AUTOMATIC NUMERICAL ELEMENT GENERATION BY BOUNDARY-FITTED CURVILINEAR COORDINATE SYSTEM

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Abstract

A method for dividing a two-dimensional multi-connected region of a complex shape into a set of triangular elements is developed. A region of a complex shape in the physical plane is divided into some simply connected subregions, and each subregion is mapped onto a square region in the transformed plane. The inverse functions of the mapping are determined by the solution to elliptic partial differential equations with the Dirichlet boundary conditions. After the square region is divided into a set of finite elements, each element is inversely mapped onto the subregions by use of the functions. The finite element data for the global region are made of those for the divided subregions.

Keywords: boundary-fitted curvilinear coordinate system, finite element method, free mesh generator.

1. Introduction

Finite Element Method (FEM) has been widely used in all fields concerned with the numerical solutions of partial differential equations: Electromagnetics, Structure mechanics, Thermodynamics and so on. This is because we can easily deal with the region of a complex shape by means of FEM. However, we have to divide the region into a set of finite elements before we use a FEM code. Although this process has been performed manually with the help of the man-machine interface program, it costs us much time and labor to carry out this process. For this reason, a method for automatic numerical element generation has been desired.

The purpose of the present study is to develop the method for dividing two-dimensional region of an arbitrary shape into a set of triangular elements. In the next section, we explain the basic idea of the method for dividing a simply connected region into linear finite elements. The
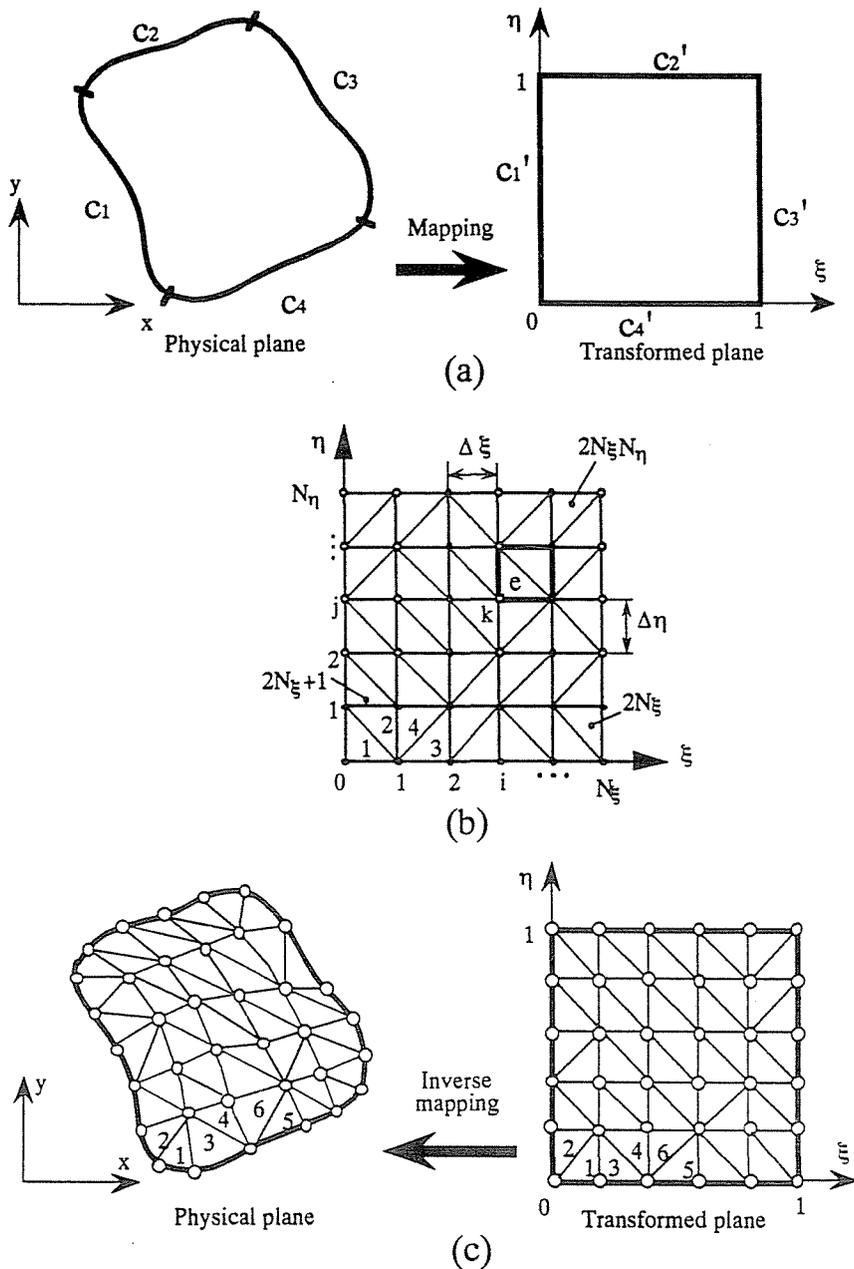
(a)

(b)

(c)

Fig. 1. Basic idea of element generation for a simply connected region
(a) Mapping a region of an arbitrary shape onto a square region
(b) Division of a square region into a set of triangular elements
(c) Inverse mapping of each element in the transformed plane
into the physical plane

treatment of a multiply connected region is presented in the third section. Conclusions are summarized in the final section.

2. Basic Idea of Element Generation for Simply Connected Region

In this section, let us consider the method for dividing a simply connected region into a set of triangular elements. The basic idea of this method is explained as follows.

First of all, a region of an arbitrary shape in xy-plane is mapped onto a square region in $\xi\eta$-plane (see *Fig. 1a*). The inverse functions of the mapping are determined by the solutions of the boundary value problem of the elliptic partial differential system. In the following, xy- and $\xi\eta$- plane are denoted by the physical and the transformed plane, respectively. Next, the square region in the transformed plane is systematically divided into a set of triangular elements (see *Fig. 1b*). Finally, by use of the inverse mapping functions, each element in the square region is inversely mapped onto the region in the physical plane (see *Fig. 1c*). Through the above process, element generation in the physical plane is completed.

To achieve the above process, the inverse mapping functions and a method for element generation in the square region are needed. We consider them in the following subsections 2.1 and 2.2. In subsection 2.3, each finite element in the transformed plane is inversely mapped onto the physical plane by use of the inverse mapping functions.

2.1 Inverse Mapping Functions

For the existence of the inverse mapping, this must be one-to-one. In other words, the inverse mapping functions, $x(\xi, \eta)$ and $y(\xi, \eta)$, must be monotonous with respect to ξ and η. In present study, the solutions to the boundary value problem of the elliptic partial differential equations, which satisfy the extremum principle (COURANT and HILBERT, 1962) are adopted as mapping functions, and the inverse mapping functions are assumed to be the solutions to the equations (THOMPSON et al., 1974, 1977, 1985)

$$\alpha \mathbf{r}_{\xi\xi} - 2\beta \mathbf{r}_{\xi\eta} + \gamma \mathbf{r}_{\eta\eta} + J^2[P(\xi, \eta)\mathbf{r}_\xi + Q(\xi, \eta)\mathbf{r}_\eta] = 0, \tag{1}$$

where

$$\mathbf{r}(\xi, \eta) = \begin{bmatrix} x(\xi, \eta) \\ y(\xi, \eta) \end{bmatrix}, \tag{2}$$

and α, β, γ and $\mathbf{J}$ are determined by

$$\alpha = \mathbf{r}_{\bar{\eta}} \cdot \mathbf{r}_{\eta}, \tag{3a}$$

$$\beta = \mathbf{r}_{\xi} \cdot \mathbf{r}_{\eta}, \tag{3b}$$

$$\gamma = \mathbf{r}_{\xi} \cdot \mathbf{r}_{\xi}, \tag{3c}$$

$$\mathbf{J} = \frac{\partial(x, y)}{\partial(\xi, \eta)}. \tag{3d}$$

In addition, the functions $P(\xi, \eta)$ and $Q(\xi, \eta)$ are assumed as

$$
P(\xi, \eta) = -\sum_{i=1}^{n} a_i^{(P)} \mathrm{sgn}\left(\xi - \xi_i^{(P)}\right) \exp\left(-c_i^{(P)}|\xi - \xi_i^{(P)}|\right)
$$
$$
-\sum_{j=1}^{m} b_j^{(P)} \mathrm{sgn}\left(\xi - \bar{\xi}_j^{(P)}\right) \exp\left\{-d_j^{(P)}\left[(\xi - \bar{\xi}_j^{(P)})^2 + (\eta - \bar{\eta}_j^{(P)})^2\right]^{1/2}\right\},
\tag{4a}
$$

$$
Q(\xi, \eta) = -\sum_{i=1}^{n} a_i^{(Q)} \mathrm{sgn}\left(\eta - \eta_i^{(Q)}\right) \exp\left(-c_i^{(Q)}|\eta - \eta_i^{(Q)}|\right)
$$
$$
-\sum_{j=1}^{m} b_j^{(Q)} \mathrm{sgn}\left(\eta - \bar{\eta}_j^{(Q)}\right) \exp\left\{-d_j^{(Q)}\left[(\xi - \bar{\xi}_j^{(Q)})^2 + (\eta - \bar{\eta}_j^{(Q)})^2\right]^{1/2}\right\},
\tag{4b}
$$

where $a_i^{(P)}$, $a_i^{(Q)}$, $b_j^{(P)}$, $b_j^{(Q)}$, $c_i^{(P)}$, $c_i^{(Q)}$, $d_j^{(P)}$ and $d_j^{(Q)}$ are all positive constants, and $\xi_i^{(P)}$, $\bar{\xi}_j^{(P)}$, $\bar{\xi}_j^{(Q)}$, $\eta_i^{(Q)}$, $\bar{\eta}_j^{(Q)}$ and $\bar{\eta}_j^{(P)}$ are constant values within interval $[0, 1]$ (THOMPSON et al., 1977, 1985). The first term in *Eq.* (4a) attracts the η-curve toward the curve $\xi = \xi_i^{(P)}$, while the se4cond term in *Eq.* (4a) attracts the η-curve toward the point $\mathbf{r}\left(\bar{\xi}_j^{(P)}, \bar{\eta}_j^{(P)}\right)$. The terms in *Eq.* (4b) have similar effects to the ξ-curve.

The boundary conditions of *Eq.* (1) are chosen such that the curves c_1–c_4 in the physical plane are mapped onto the straight lines c_1'–c_4' in the transformed plane, respectively (see *Fig. 1(a)*). By solving numerically *Eq.* (1) under the above Dirichlet boundary conditions, we can obtain the inverse mapping functions, $x(\xi, \eta)$ and $y(\xi, \eta)$.

2.2 Element Generation in the Transformed Plane

Let us divide the square region in the transformed plane into $N_\xi \times N_\eta$ small rectangles as shown in *Fig. 1(b)*. By inserting a diagonal line to each small
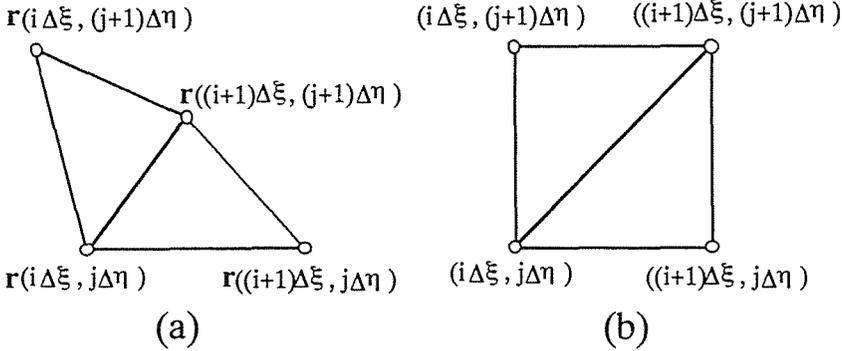
Fig. 2. A way to draw diagonal line
(*a*) Physical plane (*b*) Transformed plane

rectangle, the square region is divided into a set of triangular elements. Each rectangle has two diagonal lines, and one of them to be inserted depends on their length in the physical plane. The shorter diagonal line in the physical plane is adopted to divide a quadrilateral (see *Fig. 2a*) and its image in the transformed plane is used to separate a rectangle into two triangular elements (see *Fig. 2b*). By using the above method for inserting a diagonal line, we can avoid the appearance of deformed triangular elements to some extent.

In using a FEM code, the following data need to be prepared as input data.

  (i) total number of nodes N_{node}
  (ii) total number of elements N_{elem}
  (iii) coordinates of each node $\mathbf{r}_k \quad (k = 1, 2, \ldots, N_{\text{node}})$
  (iv) global node numbers of the g^*-th $\sigma(e, g^*) \quad (e = 1, 2, \ldots, N_{\text{elem}})$
        local node in the e-th element

In the following, we call the above data finite element data or FEM data. Let us generate the FEM data of the square region divided by the above method.

First, the total number of nodes and elements are easily evaluated by

$$N_{\text{node}} = (N_\xi + 1)(N_\eta + 1), \tag{5}$$
$$N_{\text{elem}} = 2 N_\xi N_\eta. \tag{6}$$

Next, we assign a global node number to the grid point as shown in *Fig. 3*. Here, the grid point (i, j) denotes the point $(i\Delta\xi, j\Delta\eta)$ in the transformed plane. The global node number k of the grid point (i, j) is written as

$$k = (N_\xi + 1)j + (i + 1). \tag{7}$$

By solving *Eq.* (7) with respect to i and j, we find

$$i = \text{Mod}(k - 1, N_\xi + 1), \tag{8a}$$

$$j = (k - 1)/(N_\xi + 1), \tag{8b}$$

where the symbol "/" means the division operator of integer. *Eqs.* (8a) and (8b) indicate that the (ξ, η) coordinates of each node can be determined from its global node number k.
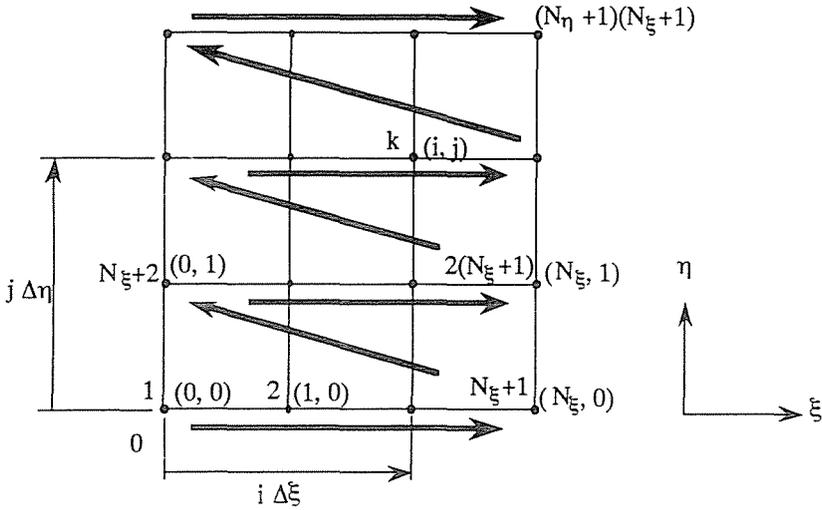

Fig. 3. The order of global node number

Finally, we assign the element number e to each element as shown in *Fig. 1b*. Even and odd number is assigned to upper and lower triangular elements, respectively. In order to give the function $\sigma(e, g^*)$, we must take account of four types of elements as shown in *Fig. 4*.

The grid point (i, j) is written in terms of the element number e as follows:

$$i = \text{Mod}(e/2 - 1, N_\xi), \tag{9a}$$

$$j = (e/2 - 1)/N_\xi, \tag{9b}$$

(for $e = 2, 4, 6, \ldots, N_{\text{elem}}$), and

$$i = \text{Mod}[(e + 1)/2 - 1, N_\xi], \tag{10a}$$

$$j = [(e + 1)/2 - 1]/N_\xi \tag{10b}$$

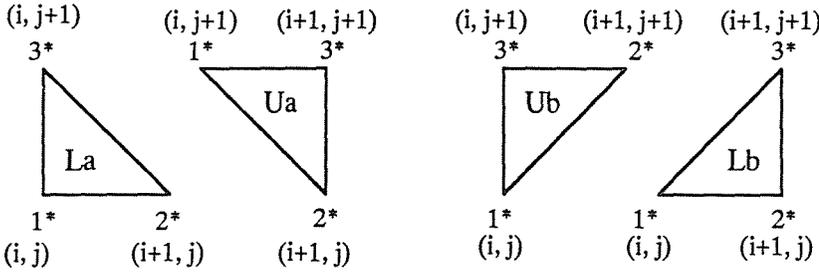(for $e = 1, 3, 5, \ldots, N_{\text{elem}} - 1$).

Fig. 4. Four types of elements for giving function $\sigma(e, g^*)$

By substituting *Eq.* (9) or (10) into *Eq.* (7), function $\sigma(e, g^*)$ is evaluated as follows.

For type La:
$$\sigma(e, 1^*) = (N_\xi + 1)\{[(e+1)/2 - 1]/N_\xi\} + \text{Mod}[(e+1)/2 - 1, N_\xi] + 1, \quad (11a)$$
$$\sigma(e, 2^*) = (N_\xi + 1)\{[(e+1)/2 - 1]/N_\xi\} + \text{Mod}[(e+1)/2 - 1, N_\xi] + 2, \quad (11b)$$
$$\sigma(e, 3^*) = (N_\xi + 1)\{[(e+1)/2 - 1]/N_\xi + 1\} + \text{Mod}[(e+1)/2 - 1, N_\xi] + 1. \quad (11c)$$

For type Ua:
$$\sigma(e, 1^*) = (N_\xi + 1)[(e/2 - 1)/N_\xi + 1] + \text{Mod}(e/2 - 1, N_\xi) + 1, \quad (12a)$$
$$\sigma(e, 2^*) = (N_\xi + 1)[(e/2 - 1)/N_\xi] + \text{Mod}(e/2 - 1, N_\xi) + 2, \quad (12b)$$
$$\sigma(e, 3^*) = (N_\xi + 1)[(e/2 - 1)/N_\xi + 1] + \text{Mod}(e/2 - 1, N_\xi) + 2. \quad (12c)$$

For type Lb:
$$\sigma(e, 1^*) = (N_\xi + 1)\{[(e+1)/2 - 1]/N_\xi\} + \text{Mod}[(e+1)/2 - 1, N_\xi] + 1, \quad (13a)$$
$$\sigma(e, 2^*) = (N_\xi + 1)\{[(e+1)/2 - 1]/N_\xi + 1\} + \text{Mod}[(e+1)/2 - 1, N_\xi] + 2, \quad (13b)$$
$$\sigma(e, 3^*) = (N_\xi + 1)\{[(e+1)/2 - 1]/N_\xi + 1\} + \text{Mod}[(e+1)/2 - 1, N_\xi] + 1. \quad (13c)$$

For type Ub:
$$\sigma(e, 1^*) = (N_\xi + 1)[(e/2 - 1)/N_\xi] + \text{Mod}(e/2 - 1, N_\xi) + 1, \quad (14a)$$
$$\sigma(e, 2^*) = (N_\xi + 1)[(e/2 - 1)/N_\xi] + \text{Mod}(e/2 - 1, N_\xi) + 2, \quad (14b)$$
$$\sigma(e, 3^*) = (N_\xi + 1)[(e/2 - 1)/N_\xi + 1] + \text{Mod}(e/2 - 1, N_\xi) + 2. \quad (14c)$$

2.3 Inverse Mapping of Elements

Now that element generation in the square region has been completed, we can map each element in the transformed plane onto the physical plane. In mapping elements onto the physical plane, the total number of nodes and elements and function $\sigma(e, g^*)$ do not undergo a change. Therefore, we have only to determine the (x, y) coordinate of the nodes. As mentioned in the previous subsection, the (ξ, η) coordinates of the node can easily be determined from its global node number k. Hence, the (x, y) coordinates of the node are obtained by the inverse mapping of the (ξ, η) coordinate.
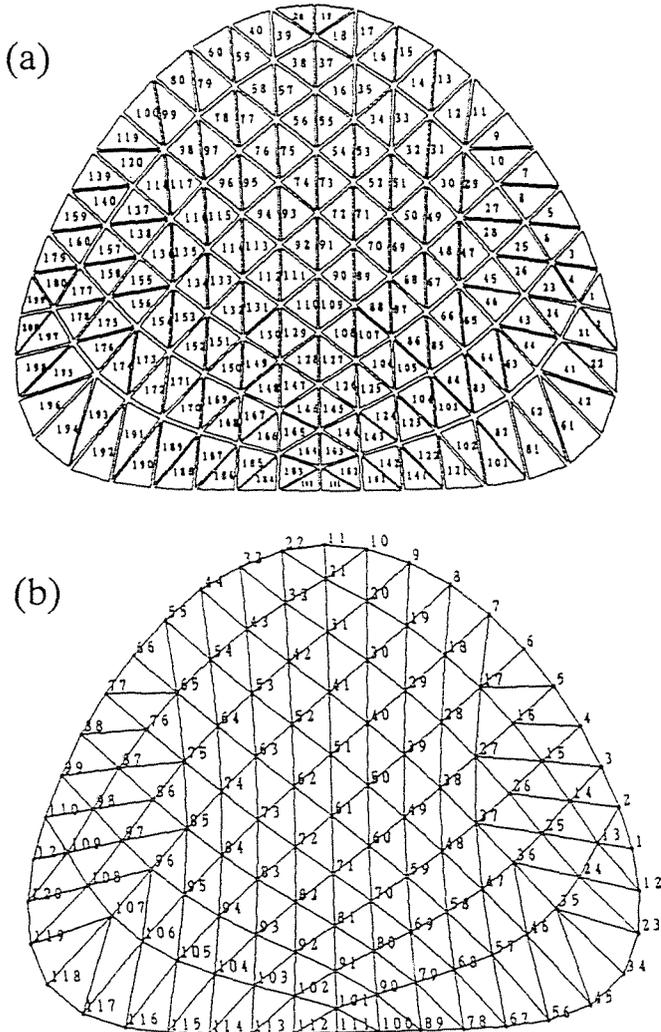
Fig. 5. A simply connected region divided into a set of triangular elements
($N_\xi = 10$, $N_\eta = 10$, $N_{\text{node}} = 121$, $N_{\text{elem}} = 200$)
(a) Element number   (b) Node number

The (x, y) coordinate of the k-th node is given by

$$\mathbf{r}_k = \mathbf{r}(\Delta\xi \text{Mod}(k - 1, N_\xi + 1), \Delta\eta[(k - 1)/(N_\xi + 1)]). \qquad (15)$$

Thus, all the finite element data in the simply connected region have been
determined by use of the above process.

Fig. 5 shows the regions divided into a set of triangular elements. Each number in *Figs. 5a* and *b* shows the element and the node number, respectively.

3. Dealing with Multi-connected Region

Generally, a multi-connected region can be divided into some simply connected subregions which do not overlap with each other (see *Fig. 6*).

Generation of FEM data in each subregion is easily performed by using the method explained in the previous section. In this section, we consider the method for making FEM data for the global region from those for the subregions.

We assume, at first, two simply connected subregions, Ω_1 and Ω_2, share their boundaries partially (see *Fig. 6b*). In addition, FEM data have been prepared as follows:
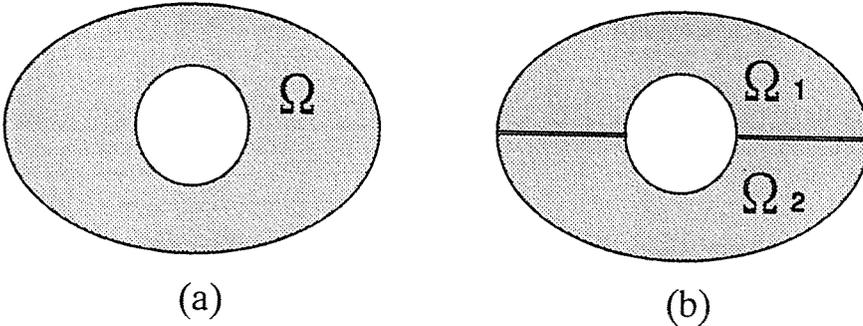

(a) (b)

Fig. 6. Basic idea to deal with multi-connected region
(*a*) A doubly connected rgion (*b*) A pair of simply connected region

For Ω_1,

$$N_{\text{node}}^{(1)}, \ N_{\text{elem}}^{(1)}, \ \mathbf{r}_k^{(1)} \ (k = 1, 2, \ldots, N_{\text{node}}^{(1)}),$$

$$\sigma^{(1)}(e, g^*) \ (e = 1, 2, \ldots, N_{\text{elem}}^{(1)}).$$

For Ω_2,

$$N_{\text{node}}^{(2)}, \ N_{\text{elem}}^{(2)}, \ \mathbf{r}_k^{(2)} \ (k = 1, 2, \ldots, N_{\text{node}}^{(2)}),$$

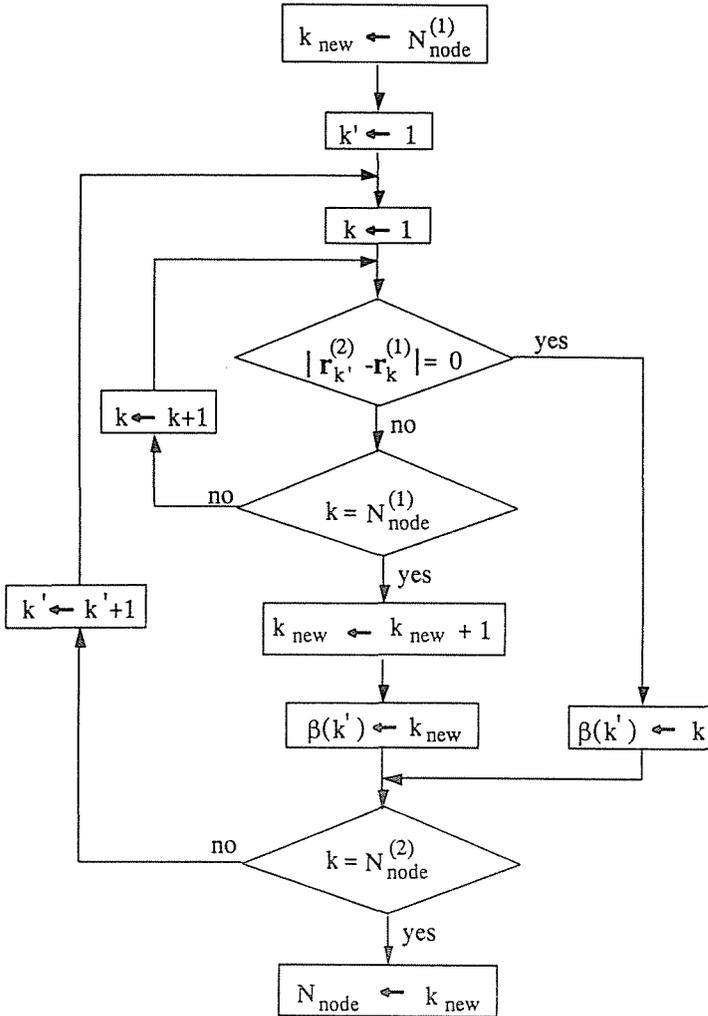$$\sigma^{(2)}(e, g^*) \ (e = 1, 2, \ldots, N_{\text{elem}}^{(2)}),$$

$$k_{\text{new}} \leftarrow N_{\text{node}}^{(1)}$$

$$k' \leftarrow 1$$

$$k \leftarrow 1$$

$$|\mathbf{r}_{k'}^{(2)} - \mathbf{r}_k^{(1)}| = 0 \quad \text{yes}$$

no

$$k \leftarrow k+1$$

$$k = N_{\text{node}}^{(1)} \quad \text{no}$$

yes

$$k' \leftarrow k'+1$$

$$k_{\text{new}} \leftarrow k_{\text{new}} + 1$$

$$\beta(k') \leftarrow k_{\text{new}}$$

$$\beta(k') \leftarrow k$$

$$k = N_{\text{node}}^{(2)} \quad \text{no}$$

yes

$$N_{\text{node}} \leftarrow k_{\text{new}}$$

Fig. 7. Flowchart to determine function $\beta(k)$ and total number of nodes N_{node} in the global region

where superscripts in parentheses denote the subregion number. Let us evaluate the FEM data in global region Ω:

$$N_{\text{node}}, \ N_{\text{elem}}, \ \mathbf{r}_k \ (k = 1, 2, \dots, N_{\text{node}}),$$
$$\sigma(e, g^*) \ (e = 1, 2, \dots, N_{\text{elem}}).$$
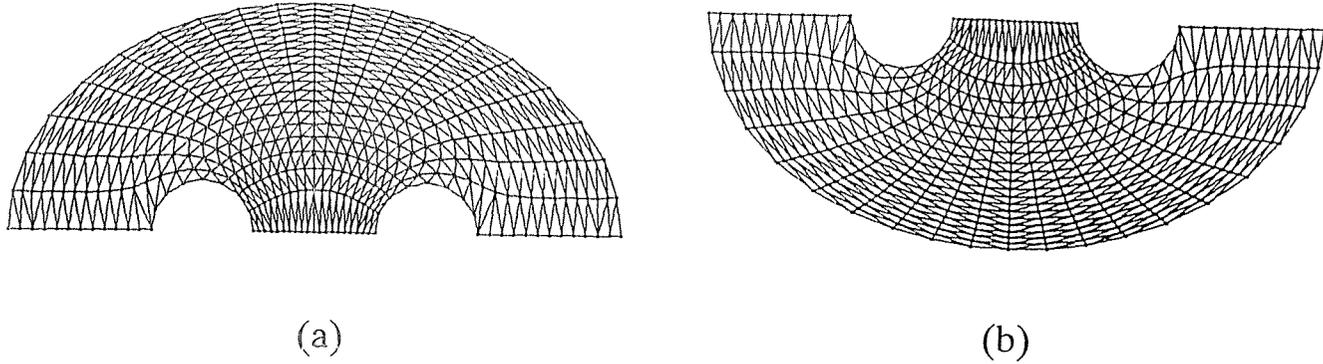
(a)

(b)

Fig. 8. A pair of simply connected regions divided into a set of first order triangular elements

(a) A simply connected region Ω_1 ($N_\xi = 20$, $N_\eta = 20$, $N_{\text{node}}^{(1)} = 441$, $N_{\text{elem}}^{(1)} = 800$)

(b) A simply connected region Ω_2 ($N_\xi = 20$, $N_\eta = 20$, $N_{\text{node}}^{(2)} = 441$, $N_{\text{elem}}^{(2)} = 800$)

First of all, the total number of elements in the global region Ω is easily obtained as follows:

$$N_{\text{elem}} = N_{\text{elem}}^{(1)} + N_{\text{elem}}^{(2)}. \tag{16}$$

Next, the node number in the global region is assigned in accordance with the following two rules:

(i) In the case that a node is contained in Ω_1, the same number as its node number in Ω_1 is assigned as the node number in the global region Ω.

(ii) In the case that a node is contained only in Ω_2 and is the k-th node of Ω_2, $\beta(k)$ is assigned to it as the node number in the global region Ω.

The function $\beta(k)$ is calculated by use of the algorithm shown in *Fig. 7*. At the same time, this algorithm determines the total number of nodes N_{node}.
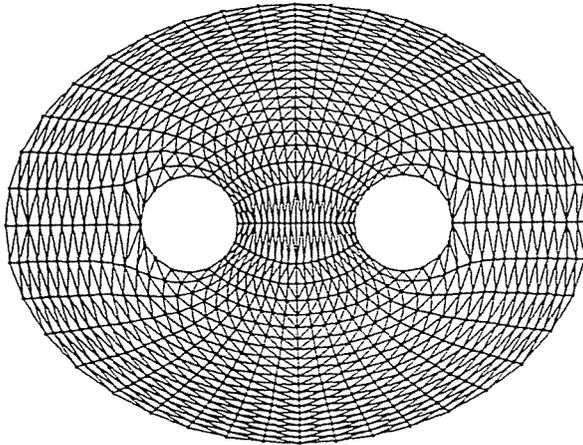

Fig. 9. Division of a triple connected region into a set of first order triangular elements ($N_{\text{node}} = 873$, $N_{\text{elem}} = 1600$)

By use of the function $\beta(k)$, the coordinates of the nodes are written as

$$r_k = r_k^{(1)} \ (1 \le k \le N_{node}^{(1)}), \tag{17a}$$

$$r_{\beta(k)} = r_k^{(2)} \ (1 \le k \le N_{node}^{(2)}), \tag{17b}$$

Finally, the element number in the global region is assigned in accordance with the following two rules:

(i) In the case that an element belongs to Ω_1, the same number as its element number in Ω_1 is assigned as the element number in the global region Ω.

(ii) In the case that an element belongs to Ω_2, the sum of $N_{elem}^{(1)}$ and the element number in Ω_2 is assigned as the element number in the global region Ω.

By using the above numbering, the function $\sigma(e, g^*)$ is given as

$$\sigma(e, g^*) = \begin{cases} \sigma^{(1)}(e, g^*) & (1 \leq e \leq N_{\text{elem}}^{(1)} \\ \beta\left(\sigma^{(2)}(e - N_{\text{elem}}^{(1)}, g^*)\right) & (1 + N_{elem}^{(1)} \leq e \leq N_{\text{elem}}). \end{cases} \quad (18)$$
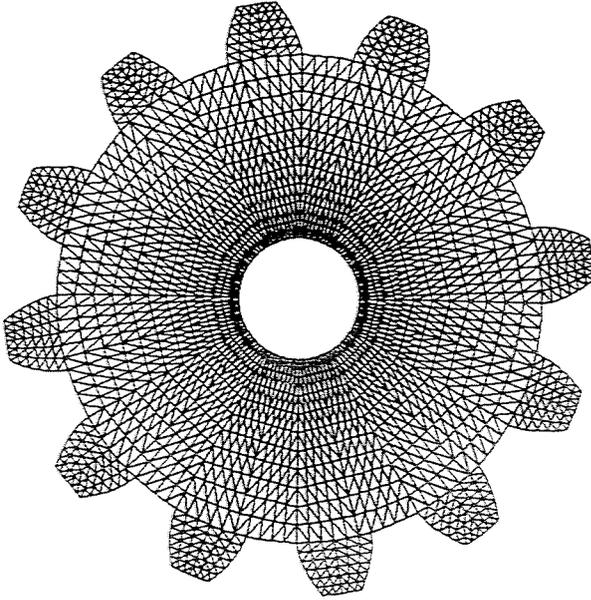

Fig. 10. A gear wheel

Thus, all the finite element data in the global region Ω have been made from the FEM data for the subregions. *Figs. 8* and *9* show the process of the method explained in present section. *Figs. 8a* and *b* show two simply connected regions divided into a set of elements. From these two regions, a triply connected region shown in *Fig. 9* is made.

By repeating the above method, more than three subregions can be assembled. A gear wheel shown in *Fig. 10* is made of 24 subregions. The
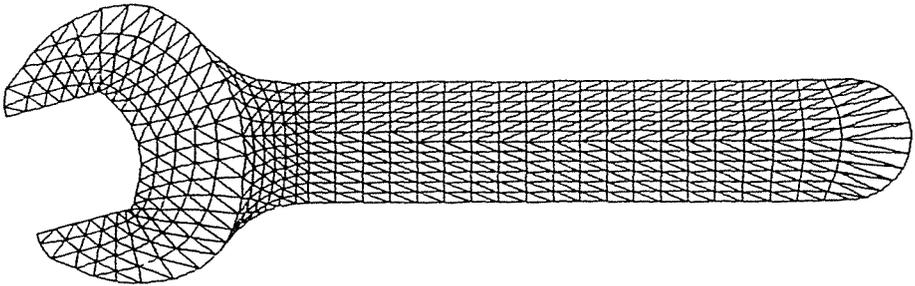
Fig. 11. A spanner

method in the present section is also effective to divide the simply connected region of an extremely complex shape such as a spanner shown in *Fig. 11.*

4. Conclusions

A method for automatic numerical element generation for two dimensional multi-connected region of a complex shape has been developed. This method is summarized as follows.

(i)   The region of a complex shape in the physical plane is divided into some simply connected subregions, and each subregion is mapped onto the square region in the transformed plane.

(ii) Each square region is divided into a set of finite elements in the transformed plane.

(iii) By use of the inverse mapping functions, each element in the square region is inversely mapped onto the subregion in the physical plane.

(iv) The finite element data for the global region are composed from those for the subregions.

Present effort is confined to linear elements in the interest of computer memory, but all techniques are immediately extendable to higher order elements. Hereafter, we should expand this method to cope with three-dimensional regions, and should develop a method to generate three-dimensional boundary element data.

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