

1. The arrangement is supposed to be infinity lengthwise of the inductor–conductor (z-direction).
2. The plate and the flux conductor are supposed to be infinity perpendicular to the length of the inductor–conductor as well (x-direction).
3. The tangential component of the magnetic field strength is supposed to be 0 at the surface of the flux conductor opposite to the plate and constant below the inductor.
4. The field impedance is supposed to be constant and independent of 'x' on the boundary plane of the ferromagnetic plate and its value is calculated from the Mac-Lean model, by a mean value H_m which will be determined later. (It will be the peak value of the sinusoidal quantity.)
5. Supposing that the air gap is small between the flux conductor–inductor and the plate, in y-direction the electromagnetic field of the air gap can be calculated by series expansion with a 2D model. Only the first four terms of the series will be taken into consideration.

3. Field Equations and Boundary Conditions in the Air Gap

Assuming that the current is sinusoidal in the inductor, or taking its first harmonic, let us formulate the Maxwell's equations by the phasors of the electromagnetic field characteristics. As a result of the assumptions, the electric field has only

z-directional component, while the magnetic field has x- and y-directional components. If the circular frequency of the sinusoidal current or its first harmonic is ω, then:

$$\frac{\partial \mathbf{E}}{\partial x} = j\omega\mu_0 \mathbf{H}_y, \tag{1}$$

$$\frac{\partial \mathbf{E}}{\partial y} = -j\omega\mu_0 \mathbf{H}_x, \tag{2}$$

$$\frac{\partial \mathbf{H}_x}{\partial y} - \frac{\partial \mathbf{H}_y}{\partial x} = 0. \tag{3}$$

The boundary conditions are (according to hypothesis 3):

$$(\mathbf{H}_x)_{y=\Delta} = H_0 g(x), \text{ where } H_0 = \frac{NI}{\ell_s}; \quad g(x) = \begin{cases} 0, & \text{if} \quad |x| \geq \dfrac{\ell_s}{2}, \\ 1, & \text{if} \quad |x| < \dfrac{\ell_s}{2}, \end{cases} \tag{4}$$

$$(\mathbf{E})_{y=0} = \mathbf{Z}_0 (\mathbf{H}_x)_{y=0}; \quad \mathbf{Z}_0 = (2+j)X_0; \quad X_0 = \frac{8}{3\pi}\frac{\rho}{\xi_m}; \tag{5a}$$

here ξ_m is the maximal penetration depth and

$$\xi_m = \sqrt{\frac{2\rho H_m}{B_0 \omega}}. \tag{5b}$$

4. Solution of the Field Equations

Let us define a dimensionless $\mathbf{f}(x)$ phasor function by the following formula:

$$(\mathbf{H}_x)_{y=0} = H_0 \mathbf{f}(x), \tag{6}$$

where H_0 was defined in (4). The substitution of (1) and (2) into *Eq.* (3) results a Laplace equation:

$$\frac{\partial^2 \mathbf{E}}{\partial x^2} + \frac{\partial^2 \mathbf{E}}{\partial y^2} = 0. \tag{7}$$

The solution can be represented by Taylor series around $y = 0$ taking the first four terms into consideration:

$$\mathbf{E} \cong (\mathbf{E})_{y=0} + \left(\frac{\partial \mathbf{E}}{\partial y}\right)_{y=0} \frac{y}{1!} + \left(\frac{\partial^2 \mathbf{E}}{\partial y^2}\right)_{y=0} \frac{y^2}{2!} + \left(\frac{\partial^3 \mathbf{E}}{\partial y^3}\right)_{y=0} \frac{y^3}{3!}. \tag{8}$$

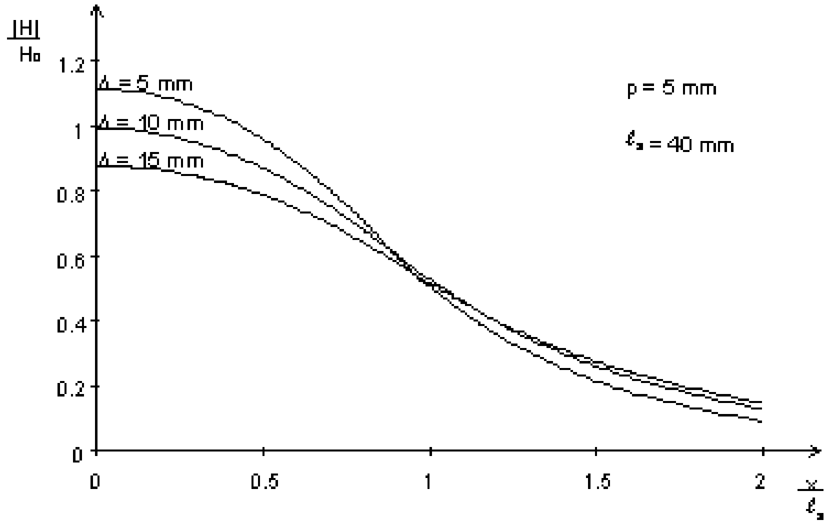

Fig. 2. Distribution of magnetic field strength

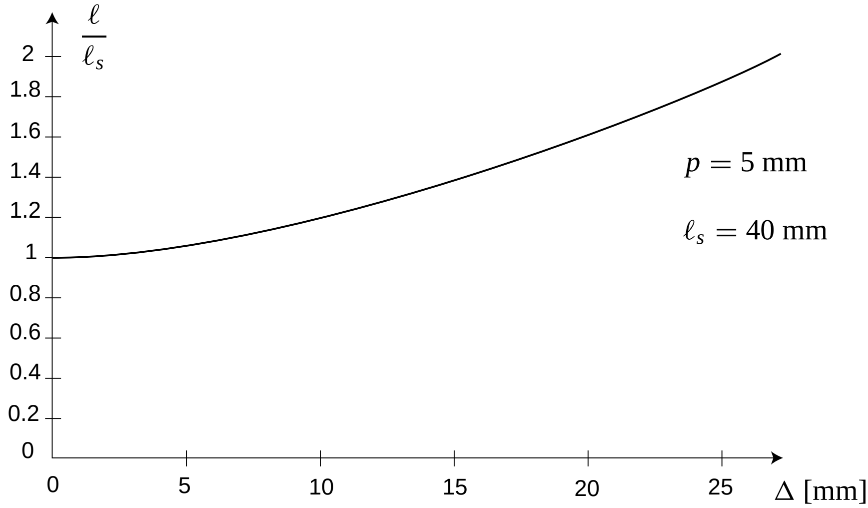

Fig. 3. Equivalent excitation length vs. the air gap

According to (5a) and (6) $(\mathbf{E})_{y=0} = (2 + j)X_0 H_0 \mathbf{f}(x)$, according to (2) and (6)

$$\left(\frac{\partial \mathbf{E}}{\partial y}\right)_{y=0} = -j\omega\mu_0 H_0 \mathbf{f}(x).$$

Based on (7)

$$\left(\frac{\partial^2 \mathbf{E}}{\partial y^2}\right)_{y=0} = -\left(\frac{\partial^2 \mathbf{E}}{\partial x^2}\right)_{y=0} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\mathbf{E})_{y=0} = -(2+j)X_0 H_0 \mathbf{f}''(x).$$

Also based on (7)

$$\left(\frac{\partial^3 \mathbf{E}}{\partial y^3}\right)_{y=0} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial \mathbf{E}}{\partial y}\right)_{y=0} = j\omega\mu_0 H_0 \mathbf{f}''(x);$$

substituting the previous equations into (8), the relation will be the following:

$$\frac{\mathbf{E}}{X_0 H_0} \cong (2+j)\mathbf{f}(x) - j\frac{\omega\mu_0}{X_0}\mathbf{f}(x)y - (2+j)\mathbf{f}''(x)\frac{y^2}{2} + j\frac{\omega\mu_0}{X_0}\frac{y^3}{6}\mathbf{f}''(x).$$

Introducing the quantity

$$p = \frac{X_0}{\omega\mu_0} \tag{9}$$

and differentiating the previous equation according to y the following formula can be written:

$$\frac{1}{X_0 H_0}\frac{\partial \mathbf{E}}{\partial y} \cong -\frac{j}{p}\mathbf{f}(x) - (2+j)\mathbf{f}''(x)y + \frac{j}{p}\mathbf{f}''\frac{y^2}{2}. \tag{10}$$

According to boundary condition (4) and *Eq.* (2)

$$\left(\frac{\partial \mathbf{E}}{\partial y}\right)_{y=\Delta} = -j\omega\mu_0 H_0 g(x). \tag{11}$$

Comparing (10) and (11), using formulation (9) it can be written:

$$\mathbf{f}(x) + \mathbf{f}''(x)\left[(1-2j)p\Delta - \frac{\Delta^2}{2}\right] = g(x),$$

or introducing formulation

$$\sqrt{p\Delta} = a, \qquad 1 - \frac{\Delta}{2p} = \chi \tag{12}$$

and inscribing the meaning of $g(x)$ according to (4)

$$(\chi - 2j)a^2\mathbf{f}''(x) + \mathbf{f}(x) = \begin{cases} 1, & \text{if} \quad |x| < \dfrac{\ell_s}{2}, \\ 0, & \text{if} \quad |x| \geq \dfrac{\ell_s}{2}. \end{cases} \tag{13}$$

Because of the symmetry, $\mathbf{f}(x)$ is an even function therefore differential equation (13) has to be solved only for $x \geq 0$. Furthermore, because of the continuity of $\mathbf{f}(x)$ and $\mathbf{f}''(x)$ and because $\mathbf{f}(x)$ is an even function $\mathbf{f}'(x) = 0$. Taking this result into consideration, the solution of differential equation (13) in the interval $0 \leq x \leq \frac{\ell_s}{2}$:

$$\mathbf{f}(x) = 1 + \mathbf{K}_1 \left(e^{\mathbf{q}\frac{x}{a}} + e^{-\mathbf{q}\frac{x}{a}} \right),$$

where $\mathbf{q}$ fulfils the characteristic equation $(\chi - 2j)\mathbf{q}^2 + 1 = 0$ and from that

$$\mathbf{q} = \sqrt{\frac{\sqrt{4+\chi^2} - \chi}{2(4+\chi^2)}} - j\sqrt{\frac{\sqrt{4+\chi^2} + \chi}{2(4+\chi^2)}} = \alpha - j\beta. \tag{14}$$

(Otherwise the formula written for $\mathbf{f}(x)$ is also correct in the interval $-\frac{\ell_s}{2} \leq x \leq 0$.)

The solution of differential equation (13) in the $\frac{\ell_s}{2} \leq x \leq \infty$, taking into account that $\lim_{x\to\infty} \mathbf{f}(x) = 0$:

$$\mathbf{f}(x) = \mathbf{K}_2 e^{-\mathbf{q}\frac{x}{a}}.$$

Constants $\mathbf{K}_1$ and $\mathbf{K}_2$ can be determined by the continuity of $\mathbf{f}(x)$ and $\mathbf{f}'(x)$ at $x = \frac{\ell_s}{2}$. The final expression for $\mathbf{f}(x)$, which determines according to (6) the distribution of $(\mathbf{H}_x)_{y=0}$, is:

$$\mathbf{f}(x) = \begin{cases} 1 - \dfrac{1}{2}\left(e^{\mathbf{q}\frac{x-\frac{\ell_s}{2}}{a}} + e^{-\mathbf{q}\frac{x-\frac{\ell_s}{2}}{a}} \right), & \text{if} \quad |x| < \dfrac{\ell_s}{2}, \\[2ex] \dfrac{1}{2}\left(e^{\mathbf{q}\frac{\ell_s}{2a}} - e^{-\mathbf{q}\frac{\ell_s}{2a}} \right) e^{-\mathbf{q}\frac{|x|}{a}}, & \text{if} \quad |x| > \dfrac{\ell_s}{2}. \end{cases} \tag{15}$$

The complex power intruding into a b wide part of the plate – using relation (5a) and supposing $\mathbf{Z}_0$ to be independent of x –

$$P + jQ = b\int_{-\infty}^{+\infty} (\mathbf{E})_{y=0}(\mathbf{H}_x)^*_{y=0}\,\mathrm{d}x = b\mathbf{Z}_0 \int_{-\infty}^{+\infty} (\mathbf{H}_x)_{y=0}(\mathbf{H}_x)^*_{y=0}\,\mathrm{d}x.$$

Finally, taking (4) and (6) into consideration

$$P + jQ = b\mathbf{Z}_0 \left(\frac{NI}{\ell_s}\right)^2 \int_{-\infty}^{+\infty} \mathbf{f}(x)\mathbf{f}^*(x)\,\mathrm{d}x.$$

The *effective length* 'ℓ' is defined in such a way, that the assumption of the magnetic field strength by a magnitude of $\frac{NI}{\ell}$ results the same complex power along the length

'ℓ' like the more accurate computation. Thus – using that $\mathbf{f}(x)$ is even –

$$b\mathbf{Z}_0\left(\frac{NI}{\ell}\right)^2\ell = P + jQ = b\mathbf{Z}_0\left(\frac{NI}{\ell_s}\right)^2\cdot 2\int_0^\infty \mathbf{f}(x)\mathbf{f}^*(x)\,\mathrm{d}x.$$

From that

$$\ell = \frac{\ell_s^2}{2\int_0^\infty \mathbf{f}(x)\mathbf{f}^*(x)\,\mathrm{d}x}. \tag{16}$$

According to (5a) and (5b) the value of impedance $\mathbf{Z}_0$ depends on the x-independent value of H_m, which is not defined yet. It is obvious to calculate this value (which is average according to x and the peak value of the sinusoidal quantity) using length 'ℓ':

$$H_m = \sqrt{2}\frac{NI}{\ell}. \tag{17}$$

In accordance with formula (5a) and (5b), notation (9) and (12), and relation (14) and (15) $\mathbf{f}(x)$ depends on H_m, therefore the computation requires iteration. The integration contained by (16) can be seen in the Appendix. The final result is:

$$\int_0^\infty \mathbf{f}(x)\mathbf{f}^*(x)\,\mathrm{d}x = \frac{\ell_s}{2} + \frac{a}{4\alpha} - \frac{a\alpha}{\alpha^2+\beta^2}$$

$$+ae^{-\alpha\frac{\ell_s}{a}}\left[\left(\frac{\alpha}{\alpha^2+\beta^2} - \frac{1}{4\alpha}\right)\cos\left(\beta\frac{\ell_s}{a}\right) - \left(\frac{\beta}{\alpha^2+\beta^2} - \frac{1}{4\beta}\right)\sin\left(\beta\frac{\ell_s}{a}\right)\right].$$

Substituting the previous result into (16) the following formula can be written:

$$\frac{\ell}{\ell_s} =$$

$$= \frac{1}{1 + \frac{a}{\ell_s}\left\{e^{-\alpha\frac{\ell_s}{a}}\left[\frac{3\alpha^2-\beta^2}{2\alpha(\alpha^2+\beta^2)}\cos\left(\beta\frac{\ell_s}{a}\right) - \frac{3\beta^2-\alpha^2}{2\beta(\alpha^2+\beta^2)}\sin\left(\beta\frac{\ell_s}{a}\right)\right] - \frac{3\alpha^2-\beta^2}{2\alpha(\alpha^2+\beta^2)}\right\}}. \tag{18}$$

Fig. 2 shows the distribution of the absolute value of the magnetic field strength on the surface of the plate at different air gaps, and at parameters $p = 5$ and $\ell_s = 40$ mm. In *Fig.* 3, the equivalent excitation length as a function of the air gap can be seen.

Conclusion

Based on *Eqs.* (5a), (5b), (9), (12), (14), (17) and (18) an iterative evaluation is given to determine the equivalent excitation length. Inserting this process into designing any induction heating arrangement the effect of the air gap can be taken into account.

References

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Appendix

Substituting $\mathbf{q}$ into (15) by $\alpha - j\beta$ according to (14), in case of $|x| \le \frac{\ell_s}{2}$

$$\mathbf{f}(x)\mathbf{f}^*(x) = \left\{1 - \frac{1}{2}\left[e^{(\alpha-j\beta)\frac{x-\frac{\ell_s}{2}}{a}} + e^{-(\alpha-j\beta)\frac{x+\frac{\ell_s}{2}}{a}}\right]\right\}$$

$$\times \left\{1 - \frac{1}{2}\left[e^{(\alpha+j\beta)\frac{x-\frac{\ell_s}{2}}{a}} + e^{-(\alpha+j\beta)\frac{x+\frac{\ell_s}{2}}{a}}\right]\right\};$$

performing the multiplication and using equivalent conversions:

$$\mathbf{f}(x)\mathbf{f}^*(x) = 1 - e^{\alpha\frac{x-\frac{\ell_s}{2}}{a}} \cos\left(\beta\frac{x-\frac{\ell_s}{2}}{a}\right) - e^{-\alpha\frac{x+\frac{\ell_s}{2}}{a}} \cos\left(\beta\frac{x+\frac{\ell_s}{2}}{a}\right)$$

$$+\frac{1}{4}e^{2\alpha\frac{x-\frac{\ell_s}{2}}{a}} + \frac{1}{4}e^{-2\alpha\frac{x+\frac{\ell_s}{2}}{a}} + \frac{1}{2}e^{-\alpha\frac{\ell_s}{a}} \cos\left(2\beta\frac{x}{a}\right),$$

so

$$\int_0^{\frac{\ell_s}{2}} \mathbf{f}(x)\mathbf{f}^*(x)\,\mathrm{d}x = \left\{x - \frac{e^{\alpha\frac{x-\frac{\ell_s}{2}}{a}}}{\frac{\alpha^2}{a^2}+\frac{\beta^2}{a^2}}\left[\frac{\alpha}{a}\cos\left(\beta\frac{x-\frac{\ell_s}{2}}{a}\right) + \frac{\beta}{a}\sin\left(\beta\frac{x-\frac{\ell_s}{2}}{a}\right)\right] -\right.$$

$$-\frac{e^{-\alpha\frac{x+\frac{\ell_s}{2}}{a}}}{\frac{\alpha^2}{a^2}+\frac{\beta^2}{a^2}}\left[-\frac{\alpha}{a}\cos\left(\beta\frac{x+\frac{\ell_s}{2}}{a}\right) + \frac{\beta}{a}\sin\left(\beta\frac{x+\frac{\ell_s}{2}}{a}\right)\right] +$$

$$\left.+\frac{a}{8\alpha}e^{2\alpha\frac{x-\frac{\ell_s}{2}}{a}} - \frac{a}{8\alpha}e^{-2\alpha\frac{x+\frac{\ell_s}{2}}{a}} + \frac{a}{4\beta}e^{-\alpha\frac{\ell_s}{a}}\sin\left(2\beta\frac{x}{a}\right)\right\}_{x=0}^{x=\frac{\ell_s}{2}}.$$

After substitution and sorting:

$$\int_0^{\frac{\ell_s}{2}} \mathbf{f}(x)\mathbf{f}^*(x)\,\mathrm{d}x = \frac{\ell_s}{2} + e^{-\alpha\frac{\ell_s}{a}}\left[\frac{\alpha a}{\alpha^2+\beta^2}\cos\left(\beta\frac{\ell_s}{a}\right) - \frac{\beta a}{\alpha^2+\beta^2}\sin\left(\beta\frac{\ell_s}{a}\right)\right]$$

$$-\frac{\alpha a}{\alpha^2+\beta^2}+\frac{a}{8\alpha}\left(1-e^{-2\alpha\frac{\ell_s}{a}}\right)+\frac{a}{4\beta}e^{-\alpha\frac{\ell_s}{a}}\sin\left(\beta\frac{\ell_s}{a}\right);$$

if $|x|>\dfrac{\ell_s}{2}$ using $\mathbf{q}=\alpha-j\beta$ according to (14),

$$\mathbf{f}(x)\mathbf{f}^*(x)=\frac{1}{4}\left[e^{(\alpha-j\beta)\frac{\ell_s}{2a}}-e^{-(\alpha-j\beta)\frac{\ell_s}{2a}}\right]\times$$

$$\times\left[e^{(\alpha+j\beta)\frac{\ell_s}{2a}}-e^{-(\alpha+j\beta)\frac{\ell_s}{2a}}\right]e^{-(\alpha-j\beta)\frac{\ell_s}{2a}}e^{-(\alpha+j\beta)\frac{\ell_s}{2a}}=$$

$$=\frac{1}{4}\left[e^{\alpha\frac{\ell_s}{a}}-e^{j\beta\frac{\ell_s}{a}}-e^{-j\beta\frac{\ell_s}{a}}+e^{-\alpha\frac{\ell_s}{a}}\right]e^{-2\alpha\frac{x}{a}}$$

it means, that

$$\mathbf{f}(x)\mathbf{f}^*(x)=\frac{1}{2}\left[\operatorname{ch}\left(\alpha\frac{\ell_s}{a}\right)-\cos\left(\beta\frac{\ell_s}{a}\right)\right]e^{-\frac{2\alpha}{a}x}.$$

Therefore

$$\int_{\frac{\ell_s}{2}}^{\infty}\overline{f}(x)\overline{f}^*(x)=\frac{1}{2}\left[\operatorname{ch}\left(\alpha\frac{\ell_s}{a}\right)-\cos\left(\beta\frac{\ell_s}{a}\right)\right]\frac{a}{2\alpha}e^{-\frac{\alpha}{a}\ell_s}=$$

$$=\frac{a}{8\alpha}\left(1+e^{-2\alpha\frac{\ell_s}{a}}\right)-\frac{a}{4\alpha}\cos\left(\beta\frac{\ell_s}{a}\right)e^{-\frac{\alpha}{a}\ell_s}.$$

Finally, because

$$\begin{aligned}
\int_0^{\infty}\mathbf{f}(x)\mathbf{f}^*(x)\,\mathrm{d}x &= \int_0^{\frac{\ell_s}{2}}\mathbf{f}(x)\mathbf{f}^*(x)\,\mathrm{d}x+\int_{\frac{\ell_s}{2}}^{\infty}\mathbf{f}(x)\mathbf{f}^*(x)\,\mathrm{d}x,\\
\int_0^{\infty}\mathbf{f}(x)\mathbf{f}^*(x)\,\mathrm{d}x &= \frac{\ell_s}{2}+\frac{a}{4\alpha}-\frac{a\alpha}{\alpha^2+\beta^2}\\
&\quad +ae^{-\alpha\frac{\ell_s}{a}}\left[\left(\frac{\alpha}{\alpha^2+\beta^2}-\frac{1}{4\alpha}\right)\cos\left(\beta\frac{\ell_s}{a}\right)\right.\\
&\quad \left.-\left(\frac{\beta}{\alpha^2+\beta^2}-\frac{1}{4\beta}\right)\sin\left(\beta\frac{\ell_s}{a}\right)\right].
\end{aligned}$$