A METHOD FOR CONVERGENCE ACCELERATION IN PSEUDO-TRANSIENT INTEGRATION OF DIFFUSION EQUATION SYSTEMS

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Abstract

The method presented in the paper ensures consistent convergence acceleration for the solution of the diffusion problem in electromagnetic field by finite difference and the pseudo-transient methods of integration of the partial differential equation system, providing it is a priori known that the solution is periodical. The effectiveness of the method is illustrated in the numerical results of an actual problem.

Keywords: finite difference method, problem of diffusion in electromagnetic field, pseudo-transient method.

1. Description of the Method

The method of finite difference is relatively often used in the solution of the electromagnetic field diffusion problem in conductive media [1], especially in cases of non-linear (e.g. ferromagnetic) media, where analytical solutions are not at hand. A comfortable way to find the permanent solution of the problem is the pseudo-transient method applying to a domain with zero initial field values the source term at its borders and carry out the computations until the stabilised solution occurs. The main inconvenience consists of the low rate of convergence leading to large values of computing time.

In case of the expression of the source term and the field equations guarantee a periodical solution, with zero mean value in each point of the considered domain, the following strategy leads to important gain in convergence speed without loss of stability or accuracy:

- carry out computations for several periods of the source term, with zero initial values (pseudo-transient method).

[1]The paper has been accomplished during a post-graduate scholarship at the Faculty of Electrical Engineering of the Technical University of Budapest.

- symmetrize the obtained solutions. This means the computation of the mean value for the solution functions in each spatial grid point and its subtraction from the actual value at the end of the last period determined. The values thus obtained are the initial values for further computations.
- continue computations for several other periods.
- repeat symmetrization if needed.

The effects of this strategy are illustrated by an actual problem.

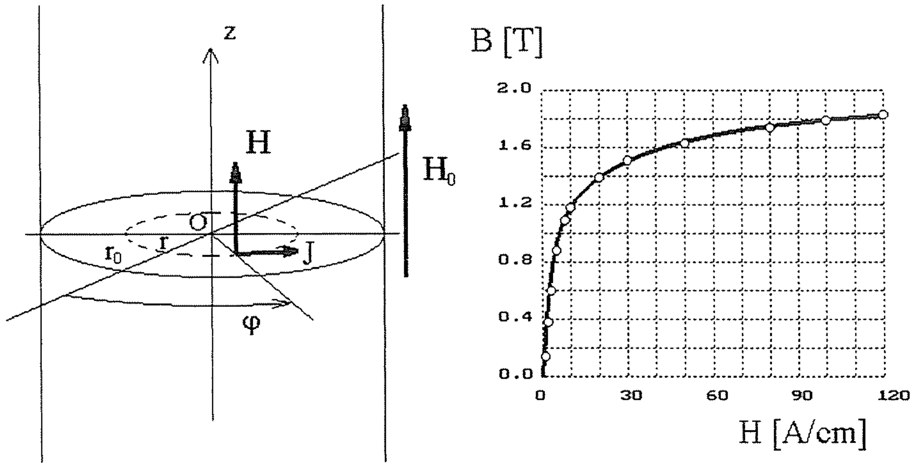

Fig. 1. Geometrical configuration and magnetisation characteristic of a cylinder

2. The Symmetrizing Technique

Let $f(t)$ be a periodical function of time, with the following feature:

$$f\left(t_0 + \frac{T}{2}\right) = -f(t_0) \ , \tag{1}$$

where T stands for the period and $a(t)$ a combination of it with a linear function:

$$a(t) = mt + n + f(t) \ . \tag{2}$$

The mean value of $a(t)$ computed for a period is:

$$\overline{a} = m\frac{T}{2} + n \ , \tag{3}$$

while the mean value of $f(t)$ is zero. This case approximates quite well the structure of the functions the intermediate solution of most eddy current problems stand of. In fact the real structure contains an exponential instead of the linear function, nevertheless, due to the low rate of decrease of the exponential, it can be replaced by a linear function during a period of the source term, providing to be far enough from the starting moment of computing process (i.e. we have computed several periods). Because of the low value of m (the decrease factor), we can approximate the mean value of $a(t)$:

$$\overline{a} \approx \frac{a_{\max} + a_{\min}}{2} , \tag{4}$$

thus simplifying the algorithm, only the extremes of $a(t)$ having to be determined, to have the approximation of the mean value. An even better result can be obtained by the following correction term instead of the mean value:

$$\Delta = \frac{a_{\max} + a_{\min}}{2} - \frac{a(t_0) - a(t_0 + T)}{2} , \tag{5}$$

including the effect of the decreasing trend in the mean value.

Applying

$$a_0 = a(t_0 + T) - \Delta \tag{6}$$

as initial value for the next period, the permanent solution of the problem is much closer to, as it is illustrated in the following example.

3. An Actual Problem

Let us consider a cylinder made of ferromagnetic material, plotted in *Fig. 1*, with electrical conductivity: $\sigma = 5 \cdot 10^6$ S/m. The magnetic permeability varies with respect to the magnetic field intensity according to:

$$\mu(H) = \mu_0 + \frac{m_1 \tan^{-1} k_1 H + m_2 \tan^{-1} k_2 H}{H} , \tag{7}$$

where $\mu_0 = 4\pi \cdot 10^{-7}$ H/m is the permeability of free space and the coefficients $m_1 = 0.85$ T; $k_1 = 0.00328$ m/A; $m_2 = 0.4$ T; $k_2 = 0.00024$ m/A have been determined based on the minimising of error square by means of a non-linear optimisation algorithm [5] in a manner that the model thus obtained would be fitted to the measured data of the anhysteretic magnetisation curve of a certain material (steel). In *Fig. 1* the circles stand for measured points of the magnetisation curve and the plotted curve was obtained by *Eq.* (7).

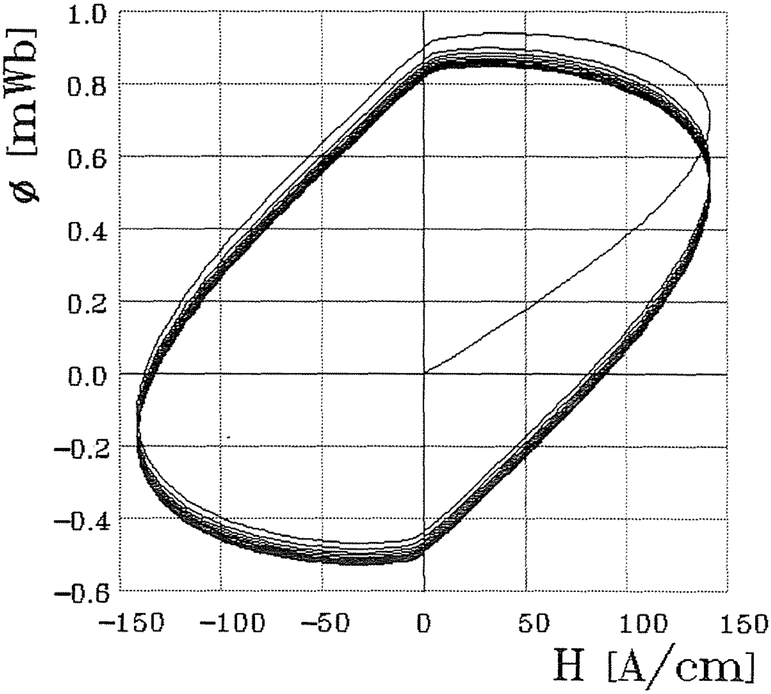

Fig. 2. Pseudo-transient magnetisation loop as computed without convergence acceleration

A magnetic field of maximum intensity $H_{0m} = 10^5 \cdot \sqrt{2}$ A/m and frequency $f = 500$ Hz is applied at the surface of the cylinder with radius $r_0 = 20$ mm.

In the eddy current simulation; the governing equations of the electromagnetic field are [2]:

$$\begin{aligned} &\text{rot } \mathbf{H} = \mathbf{J}, && \\ &\text{rot } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, && \mathbf{J} = \sigma \mathbf{E} \,, \\ &\text{div } \mathbf{B} = 0, && \mathbf{B} = \mu(H)\mathbf{H}, \end{aligned} \tag{8}$$

with the non-linear magnetic permeability $\mu(H)$ given in *Eq.* (7) . Applying a z- directed external magnetic field at the surface a magnetic field results parallel to the axis and azimuthal oriented current density in the material:

$$\mathbf{H} = \mathbf{k}H(r,t) \; ; \quad \mathbf{J} = \mathbf{u}_\varphi J(r,t) \; . \tag{9}$$

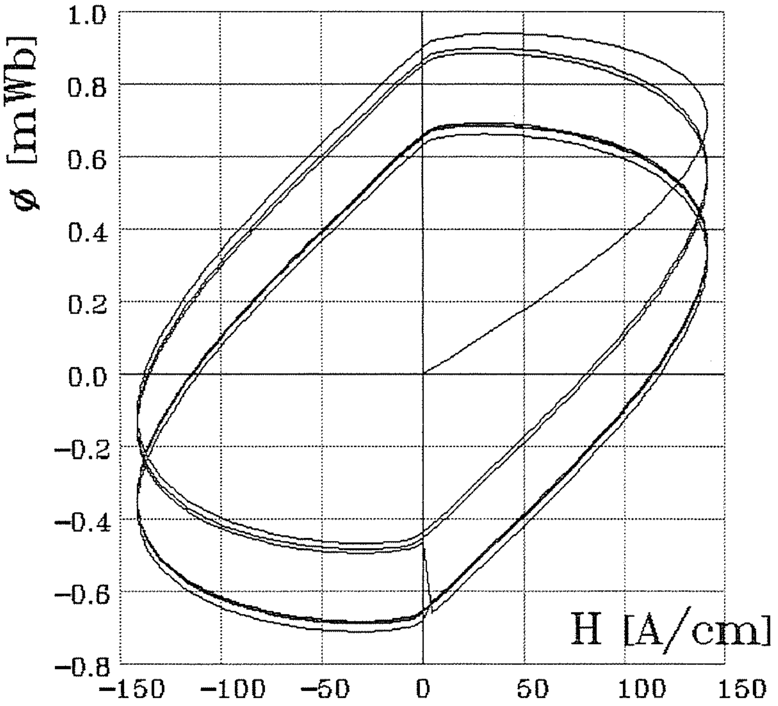

Fig. 3. Pseudo-transient magnetisation loop as computed with the simple convergence acceleration scheme

The equations of the electromagnetic field yield the following differential equation, written in cylindrical co-ordinates [3]:

$$\frac{\partial H}{\partial t} = \frac{\frac{\partial^2 H}{\partial r^2} + \frac{1}{r}\frac{\partial H}{\partial r}}{\sigma\left(\mu_0 + \frac{m_1 k_1}{1+k_1^2 H^2} + \frac{m_2 k_2}{1+k_2^2 H^2}\right)} \; ; \quad r \in [0, r_0] \, , \quad t \geq 0 \, , \qquad (10)$$

with the following boundary conditions:

$$\frac{\partial H}{\partial r}(0,t) = 0, \qquad t \geq 0 \, . \qquad (11)$$
$$H(r_0, t) = H_{0m} \sin 2\pi f t.$$

According to the solution in time domain obtained by the pseudo-transient method the initial condition is selected as:

$$H(r, 0) = 0 \; ; \quad r \in [0, r_0] \, . \qquad (12)$$

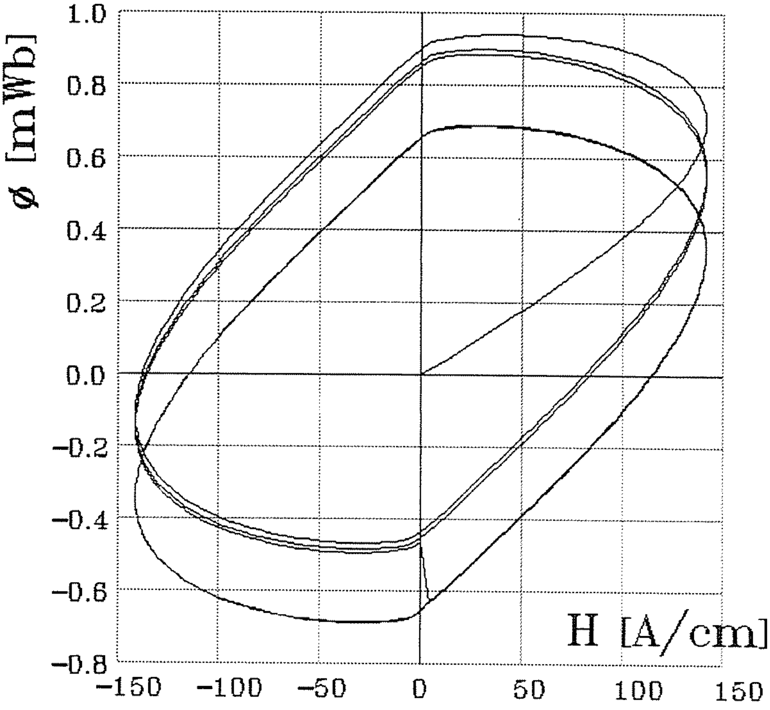

Fig. 4. Pseudo-transient magnetisation loop as computed with the refined convergence acceleration scheme

Formulating the solution by finite difference method, discretisation is performed over a grid selected as:

$$\begin{matrix} r = j\frac{r_0}{n} & ; & j = \overline{0, n} & ; & n = 200 \\ t = l\tau & ; & l \geq 0 & ; & \tau = 5 \cdot 10^{-7}\mathrm{s} \end{matrix} . \tag{13}$$

A three time-level discretisation scheme had been chosen, with the following approximations of the partial differentials:

$$\begin{aligned} \frac{\partial H}{\partial t}(r,t) &\approx \frac{H_j^{l+1} - H_j^l}{\tau} \\ \frac{\partial H}{\partial r}(r,t) &\approx \frac{1}{3} \cdot \frac{H_{j+1}^{l+1} - H_{j-1}^{l+1} + H_{j+1}^l - H_{j-1}^l + H_{j+1}^{l-1} - H_{j-1}^{l-1}}{h} , \\ \frac{\partial^2 H}{\partial r^2}(r,t) &\approx \frac{1}{3} \cdot \frac{H_{j+1}^{l+1} - H_{j-1}^{l+1} - 2H_j^{l+1} + H_{j+1}^l + H_{j-1}^l - 2H_j^l + H_{j+1}^{l-1} + H_{j-1}^{l-1} - 2H_j^{l-1}}{h^2} , \end{aligned} \tag{14}$$

where

$$H_j^l = H(jh, l\tau) \; ; \quad h = \frac{r_0}{n} \; . \tag{15}$$

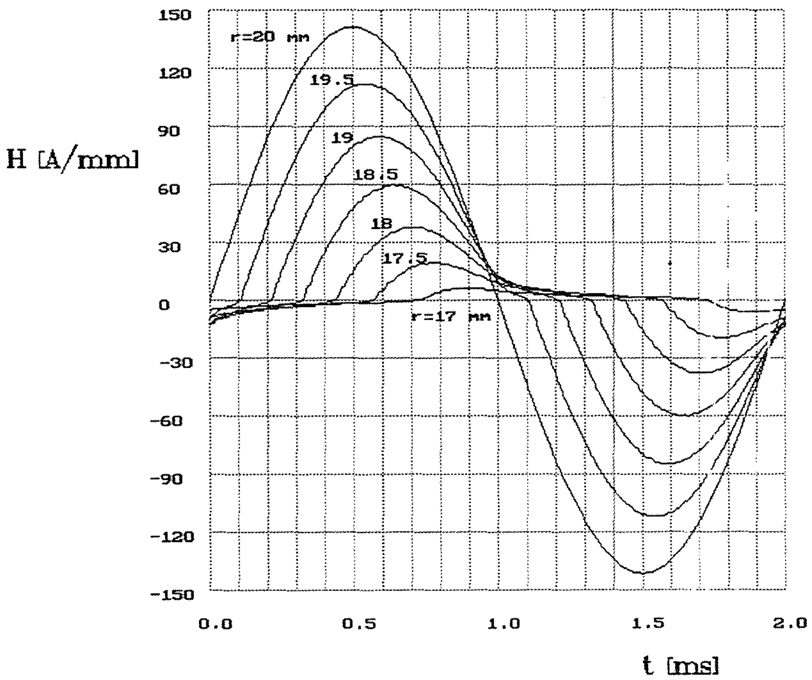

Fig. 5. Magnetic field intensity with respect to time and radius

The reason for selection of a three-level scheme is the presence of the variable coefficient in the denominator of the right side of *Eq.* (10). Approximating it with a central value in both spatial and time direction, the following solution can be applied:

$$\frac{\partial \mu}{\partial H}(r,t) \approx \mu_0 + \frac{m_1 k_1}{1+k_1^2(H_j^l)^2} + \frac{m_2 k_2}{1+k_2^2(H_j^l)^2} \,. \tag{16}$$

The system resulted can easily be solved by Gaussian elimination [4] at every time level. Due to the cylindrical symmetry, at the axis of the cylinder the condition

$$\lim_{r\to 0}\left(\frac{\partial^2 H}{\partial r^2} + \frac{1}{r}\frac{\partial H}{\partial r}\right) = 2\frac{\partial^2 H}{\partial r^2}(0,t) \tag{17}$$

has to be followed.

The condition of convergence, the criteria for stopping computations, is:

$$H(r,t+T) = H(t) \ ; \quad r \in [0, r_0] \,. \tag{18}$$

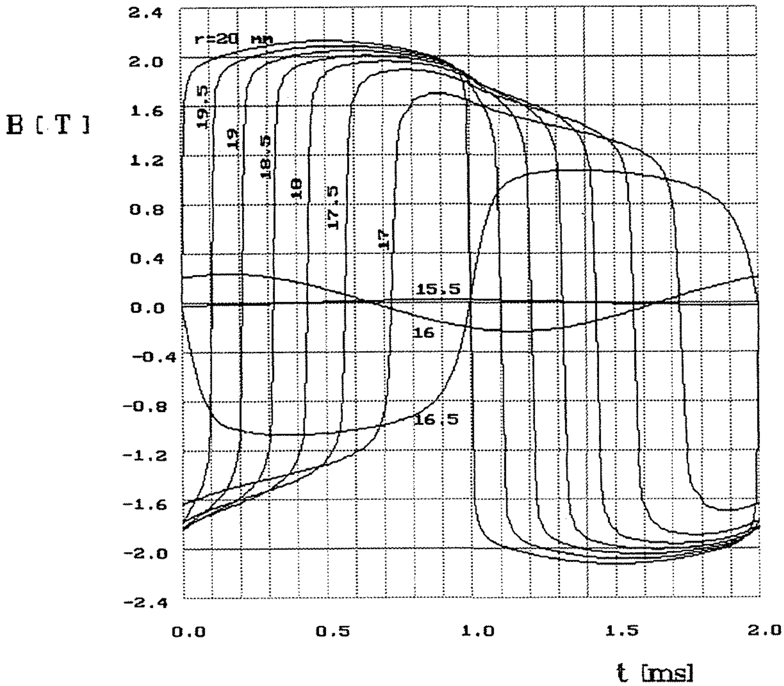

Fig. 6. Magnetic flux density with respect to time and radius

With the calculated values of the magnetic field intensity the corresponding flux densities can be computed and integrated over the cross-section of the cylinder, to find the total magnetic flux at each temporal grid level

$$\Phi(t) = \Phi^l \approx 2\pi h^2 \left(\frac{1}{8} B_0^l + \sum_{j=1}^{n-1} j B_j^l + \frac{n}{2} B_n^l \right) . \tag{19}$$

Providing the flux density functions are periodical with zero mean value, the total flux is expected to be a similar function of time as well. Plotting the total flux with respect to the field intensity at the surface of the cylinder, information can be obtained regarding the level of convergence achieved by the level of symmetry of the resulting loop with respect to the co-ordinate axes.

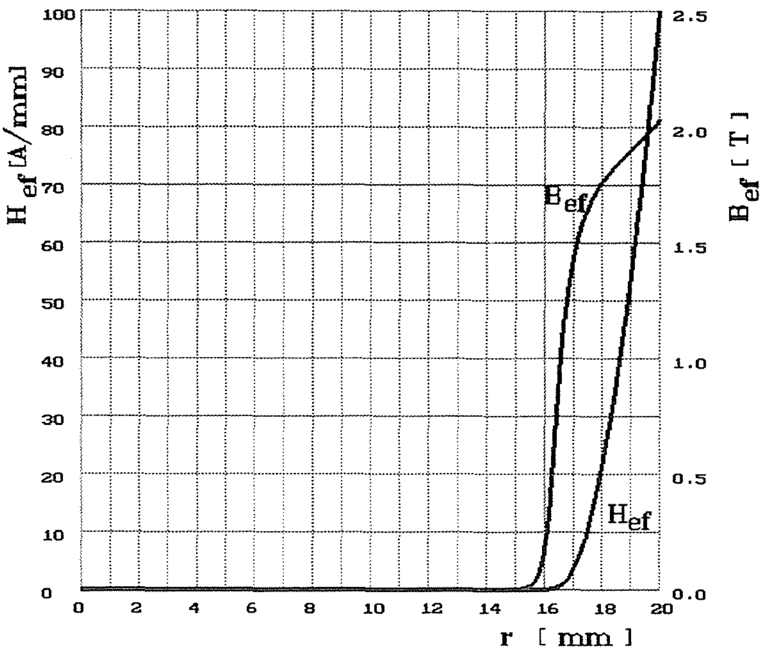

Fig. 7. Effective values of magnetic field intensity and flux density with respect to radius

4. Numerical Results

Computations were carried out for the example given above in three cases. First, without any convergence-accelerating procedure, ten periods of the source field lead to the result shown on *Fig.* 2 where the extremely low rate of convergence can be observed.

Then the simple accelerating procedure was applied after the third and following periods, just by subtracting the mean values of the field intensity functions computed for each period from the actual values at the end of the respective periods. In *Fig. 3* an oscillating process can be observed around the symmetrical position of the loop. The permanent solution could be reached after six seven periods, a serious improvement regarding the situation presented in *Fig.* 2.

Further improvement can be seen in *Fig. 4* obtained by the refined accelerating process, taking into account the decreasing tendency of the mean values as well. With this strategy the permanent solution can be reached as soon as the fourth period of the source field, while the track of the operating point follows in the fifth period the loop plotted during the fourth period.

The resulting wave-forms of the magnetic field intensity and flux density functions with respect to time obtained by means of the refined acceleration scheme can be seen on *Figs. 5* and *6*, for different depths in the cylinder. The symmetry of these wave-forms stated in *Eq.* (1) can be noticed as well as the fact that their mean value is zero. Thus one can conclude that a fairly sufficient convergence level has indeed been achieved. Although it is not represented in *Figs. 5* and *6*, due to scale incompatibility, the wave-forms at much smaller radius, closer to the axis, maintain the symmetry mentioned above. Computing the effective values of field intensity and flux density at different depths, and plotting the graph given in *Fig. 7*, the skin-effect occurring under the conditions given at the beginning of point 3 can be observed.

5. Conclusions

The method described in point 1 leads to substantial gain in convergence speed applying the pseudo-transient algorithms for solution of diffusion problems in time domain. Implementation of the method into the existing program source does not require much supplementary programming effort and does not affect the accuracy of the obtained results. Combined with an appropriate finite difference algorithm it provides a comfortable and fast code for simulation of electromagnetic field in both linear and non-linear media.

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