FUZZY SUBOPTIMAL FEEDBACK CONTROLLER

Piotr KULCZYCKI* and László T. KÓCZY**

* Faculty of Electrical Engineering
Cracow University of Technology
ul. Warszawska 24
PL–31-155 Cracow, Poland
** Department of Telecommunications and Telematics
Technical University of Budapest
H–1521 Budapest, Hungary

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Abstract

The paper concerns the time-optimal control of objects described by a differential equation representing the second law of Newtonian mechanics and taking into account a discontinuous model of resistances to motion. Such a task has broad technical applications, especially in robotics. A fuzzy approach is used to design a suboptimal closed-loop control structure, convenient in practice thanks to its many advantages, especially in respect to robustness.

Keywords: time-optimal control, fuzzy dynamical system, industrial manipulators and robots, suboptimal feedback controller, fuzzy decision rules.

1. Introduction

Consider the following differential equation:

$$\ddot{y}(t) = h(\cdot) + u(t) . \tag{1}$$

If u represents a control and y denotes the position of an object, then the above dependence expresses the second law of Newtonian mechanics, with the model of resistances to motion given by the function h. Thus, an equation of type (1) provides a useful basis for the analysis of an exceptionally broad class of technical devices often encountered in engineering practice, particularly industrial manipulators and robots. The time-optimal control that yields the minimum operation time for such plants has a direct impact on their efficiency.

In this paper a concept based on fuzzy logic (KACPRZYK, 1986; KLIR and FOLGER, 1988) is introduced. The function h will be proposed in a form inspired by the physical perspective:

$$h(\cdot) = -V \operatorname{sgn}(\dot{y}(t)) . \tag{2}$$

The sgn mapping describes the discontinuous nature – with respect to velocity – of friction phenomena; however, the fuzzy set V regards – as a fuzzy uncertainty – the dependence of motion resistances on a number of factors, e.g. position, also velocity, or even temperature and disturbances, which are usually omitted in the traditional approach due to the necessity to simplify the model. By its very nature, the fuzzy approach offers the possibility to describe a complex reality with a precision that exceeds classical modelling techniques. Allowing for a certain discomfort resulting from the uncertainty introduced into the model, one may achieve a characteristic that is essential in modern engineering: robustness of the designed control system.

Finally, let:

(A) $[x_0, y_0]^T \in \mathbf{R}^2$ and $[x_T, y_T]^T \in \mathbf{R}^2$ represent initial and target states, respectively;

(B) V denote a fuzzy set with a support such that $\mathrm{supp}(V) \subset [0, 1)$;

(C) the difference equation

$$X_{j+1} = X_j + Y_j \,, \tag{3}$$

$$Y_{j+1} = Y_j - V \,\mathrm{sgn}\,(Y_j) + u_j \qquad \text{for } j = 0, 1, \ldots \tag{4}$$

with the initial condition

$$\begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \tag{5}$$

describe the dynamics of a system with the fuzzy state $[X_j, Y_j]^T$, submitted to a control u_j with values limited to the interval $[-1, 1]$.

The goal of this paper is to design a useful suboptimal (in respect to time) feedback controller, whose values – due to practical requirements – directly depend only on the valid state of the object, obtained by a real-time measurement process.

2. Some Auxiliary Considerations

In the following section, an auxiliary task will be considered. Let at this point the fuzzy set V be reduced to a real number, denoted hereafter by w. Due to assumption (B), $w \in [0, 1)$.

Suppose that $[x_+, y_+]^T$ and $[x_-, y_-]^T$ are unique solutions (KULCZYCKI, 1996) of the ordinary differential equation related to system (3)–(4):

$$\dot{x}(t) = y(t) \,, \tag{6}$$

$$\dot{y}(t) = u(t) - w \,\mathrm{sgn}\,(y(t)) \,, \tag{7}$$

with the condition $[x_+(0), y_+(0)]^T = [x_-(0), y_-(0)]^T = [x_T, y_T]^T$, defined on the interval $(-\infty, 0]$, and generated by the control $u \equiv +1$ or $u \equiv -1$, respectively. Moreover, consider

$$K_+ = \{[x_+(t), y_+(t)]^T \quad \text{for } t < 0\}, \tag{8}$$
$$K_- = \{[x_-(t), y_-(t)]^T \quad \text{for } t < 0\}; \tag{9}$$

therefore, these are the sets of all states which can be brought to the target $[x_T, y_T]^T$ by the control $u \equiv +1$ or $u \equiv -1$, respectively. Let also:

$$R_+ = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \text{ such that there exists } \begin{bmatrix} x^* \\ y^* \end{bmatrix} \in K, \text{ with } x < x^* \text{ and } y = y^* \right\}, \tag{10}$$

$$R_- = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \text{ such that there exists } \begin{bmatrix} x^* \\ y^* \end{bmatrix} \in K, \text{ with } x > x^* \text{ and } y = y^* \right\}, \tag{11}$$

where $K = K_- \cup \{[x_T, y_T]^T\} \cup K_+$. The time-optimal control is expressed by the following formula (KULCZYCKI, 1992):

$$u(t) = u_r(x(t), y(t)) = \begin{cases} -1 & \text{if } [x(t), y(t)]^T \in (R_- \cup K_-), \\ 0 & \text{if } [x(t), y(t)]^T = [x_T, y_T]^T, \\ +1 & \text{if } [x(t), y(t)]^T \in (R_+ \cup K_+), \end{cases} \tag{12}$$

and the set K constitutes a switching curve (*Fig. 1,2*).

In the time-optimal feedback controller equations, i.e. formulas (8)–(12), the parameter w intervenes, because it influences the form of the trajectories $[x_+, y_+]^T$, $[x_-, y_-]^T$ and therefore also the shape of the switching curve K. But in the fuzzy system, its value is of course not uniquely defined. The analysis of the system sensitivity to the value of the parameter (KULCZYCKI, 1992), which is briefly presented in the following, will then be of great importance. Thus, the value of the parameter w occurring in the object is still denoted as w; however, the value assumed in feedback controller equations will be marked by W.

The case where the second coordinate of the target state is equal to zero, i.e. with $y_T = 0$, will be considered first.

If $W = w$, the control is time-optimal (*Fig. 1*). The state of the system is brought to the switching curve, and being permanently included in this curve hereafter, it reaches the target in a minimal and finite time.

The trajectory representative for the case $W > w$ is shown in *Fig. 3*. As a result of its having oscillations around the target, over-regulations occur in the system. The target is reached in a finite time.

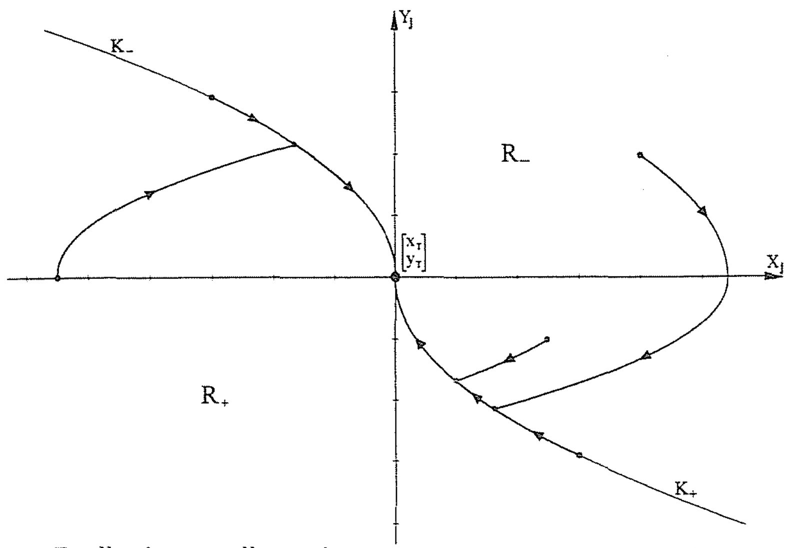

Fig. 1. Feedback controller and representative trajectories in the case $y_T = 0$

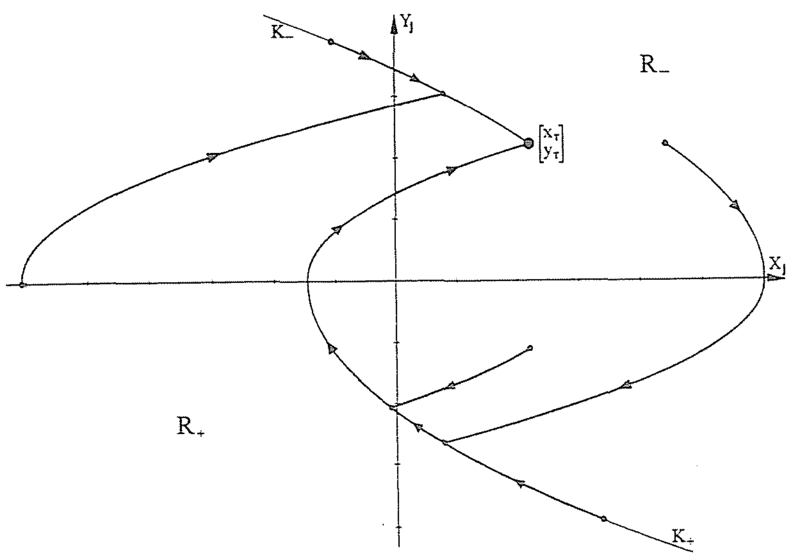

Fig. 2. Feedback controller and representative trajectories in the case $y_T \neq 0$

Fig. 4 shows trajectories representative for the case $W < w$. After the switching curve is crossed, the sliding trajectories (SLOTINE and LEE, 1991) appear in the system. Here, too, the target is reached in a finite time.

In both of the last two cases, i.e. with $W \neq w$, the time to reach the target increases from the optimal more or less proportionally to the difference between the values W and w.

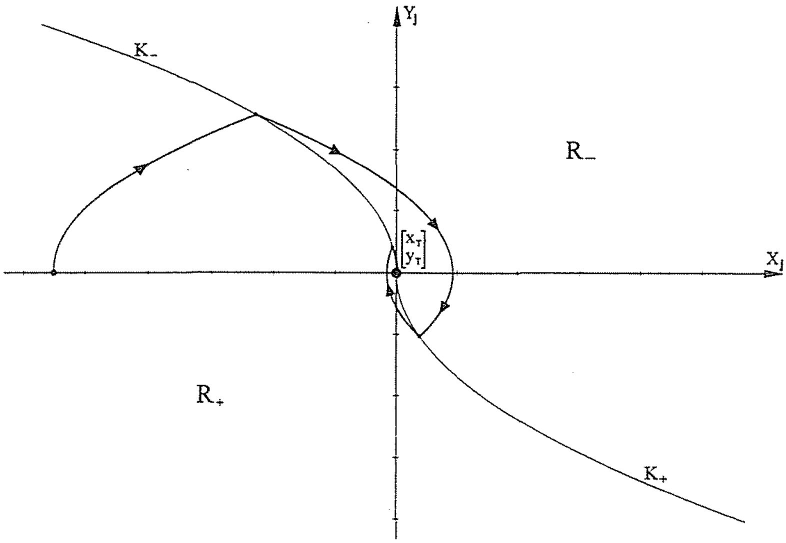

Fig. 3. Trajectory representative for $W > w$ in the case $y_T = 0$

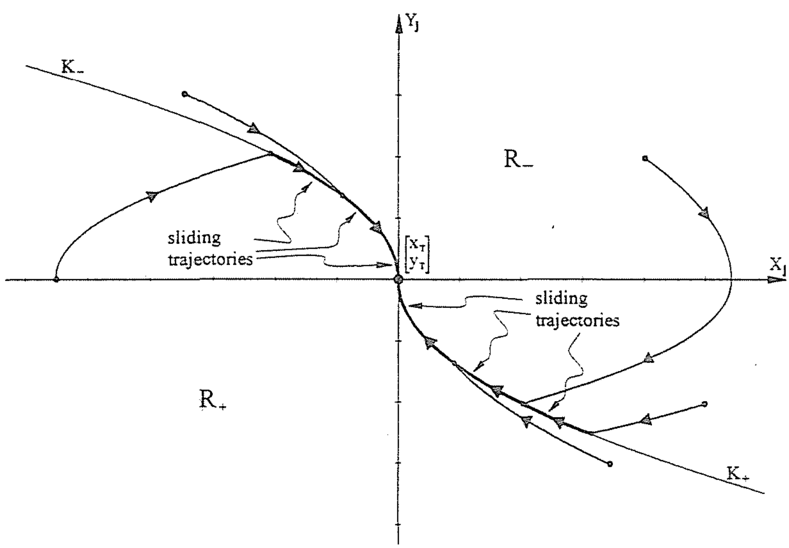

Fig. 4. Trajectories representative for $W < w$ in the case $y_T = 0$

The remaining case $y_T \neq 0$ will be presented now. Let in particular $y_T > 0$; investigations for $y_T < 0$ can be made analogously.

If $W = w$ (*Fig. 2*), the considerations are identical as before for $y_T = 0$.

In the case $W > w$ (*Fig. 5*), the trajectories occurring in the system create a limit cycle: the target is not reached.

Finally, in the case $W < w$ (*Fig. 6*) only some of the trajectories (marked on *Fig. 6* with arrows) reach the target in a finite time. Other trajectories attain only the end point $[x_E, y_E]^T$, which is the intersection of the axis x and the switching curve; the state does not reach then the target. Sliding trajectories occur on the switching curve.

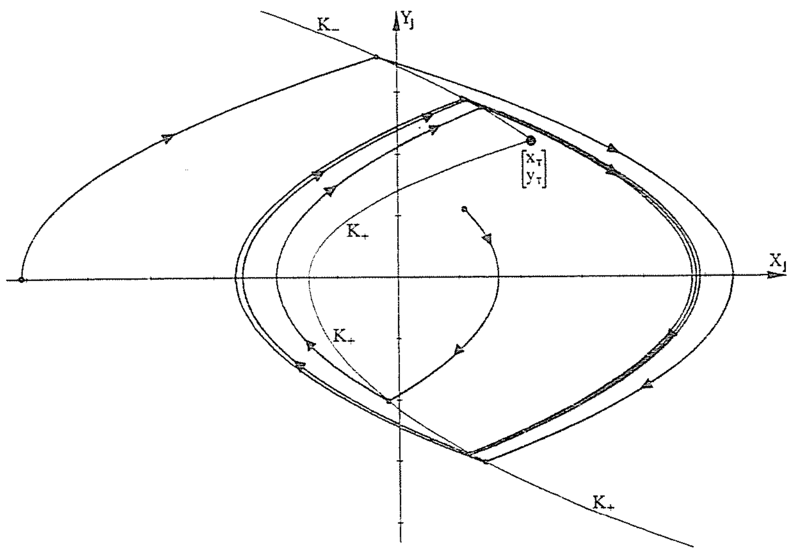

Fig. 5. Trajectories representative for $W > w$ in the case $y_T \neq 0$

3. Sub-Optimal Feedback Controller for a Fuzzy System

In this section the fuzzy system (3)–(5), which is the subject of the present paper, will be considered. The parameter w, introduced in the previous section, happens to be a fuzzy set in the problem worked out here. A fuzzy set naturally cannot be used directly to define a control in a real system. For this reason, some elements of fuzzy decision theory (KACPRZYK, 1986) will be used. Its aim is to make the optimal selection of one element from all possible decisions on the basis of a membership function.

Let the following be given: a fuzzy set Z (with the membership function $\mu_Z : \mathbf{R} \to [0, \infty)$) representing the state of reality, a non-empty set D of possible decisions and a loss function

$$l : D \times \mathbf{R} \to \mathbf{R} \cup \{\pm\infty\} , \tag{13}$$

where the values $l(d, z)$ can be interpreted as losses occurring in the hypothetical case when the fuzzy set Z is reduced to the real number z, and the

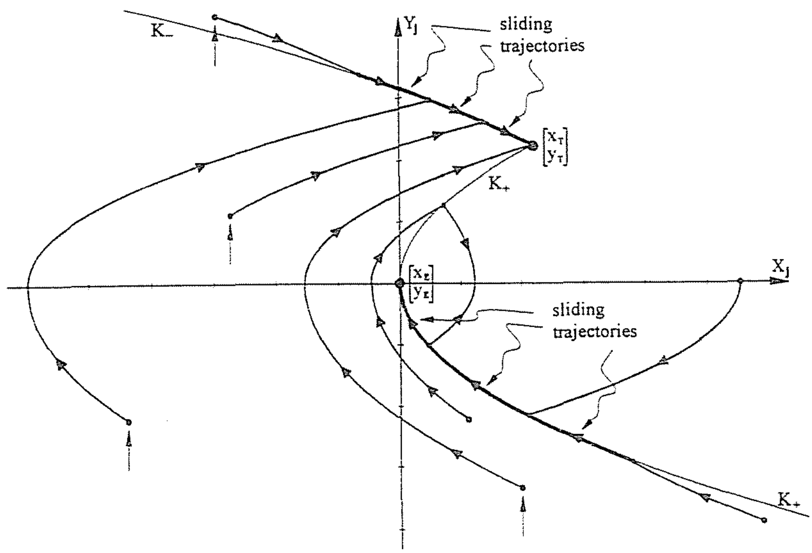

Fig. 6. Trajectories representative for $W < w$ in the case $y_T \neq 0$

decision d has been made. Denote by $l_m : D \to \mathbf{R} \cup \{\pm\infty\}$ the minimax loss function

$$l_m(d) = \sup_{z \in \operatorname{supp}(Z)} l(d, z) \,. \tag{14}$$

If additionally for every $d \in D$ the integral $\int_{\mathbf{R}} l(d,z)\mu_Z(z)\,\mathrm{d}z$ exists, suppose also the Bayes loss function $l_b : D \to \mathbf{R} \cup \{\pm\infty\}$ defined as

$$l_b(d) = \int_{\mathbf{R}} l(d,z)\mu_Z(z)\,\mathrm{d}z \,. \tag{15}$$

Every element $d_m \in D$ such that

$$l_m(d_m) = \inf_{d \in D} l_m(d) \tag{16}$$

is called a minimax decision, and analogously, every element $d_b \in D$ such that

$$l_b(d_b) = \inf_{d \in D} l_b(d) \tag{17}$$

is called a Bayes decision. The above procedures for obtaining these elements are said to be minimax and Bayes rules, respectively. The main difference between the above rules is in their interpretation. This results directly from the forms of the functions l_m and l_b: the 'pessimistic' minimax rule assumes the occurrence of the most unfavorable state of reality and counteracts it, while the Bayes rule is more flexible.

In the problem of a time-optimal control investigated here, the parameter $W \in D \subset [0,1)$ assumed in the feedback controller equations will be treated as a decision, while the number V occurring in system (3)–(4), as the fuzzy state of reality. The loss function is defined for $(W, w) \in D \times \mathbf{R}$, and its values are related to the time to reach the target, if in the feedback controller equations the parameter W was assumed, but hypothetically in the object the value w is occurring.

Again the case $y_T = 0$ will be considered first. The following suggestions for the determination of the value of the parameter W result from the analysis of the auxiliary problem presented in the previous section.

If over-regulations can be allowed, it is worthwhile using the Bayes rule with real values for the loss function. Such a choice is possible because the determination of the parameter W value that is either smaller, equal, or greater than w allows the system state to be brought to the target in a finite time. (However, this time increases more or less proportionally to the difference between the values W and w.)

If over-regulations are not allowed, this determination should be carried out on the basis of the minimax rule, assuming infinite values of the loss function for $W > w$. This enables the over-regulations to be avoided, because they occur only if $W > w$.

Let now $y_T \neq 0$. The case $y_T > 0$ will be considered; investigations for $y_T < 0$ are analogous.

The case $W = w$ is impossible to obtain in practice. However, the determination of the parameter W value that is either greater or smaller than w precludes reaching the target from any initial state, because of the occurrence of the cycle (*Fig. 5*) or existence of the end point (*Fig. 6*). In the proposed feedback controller, the switching curve K will be divided into three parts. The division points will be the target and the point of intersection with the axis x. For every part there will be differently determined values of the parameter W, which for particular parts are defined in the following as W_1, W_2 and W_3.

The value of the parameter W_1, i.e. the one which defined the part of the switching curve K_-, or K for $y \in [y_T, \infty)$ (see also *Fig. 8*), should be determined using the minimax rule with infinite values of the loss function for $W > w$. This choice is made in order to avoid the generation of a limit cycle, which appears when the value of the parameter W_1 is greater than w. If, however, this value is smaller than w, the state of the system is brought to the target in a finite time.

For the determination of the value of the parameter W_2 defining the part K_+ for $y \in [0, y_T]$, it is necessary to apply the minimax rule with infinite values of the loss function for $W < w$. This is because an overly large value of the parameter W_2 allows the state to be brought to the part defined by the parameter W_1, which, as was demonstrated above, can be successfully determined. An overly small one, however, causes the occurrence of the end point, whose existence is not admissible from the point of view of utility.

Finally, the value of the parameter W_3 defining the part K_+ for $y \in (-\infty, 0]$ can be obtained using the Bayes rule with real values of the loss function. Both overly small and large values of this parameter are acceptable, because this allows the state to be brought to the parts defined by the parameters W_1 and W_2, which can be successfully determined as shown above.

Suppose, as an example, that the fuzzy set V has the support of the form $\text{supp}(V) = [w_*, w^*] \subset [0, 1)$, and moreover, let its membership function μ_V be continuous and positive in the interval (w_*, w^*). The loss function (13) will be described by the following formula:

$$l(W, w) = \begin{cases} -p(W - w) & \text{if} \quad W - w \leq 0\,, \\ q(W - w) & \text{if} \quad W - w \geq 0\,, \end{cases} \tag{18}$$

where $p, q \in \mathbf{R} \cup \{\infty\}$; however, only one of them can be infinite. In this case, let $\infty \cdot 0 = 0$.

According to the above assumptions, it is accepted that $D = [w_*, w^*]$.

With the fixed value of the parameter W, from the definitions of minimax and Bayes loss functions – (14) and (15) – the following results, respectively:

$$l_m(W) = \max(\{-p(W - w^*), q(W - w_*)\})\,, \tag{19}$$

$$l_b(W) = \int_{w_*}^{W} q(W - w)\mu_V(w)\,\mathrm{d}w - \int_{W}^{w^*} p(W - w)\mu_V(w)\,\mathrm{d}w\,. \tag{20}$$

If $p = \infty$, then from *Eq.* (19) it can be obtained that the infimum of the function l_m on the set D is realized by

$$W = w^* \tag{21}$$

and if $q = \infty$, then this infimum is assumed for

$$W = w_*\,. \tag{22}$$

The values W indicated by formulas (21) and (22) constitute the desired minimax decision with infinite values of the loss function for $W < w$ and $W > w$, respectively.

However, with real positive values p and q, the function l_b is differentiable in the set (w_*, w^*); therefore one obtains (RUDIN, 1974; Theorem 6.20):

$$l'_b(W) = p\int_{w^*}^{W} \mu_V(w)\,\mathrm{d}w + q\int_{w_*}^{W} \mu_V(w)\,\mathrm{d}w\,, \tag{23}$$

and analogously

$$l_b''(W) = (p+q)\mu_V(w)\,. \tag{24}$$

Using formula (23), the equivalence of the following conditions can be proved by elementary transformations:

$$l_b'(W) \;=\; 0\,, \tag{25}$$

$$\int_{w_*}^{W} \mu_V(w)\,\mathrm{d}w \;=\; \frac{p}{p+q}\int_{w_*}^{w^*} \mu_V(w)\,\mathrm{d}w\,. \tag{26}$$

Formula (24) implies that the function l_b'' is positive in the set (w_*, w^*); therefore, the function l_b is here strictly convex. Because $0 < \frac{p}{p+q} < 1$, *Eq.* (26), equivalent to condition (25), is fulfilled only in one point; in this point, then, the function l_b assumes its minimum, global in the set $D = [w_*, w^*]$ due to the continuity of this function in the points w_* and w^*.

The value W that fulfills condition (26) constitutes the desired Bayes decision with real values of the loss function. To obtain its value one can use the kernel estimators technique, according to the algorithm presented in papers (KULCZYCKI and SCHIØLER, 1994; SCHIØLER and KULCZYCKI, 1997).

To summarize, in accordance with the considerations stated before, if the values of the parameters W or W_1, W_2, W_3 should be determined due to the minimax rule with infinite values of the loss function for $W < w$ or $W > w$, or the Bayes rule with real values of this function, then they can be obtained from formulas (21), (22) and (26), respectively.

If one possesses the obtained values W or W_1, W_2, W_3, the feedback controller equations can be calculated. Thus, the equations of the switching curve K take on the form

$$x \;=\; \frac{y^2 - y_T^2}{2(-1-W_1)} + x_T \qquad \text{for } y \in [y_T, \infty)\,, \tag{27}$$

$$x \;=\; \frac{y^2 - y_T^2}{2(1-W_2)} + x_T \qquad \text{for } y \in [0, y_T]\,, \tag{28}$$

$$x \;=\; \frac{y^2}{2(1+W_3)} - \frac{y_T^2}{2(1-W_2)} + x_T \qquad \text{for } y \in (-\infty, 0]\,, \tag{29}$$

in the case $y_T > 0$. (Formula (27) defines the set K_-, while dependencies (28) and (29), the set K_+.) For $y_T < 0$, the equations are analogous. If $y_T = 0$, one should substitute, in formulas (27) and (29), $W = W_1 = W_3$ (dependence (28) has no meaning here). The sets R_- and R_+ constitute adequate areas resulting from the section of the plane $\mathbf{R}^2$ by the curve K, according to formulas (10) and (11). For the sets K_-, K_+, R_-, R_+ obtained

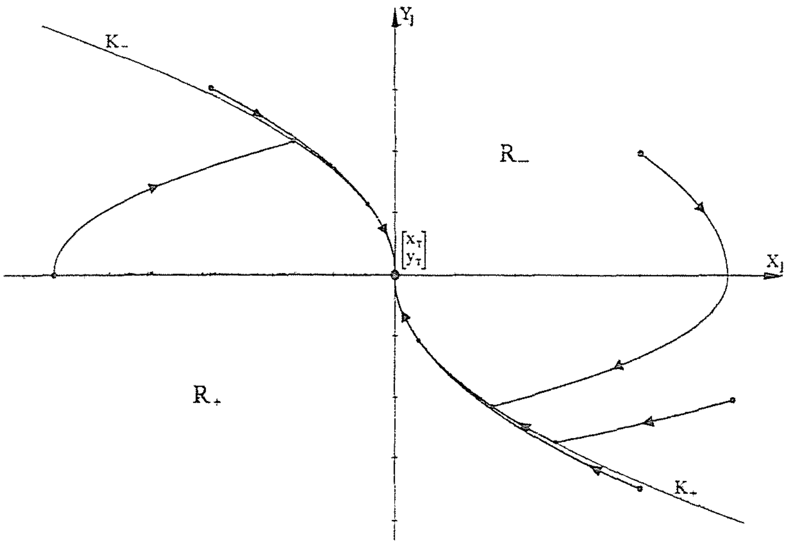

Fig. 7. Fuzzy feedback controller and empirically obtained trajectories in the case $y_T = 0$

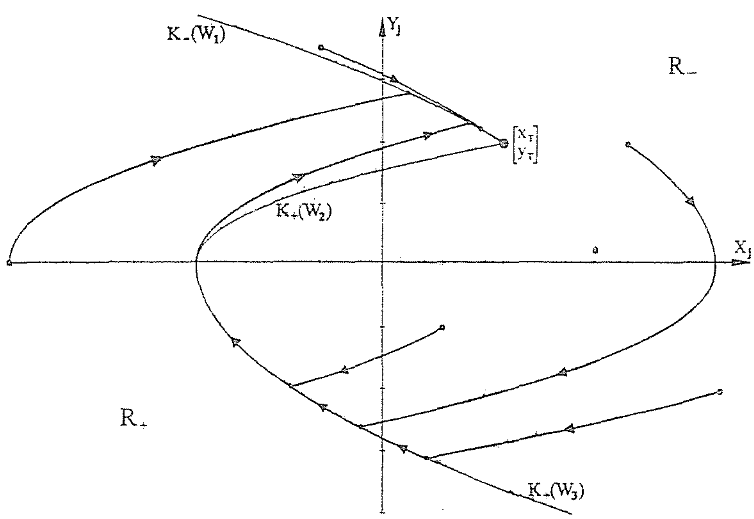

Fig. 8. Fuzzy feedback controller and empirically obtained trajectories in the case $y_T \neq 0$

above, the value of the control is simply defined by equation

$$u(t) = \begin{cases} -1 & \text{if the state of the object belongs to the set } R_- \cup K_- \,, \\ 0 & \text{if the state of the object equals } [x_T, y_T]^T \,, \\ +1 & \text{if the state of the object belongs to the set } R_+ \cup K_+ \,. \end{cases} \tag{30}$$

Figs. 7 and *8* provide an illustration of the results obtained above.

4. Conclusions

In this paper a fuzzy approach to solve the time-optimal control problem has been investigated. Theoretical considerations led to the design of a closed-loop control system, convenient in engineering practice. A large number of completed empirical examinations confirmed the presented material, and proved the correct operation of the system (see *Fig. 7* and *8*). The target state was reached in every case with the assumed precision of the scale 0.1–0.5% of the initial state norm. If it was assumed that over-regulations were unacceptable, they did not occur during the control process. The constructed control system turned out to be only slightly sensitive to the inaccuracy resulting from identification and the occurrence of perturbations. This should be emphasized as a very valuable property of uncertain, especially fuzzy, control systems.

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