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MODELLING, FLATNESS AND SIMULATION OF A CLASS OF CRANES[1]

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Abstract

A unified framework for the modelling of a class of cranes is presented. The dynamic equations are obtained using Lagrange multipliers associated to geometric constraints between generalized coordinates. This approach provides a simple way to show differential flatness for all cranes of the class and to generate a compact numerical simulation software. Examples illustrate the approach.

Keywords: crane modelling, flatness, underactuated mechanical systems, constrained Lagrangian systems.

1. Introduction

Many different types of cranes are used in various industries like construction or naval transport where both economic and security improvements are needed [5, 6, 10, 11].

In spite of different structures, numerous types of cranes carrying the load using cables and pulleys have similar mechanical properties. In particular, they are all underactuated systems showing oscillatory behaviour. These mechanical similarities suggest that modelling of cranes with different structures may be carried out using a unified framework. This has particular interest if one can prove general properties for all elements of the considered set of cranes.

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In this paper, the flatness property [1, 2, 3] is proven to hold to the modelled class of cranes. This property is useful for both motion planning purposes and for closed loop control, aspects that are not treated here in details due to space limitations (the reader may find several applications in the literature, see [8] and its bibliography).

The modelling framework is also useful to simulate the nonlinear dynamics of cranes in the considered class without need to obtain a state-space representation, that would require complicated algebraic operations. Also, testing control algorithms might be significantly simplified with this approach.

We start with an introductory example. Section 2 gives a general definition of a crane together with a method to obtain its dynamics using Lagrange multipliers. Flatness is proven in Section 3 and a simulation method is proposed in Section 4. Two more examples are given in Section 5.

Example 1 The crane is depicted in *Fig. 1*.

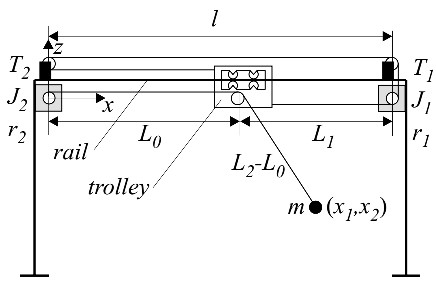

Fig. 1. Overhead crane with its two winches and its trolley

It comprises a working load with mass m whose position is denoted by (x_1, x_2); two motors, one located at the origin and a second one located at the end at a distance l. The first (resp. second) motor has inertia J_1 (resp. J_2) and a pulley of radius r_1 (resp. r_2). Both motors are torque controlled so they deliver a direct force T_1 (resp. T_2); a main pulley mounted on a trolley moving on a rail and actuated through cables by the first motor, L_1 denoting the distance between the main pulley and the motor; the motor at the origin winches the main cable with length L_2 passing through the pulley on the trolley before ending attached to the load. Let us denote by m_2 (resp. m_1) the total inertia with respect to the variable L_2 (resp. L_1): $m_2 = J_2/r_2^2$ (resp. $m_1 = m' + J_1/r_1^2$ where m' is the mass of the trolley). The kinetic and gravitational potential energy read: $W_k = \frac{1}{2}m_2\dot{L}_2^2 + \frac{1}{2}m_1\dot{L}_1^2 + \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2)$, $W_p = mgx_2$. Here, the vector of generalized coordinates is $q = (q_1, q_2, q_3, q_4) = (x_1, x_2, L_1, L_2)$. The rope length between the main pulley and the load equals $L_2 - L_0 = L_1 + L_2 - l$ and the following constraint is valid: $C = \frac{1}{2}\left((x_1 - l + L_1)^2 + x_2^2 - (L_1 + L_2 - l)^2\right) = 0$. The Lagrangian reads $\mathcal{L} = W_k - W_p$. Denote by $\tau_{q_i} = (\tau_{x_1}, \tau_{x_2}, \tau_{L_1}, \tau_{L_2})^T$ the internal force acting on the system to realize the above constraint C. We prove that (see Theorem 1 below):

$\tau_{q_i} = \lambda \frac{\partial C}{\partial q_i}$, $\quad i = 1 \ldots 4$, where λ is the Lagrange multiplier. Then the dynamics read:

$$\begin{aligned} m\ddot{x}_1 &= \lambda(x_1 - l + L_1), & m_1\ddot{L}_1 &= \lambda(x_1 - L_2) + T_1, \\ m\ddot{x}_2 &= \lambda x_2 - mg, & m_2\ddot{L}_2 &= -\lambda(L_1 + L_2 - l) + T_2, \end{aligned} \tag{1}$$

subject to Constraint C.

Note that another crane with the same configuration variables but with a different geometry would lead to the same left hand sides as in (1). Moreover, the geometry appears only through the Lagrange multiplier λ and derivatives of the geometric constraint as can be seen on the right hand sides of (1). Thus, this method provides a unifying modelling framework for cranes with various geometries, an easy way to prove the flatness property of the crane and a well adapted numerical simulation approach.

2. General Formulation for 2D and 3D Cranes

2.1. Crane Description

Let p be the dimension of the working space with $p \in \{2, 3\}$.

Definition 1 (crane) A crane is constituted by the following elements: i) a rigid articulated actuated mechanical system with $d \in \{0, 1\}$ degrees of freedom, ii) motors, iii) cables, iv) pulleys, v) a load, and enjoys the following topographic properties:

1. Let $s + 1$ be the number of motors fixed on the articulated structure.
2. There are as many cables as motors.
3. A motor is linked to a pulley or to the load with a cable.
4. s cables end on a unique pulley, called the main pulley. If $s = 0$ there is no main pulley. Every other pulley is fixed to the structure.
5. There is a unique cable going through the main pulley and ending on the load.
6. Between the load and the main pulley there is no other pulley.

Moreover, the following physical property is assumed. The main pulley moves in a manifold of dimension $n \in (p-1, p)$. When $n = p$, the main pulley can move in every direction of the working space. If $n = p - 1$, it can move in a $p - 1$ dimensional manifold (corresponding to a one dimensional geometric constraint, for example when the main pulley is constrained to move along a rail) assumed to be transversal to the gravitational field.

Any mechanical structure satisfying this definition will be referred to as a crane. The parameters s, d, n are specific to the given crane. For the planar ($p = 2$) example presented in the introduction we have: $s = 1$, $d = 0$, $n = 1$.

Let us enumerate and order the fixed pulleys along each cable starting from the motor winching the cable to the main pulley or to the load. This is possible due to the previous definition. Denote by r_i the number of fixed pulleys along the ith cable ($i = 1 \dots s + 1$).

2.2. Crane Modelling

We present here a Lagrangian approach to the crane modelling. Hence, we start with the choice of generalized coordinates, then express the Lagrangian and the geometric constraints. The model is given in Theorem 1 below.

Consider an inertial base frame such that its pth axis is pointed in the direction opposite to g, the gravity acceleration. We introduce the following coordinates:

1. position of the working load: $(x_1, \dots, x_p)$,
2. position of the main pulley (if it exists): $(x_{01}, \dots, x_{0p})$,
3. positions of the motors: $(x_{i1}, \dots, x_{ip})$ for $i = 1 \dots s + 1$,
4. positions of the fixed pulleys: $(w_{ij1}, \dots, w_{ijp})$ for $i = 1 \dots s+1$ and $j = 1 \dots r_i$,
5. cable lengths: L_i for $i = 1 \dots s + 1$,
6. cable length L_0 between the main pulley (if it exists) and the motor winching the working load.

The load mass is m and the main pulley mass is m_0. To each motor fixed on the structure there is a corresponding equivalent mass m_i, $i = 1 \dots s + 1$. The coordinate L_0 is not associated to any mass. We assume that the rigid body with at most one degree of freedom has an equivalent mass M and its coordinates coincide with the ones of the motor winching the load, namely $(x_{(s+1)1}, \dots, x_{(s+1)p})$.

The reader can easily check that all fixed pulleys along each cable can be virtually eliminated by placing the corresponding motor at the position of the last pulley with an equivalent mass obtained by adding to its own equivalent mass the sum of the equivalent masses of all the pulleys removed. Each cable length is then reduced by the sum of the constant cable distances between the pulleys removed along that cable. For notational convenience, L_i's stand for these new lengths. Because of space limitations we suppose the following.

Assumptions 1

(A1) The main pulley is present. Consequently, $s \geq 1$.

(A2) The angular velocities of the fixed pulleys are small enough to neglect their quadratic effects w.r.t. the structure. We suppose that all the motors are located on the structure along a line determined by the origin of the base frame and by the position of the motor winching the load: $x_{ji} = \alpha_j x_{(s+1)i}$ for $j = 1 \dots s$ and $i = 1 \dots p$.

(A3) If the main pulley moves along a rail, the rail coincides with the above line. Let us introduce a parameter c such that $c = 1$ if the rail is present and $c = 0$ otherwise.

(A4) The crane has no redundant actuator or motor: $s = p - d - c$. (Recall that d is the number of degrees of freedom of the articulated structure, $s + 1$ is the number of motors winching cables, and p is the dimension of the working space).

(A5) If $d = 1$ the origin of the base frame is on the joint axis of the articulated mechanical structure. The articulated mechanical structure consists of either a rotational joint, to which case the joint axis is collinear with g, or a prismatic joint, to which case the joint axis is orthogonal to g. This assumption eliminates the variable $x_{(s+1)p}$. (The vertical position of the motor winching the load remains constant.)

Table 1. Parameter values compatible with the assumptions

p	d	c	s	$d+s+1$
2	0	0	2	3
2	0	1	1	2
3	1	0	2	4
3	1	1	1	3

The number of actuators (i.e. the actuator of the articulated structure and the motors winching the cables taken together) equals $s+d+1$. *Table1* gives the possible values of the parameters p, d, c and s compatible with the assumptions.

The Lagrangian reads:

$$\mathcal{L}=\frac{1}{2}\left(m\sum_{i=1}^{p}\dot{x}_i^2+m_0\sum_{i=1}^{p}\dot{x}_{0i}^2+M\sum_{i=1}^{p}\dot{x}_{(s+1)i}^2+m_i\sum_{i=1}^{s+1}\dot{L}_i^2\right)-g(mx_p+m_0x_{0p}). \quad (2)$$

Constraints on the cable lengths are present either due to cables terminating at the main pulley:

$$C_j(x_{01},\ldots,x_{0p},x_{(s+1)1},\ldots,x_{(s+1)p-1},L_j)=0, \quad j=1\ldots s, \quad (3)$$

or due to the cable terminating at the working load, one for the total length between the main pulley and the corresponding motor, and one for the length between the load and the main pulley:

$$C_{s+1}(x_{01},\ldots,x_{0p},x_{(s+1)1},\ldots,x_{(s+1)p-1},L_0)=0, \quad (4)$$

$$C_{s+2}(x_{01},\ldots,x_{0p},x_1,\ldots,x_p,L_0,L_{s+1})=0. \quad (5)$$

An additional constraint is imposed by the motion compatible with the degree of freedom of the structure. In view of the above assumptions, the following constraint exists only if $p=3$:

$$C_{s+3}(x_{(s+1)1},\ldots,x_{(s+1)p-1})=0\ . \quad (6)$$

The motion of the main pulley along the rail (if it is present) is of the form:

$$C_{s+p+k}(x_{0k},x_{0p},x_{(s+1)k})=0, \quad k=1\ldots p-1. \quad (7)$$

Denote by l the total number of constraints. If (7) is present, $l = s + 2p - 1$ and $l = s + p$ otherwise.

Here, the functions $C_1, \ldots, C_l$ are quadratic functions of all their arguments. Moreover, $C_1, \ldots, C_{s+2}$ contain no product involving L_j, for $j = 0 \ldots s+1$. Their exact form is not needed in the sequel (see Remark 2 below).

In place of obtaining an explicit differential model, we prefer an implicit formulation with additional variables, known as *Lagrange multipliers*.

Theorem 1 *Assume that the constraints are independent in an open subset of the generalized coordinate space. The dynamical model associated to a crane corresponding to Definition 1 reads:*

$$m\ddot{x}_i = \lambda_{s+2}\frac{\partial C_{s+2}}{\partial x_i} - \delta_{ip} mg, \quad i = 1 \ldots p, \tag{8}$$

$$m_0\ddot{x}_{0i} = \sum_{j=1}^{l} \lambda_j \frac{\partial C_j}{\partial x_{0i}} - \delta_{ip} m_0 g, \quad i = 1 \ldots p, \tag{9}$$

$$0 = \sum_{j=1}^{l} \lambda_j \frac{\partial C_j}{\partial L_0}, \tag{10}$$

$$m_i \ddot{L}_i = \sum_{j=1}^{l} \lambda_j \frac{\partial C_j}{\partial L_i} + T_i, \quad i = 1 \ldots s+1, \tag{11}$$

$$M\ddot{x}_{(s+1)i} = \sum_{j=1}^{l} \lambda_j \frac{\partial C_j}{\partial x_{(s+1)i}} + F_i(T_{s+2}), \quad i = 1 \ldots p-1 \tag{12}$$

subject to Constraints (3)–(7), where $\delta_{ip} = 1$ if $i = p$ and $\delta_{ip} = 0$ otherwise. $T_1, \ldots, T_{s+1}$ are the torques produced by the motors on the structure and T_{s+2} the one produced by the structure actuator.

Proof 1 We compute $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = F_q + \tau_q$ where $q = (x_1, \ldots, x_p, x_{01}, \ldots, x_{0p}, L_0, L_1, \ldots, L_{s+1}, x_{(s+1)1}, \ldots, x_{(s+1)p-1})^T$, F_q are the external generalized forces and τ_q are the constraint forces. We have

$$F_q = (\underbrace{0, \ldots, 0}_{2p+1}, T_1, \ldots, T_{s+1}, F_1(T_{s+2}), \ldots, F_{p-1}(T_{s+2}))^T.$$

Taking total differential of the constraints leads to $\sum_{j=1}^{\dim q} \frac{\partial C_i}{\partial q_j} dq_j = 0$, $i = 1 \ldots l$, expressing that virtual displacements are in ker dC, where dC is the matrix whose entries are $\frac{\partial C_i}{\partial q_j}$. Since the constraint forces compatible with the virtual displacements are workless we have $\sum_{i=1}^{\dim q} \tau_i dq_i = 0$. Therefore τ_i is a linear combination of the lines of dC:

$$\tau_i = \sum_{j=1}^{l} \lambda_j \frac{\partial C_j}{\partial q_i}, \qquad i = 1 \ldots \dim q \tag{13}$$

and the theorem is proved.

Remark 1 As announced in the introductory example, the left hand side $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}$ of the model (8)–(12) is independent of the specific topography of the crane, whereas the right hand side consists of the exterior forces F_q plus gravity terms $\frac{\partial \mathcal{L}}{\partial q}$ and the terms given by (13) which sum up the topographic specificity.

Remark 2 The exact form of the constraints C_j, $j = 1 \ldots l$ are:

$$C_j = \frac{1}{2}\sum_{i=1}^{p}(x_{0i} - \alpha_j x_{(s+1)i})^2 - \frac{1}{2}L_j^2 = 0, \qquad j = 1 \ldots s, \tag{14}$$

$$C_{s+1} = \frac{1}{2}\sum_{i=1}^{p-1}(x_{0i} - x_{(s+1)i})^2 - \frac{L_0^2}{2} = 0, \tag{15}$$

$$C_{s+2} = \frac{1}{2}\sum_{i=1}^{p}(x_i - x_{0i})^2 - \frac{(L_{s+1} - L_0)^2}{2} = 0, \tag{16}$$

$$C_{s+3} = \begin{cases} \frac{1}{2}\sum_{i=1}^{p-1} x_{(s+1)i}^2 - r^2 = 0 & \text{for rotational joint,} \\ t_1 x_{(s+1)2} - x_{(s+1)1} t_2 = 0 & \text{for prismatic joint,} \end{cases} \tag{17}$$

$$C_{s+p+k} = x_{0k}x_{(s+1)p} - x_{(s+1)k}x_{0p} = 0 \qquad k = 1 \ldots p-1, \tag{18}$$

where $t = (t_1, \ldots, t_p)^T$ is the vector of joint axis of the articulated structure and r is the constant distance between the joint axis and the motor winching the load in the case of rotational joint. Note that these formulas are not needed to state and prove our main results.

3. Flatness

For completeness, let us give first the definition of differentially flat systems.

Definition 2 (flatness) The system

$$\dot{x} = f(x, u) \tag{19}$$

with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ is differentially flat if one can find a set of variables, called flat output,

$$y = h(x, u, \dot{u}, \ddot{u}, \ldots, u^{(r)}), \qquad y \in \mathbb{R}^m \tag{20}$$

with r finite integer, such that

$$\begin{aligned} x &= \alpha(y, \dot{y}, \ddot{y}, \ldots, y^{(q)}), \\ u &= \beta(y, \dot{y}, \ddot{y}, \ldots, y^{(q+1)}), \end{aligned} \tag{21}$$

with q a finite integer, and such that the system equations

$$\frac{d\alpha}{dt}(y, \dot{y}, \ddot{y}, \ldots, y^{(q+1)}) = f(\alpha(y, \dot{y}, \ddot{y}, \ldots, y^{(q)}), \beta(y, \dot{y}, \ddot{y}, \ldots, y^{(q+1)}))$$

are identically satisfied.

Assume that we exclude trajectories in free fall, namely such that $\ddot{x}_p = -g$, and such that $\frac{\partial C_{s+2}}{\partial x_p} \neq 0$.

Theorem 2 *Cranes defined by Definition 1 and satisfying (A1)–(A5) are differentially flat. The flat output can be chosen as* $(x_1, \ldots, x_p)$, *the coordinates of the load, and* $s+d+1-p$ *coordinates of the main pulley.*

Proof 2 In view of the assumptions we need to distinguish the four cases of *Table1*. We provide the proof for $p = 3$, the simplest cases with $p = 2$ are left to the reader. (Recall that $p = 2$ implies $d = 0$.)

Assume first that $s = 2 = p - 1$ and consider $(x_1, \ldots, x_p, x_{0p})$ as a candidate flat output. Combining the pth equation of (8) and (5) and the fact that the C_i's contain no cross-terms involving L_0, L_{s+2} by assumption, one obtains λ_{s+2} as a function of x_p, $\ddot{x}_p$ and x_{0p} since $\frac{\partial C_{s+2}}{\partial x_p} \neq 0$. Next, as long as $\lambda_{s+2} \neq 0$ which is guaranteed by the assumption that $\ddot{x}_p = -g$, the first $p-1$ equations of (8) express the remaining coordinates $x_{01}, \ldots, x_{0(p-1)}$ as functions of x_j, $\ddot{x}_j$, $j = 1 \ldots p$, and x_{0p}. Next, we use the $2p+1$ equations (4)–(6), (9) and (10) to express the $2p+1$ variables L_0, L_{s+1}, $x_{(s+1)1}, \ldots, x_{(s+1)p-1}$, $\lambda_1, \ldots, \lambda_p$ as functions of $x_{01}, \ldots, x_{0p}$, $x_1, \ldots, x_p$, λ_{s+2} and derivatives up to order 2, which in turn can be expressed as functions of $x_1, \ldots, x_p, x_{0p}$ and derivatives up to order 4. Now, by (3), one can express $L_1, \ldots, L_s$ as functions of the previous ones. By (11), $T_1, \ldots, T_p$ are also obtained as functions of the previous ones and derivatives up to order 6, and finally, T_{s+2} and λ_{s+3} are obtained in a similar way by (12), which proves that $(x_1, \ldots, x_p, x_{0p})$ is a flat output.

Consider now the case with $s = c = 1$ (i.e. the rail constraints (7) are present). First, we use the $2p$ equations (6)–(7) and (8) to express $2p$ variables $x_{01}, \ldots, x_{0p}$, λ_{s+2}, $x_{(s+1)1}, \ldots, x_{(s+1)p-1}$ in function of x_j, $\ddot{x}_j$, $j = 1 \ldots p$. We proceed using *Eqs.* (4), (3), (5) and (10) to express the cable lengths L_0, L_1, L_2 and λ_{s+1} in function of x_j, $\ddot{x}_j$, $j = 1 \ldots p$. Next, we use *Eq.* (9) to obtain λ_s, λ_{s+p+1}, λ_{s+p+2} as functions of $x_1, \ldots, x_p$ and their derivatives up to order 4. Finally, we use *Eqs.* (11) and (12) to express $T_1 \ldots T_{s+2}$ and λ_{s+3} in function of $x_1, \ldots, x_p$ and their derivatives up to order 4 which proves that $x_1, \ldots, x_p$ is a flat output.

4. Simulation

Dynamical simulation of a system consists of numerically integrating its state equations. For the cranes we advocate to integrate the extended state equations without reducing them by choosing a particular set of independent coordinates. The system to be integrated (8)-(12) being affine w.r.t. $\Lambda = (\lambda_1, \ldots, \lambda_l)^T$ the vector of Lagrange multipliers, it is of the form

$$\ddot{q} = F(q, \dot{q})\Lambda + F_0(q, \dot{q}), \tag{22}$$

where q stands for the vector of generalized coordinates. For this system to be well determined, expressions of λ_i as functions of q and $\dot{q}$ need to be obtained. To do so, we differentiate twice the constraints $C_j(q)$, $j = 1, \ldots, l$ which gives, in matrix form $A(q, \dot{q}) + \frac{\partial C}{\partial q}\ddot{q} = 0$, with $A(q, \dot{q}) = \left(\dot{q}^T \left(\frac{\partial^2 C_1}{\partial q^2}\right)\dot{q}, \ldots, \dot{q}^T \left(\frac{\partial^2 C_l}{\partial q^2}\right)\dot{q}\right)^T$. We then replace $\ddot{q}$ by its expression given by (22) to yield $\frac{\partial C}{\partial q}F(q, \dot{q})\Lambda = -A(q, \dot{q}) - \frac{\partial C}{\partial q}F_0(q, \dot{q})$. It can be shown that $\frac{\partial C}{\partial q}F(q, \dot{q})$ is always an invertible matrix and thus

$$\Lambda = -\left(\frac{\partial C}{\partial q}F(q, \dot{q})\right)^{-1}\left(A(q, \dot{q}) + \frac{\partial C}{\partial q}F_0(q, \dot{q})\right). \tag{23}$$

Eq. (22) with Λ replaced by (23) is then integrated using a standard algorithm. Numerical simulations show that the constraints are satisfied throughout the integration process once the initial conditions satisfy them.

5. Examples

Let us illustrate our approach with two more examples where the load can move in a three dimensional working space ($p = 3$) in contrast with the introductory example where the motion of the load is restricted in a vertical plane.

The constraints can easily be obtained using *Eqs.* (14)–(18) and the notations of *Fig. 2*.

Example 2 3D Cantilever Crane. The first crane in *Fig. 2* comprises a trolley (main pulley) restricted to move along a rail. The rail rotates around its vertical axis. Thus, we have the following parameters: $n = 2$, $p = 3$, $d = s = c = 1$. The generalized coordinates are $q = \{x_1, x_2, x_3, x_{21}, x_{22}, x_{01}, x_{02}, x_{03}, L_0, L_1, L_2\}$. The constraints read:

$$\frac{1}{2}\left((x_{01} - x_{21})^2 + (x_{01} - x_{22})^2 - L_0^2\right) = 0, \qquad \frac{1}{2}\left(x_{21}^2 + x_{22}^2 - r^2\right) = 0,$$
$$\frac{1}{2}\left((x_{01} - \alpha_1 x_{21})^2 + (x_{02} - \alpha_1 x_{22})^2 - L_1^2\right) = 0, \qquad \frac{1}{2}\left(x_{01}x_{23} - x_{03}x_{21}\right) = 0,$$
$$\frac{1}{2}\left(x_{02}x_{23} - x_{03}x_{22}\right) = 0,$$
$$\frac{1}{2}\left((x_1 - x_{01})^2 + (x_2 - x_{02})^2 + (x_3 - x_{03})^2 - (L_2 - L_0)^2\right) = 0.$$

The model is thus given by Theorem 1. One can prove, using Theorem 2, that (x_1, x_2, x_3) is a flat output (see also [4]).

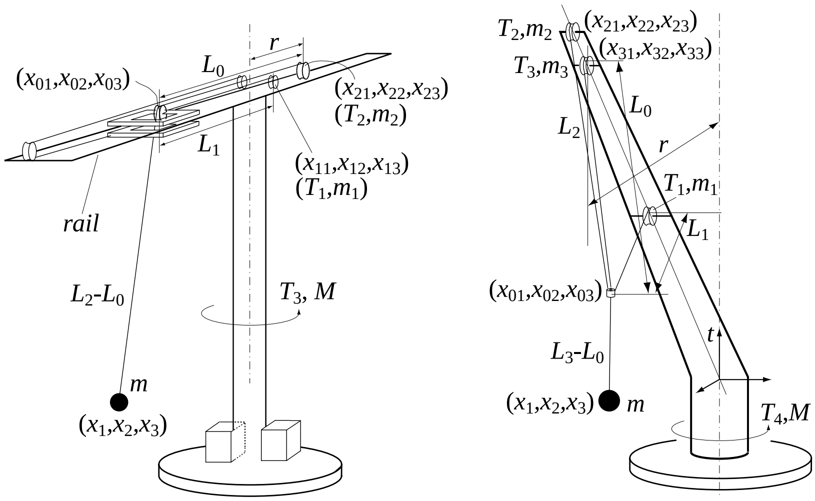

Fig. 2. 3D Cantilever and 3D US-Navy crane

Example 3 3D US-Navy Crane. The main pulley whose coordinates are x_{01}, x_{02}, x_{03} in *Fig. 2* can move in every direction: Pulley no. 1 (resp. 2) produces horizontal (resp. vertical) deviations. Motor no. 4 rotates the whole setup around the vertical axis. The hoisting cable passes through the main pulley to hoist the load. It is actuated by motor no. 3. The parameters are $n = 3$, $p = 3$, $d = 1$, $c = 0$, $s = 2$ and the vector of generalized coordinates is $q = \{x_1, x_2, x_3, x_{31}, x_{32}, x_{01}, x_{02}, x_{03}, L_0, L_1, L_2, L_3\}$. The constraints read:

$$
\begin{aligned}
\tfrac{1}{2}\left((x_1 - x_{01})^2 + (x_2 - x_{02})^2 + (x_3 - x_{03})^2 - (L_3 - L_0)^2\right) &= 0,\\
\tfrac{1}{2}\left((x_{01} - \alpha_1 x_{31})^2 + (x_{02} - \alpha_1 x_{32})^2 + (x_{03} - \alpha_1 x_{33})^2 - L_1^2\right) &= 0,\\
\tfrac{1}{2}\left((x_{01} - \alpha_2 x_{31})^2 + (x_{02} - \alpha_2 x_{32})^2 + (x_{03} - \alpha_2 x_{33})^2 - L_2^2\right) &= 0,\\
\tfrac{1}{2}\left((x_{01} - x_{31})^2 + (x_{02} - x_{32})^2 + (x_{03} - x_{33})^2 - L_0^2\right) &= 0,\\
\tfrac{1}{2}\left(x_{31}^2 + x_{32}^2 - r^2\right) &= 0.
\end{aligned}
$$

Again, the model is given by Theorem 1. One can prove, using Theorem 2, that (x_1, x_2, x_3, x_{03}) is a flat output (see also [9, 8]). Simulation results for this type of crane using proportional-derivative type feedbacks on the angular positions of the motors are reported in [7].

6. Conclusion

We have shown in this paper that a large class of cranes and weight handling equipments can be modelled in a unified way, using Lagrange multipliers to describe the geometric constraints. The main advantage of this approach can be seen in two applications, namely detecting the flatness property and computing the flat output on the one hand and simulating the system without need to express it in explicit form to achieve simpler computation though with a larger number of variables on the other hand.

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