PERIODICA POLYTECHNICA SER. EL. ENG. VOL. 38, NO. 2, PP. 71–77 (1994)

ON THE COMPUTATION OF TM AND TE MODE ELECTROMAGNETIC FIELDS

István VÁGÓ

Attila u. 23
1013 Budapest,
Hungary

Received: June 3, 1994

Abstract

The TM and TE mode fields can be derived either from the magnetic vector potential **A** or from the electric vector potential **F**. **A** and **F** will be defined with the aid of two scalar quantities each. The relationship between the vector potentials **A** and **F** describing the same electromagnetic field will be stated. The sum of the TM and TE mode fields is the general solution of the Maxwell equations.

Keywords: electromagnetic field, TM and TE modes, magnetic vector potential, electric vector potential.

Introduction

The solution of electromagnetic field problems is known to be obtainable from a magnetic vector potential **A** and an electric vector potential **F** both of which have a longitudinal component only. Then the field derived from **A** is of TM mode while that obtained from **F** is of TE mode.

It is shown in what follows how both the TM and the TE mode fields can be derived from either the magnetic potential **A** or from the electric potential **F**. In the discussion the vector potentials **A** and **F** will be defined with the aid of two scalar quantities each. The relationship between the vector potentials **A** and **F** describing the same electromagnetic field will be stated.

Computation of the Electromagnetic Field from a Magnetic and an Electric Vector Potential

The magnetic vector potential **A** is defined by the relationship

$$\mathbf{B} = \operatorname{curl} \mathbf{A} \ , \tag{1}$$

where **B** is the vector of magnetic flux density. **A** can be shown to satisfy the wave equation

$$\Delta \mathbf{A} - \mu\sigma \frac{\partial \mathbf{A}}{\partial t} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = 0 , \tag{2}$$

where μ is the permeability, ε the permittivity and σ is the conductivity of the medium.

The relationship between the electric field intensity **E** and the vector potential **A** can be written as

$$\mu\sigma \mathbf{E} + \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} = \text{curl curl } \mathbf{A} , \tag{3}$$

hence making use of the above wave equation:

$$\mu\sigma \mathbf{E} + \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} = -\left(\mu\sigma \frac{\partial \mathbf{A}}{\partial t} + \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) + \text{grad div } \mathbf{A} . \tag{4}$$

In current-free regions, **E** can be explicitly written with the aid of **A** as

$$\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\mu\varepsilon} \text{curl curl } \mathbf{A} . \tag{5}$$

The electromagnetic field can also be derived from the electric vector potential **F** defined by the relationship

$$\mathbf{D} = \text{curl } \mathbf{F} , \tag{6}$$

where **D** is the vector of displacement. **F** also satisfies the wave equation:

$$\Delta \mathbf{F} - \mu\sigma \frac{\partial \mathbf{F}}{\partial t} - \mu\varepsilon \frac{\partial^2 \mathbf{F}}{\partial t^2} = 0 . \tag{7}$$

The expression

$$\frac{\partial \mathbf{H}}{\partial t} = -\frac{1}{\mu\varepsilon} \text{curl curl } \mathbf{F} \tag{8}$$

yields the magnetic field intensity **H**.

If the only nonzero component of the vector potential is one pointing in a special direction (in case of wave propagation, the direction of the propagation), then the vector potential is of logitudinal direction and it can be written as

$$\mathbf{A} = A_l \mathbf{e}_l \tag{9}$$

or

$$\mathbf{F} = F_l \mathbf{e}_l, \tag{10}$$

where $\mathbf{e}_l$ is the unit vector in the longitudinal direction. In the electromagnetic field derived from such an $\mathbf{A}$, the magnetic field is of a direction perpendicular to it, i.e. of transversal direction. The solution is of TM mode:

$$\mathbf{B}_{\mathrm{TM}} = \text{curl } (A_l \mathbf{e}_l) = \text{grad } A_l \times \mathbf{e}_l. \tag{11}$$

The electric field computed from $\mathbf{F}$ defined in (10) is of transversal direction. A solution of TE mode can be hence derived:

$$\mathbf{D}_{\mathrm{TE}} = \text{grad } F_l \times \mathbf{e}_l. \tag{12}$$

The superposition of the TM and TE mode fields is the general solution, i.e. the general solution can be derived from two scalar quantities: from A_l and F_l.

In the following, it will be examined how a TE mode solution can be obtained from the vector potential $\mathbf{A}$ and a TM mode one from $\mathbf{F}$. To this end, the vector potential $\mathbf{A}$ is decomposed into a transversal part $\mathbf{A}_\tau$ and a longitudinal part $\mathbf{A}_l$;

$$\mathbf{A} = \mathbf{A}_\tau + \mathbf{A}_l. \tag{13}$$

According to (11), a TM mode solution is obtained from $\mathbf{A}_l \mathbf{A}_\tau$ will be selected to yield a TE mode field. In view of (4), this is ensured if

$$\text{div } \mathbf{A}_\tau = 0. \tag{14}$$

Such a choice constitutes no limitation since, according to (1) and (5), the same field can be obtained with different choices of div A_τ. So, in view of (4):

$$\mathbf{E}_{\mathrm{TE}} + \frac{\varepsilon}{\sigma}\frac{\partial \mathbf{E}_{\mathrm{TE}}}{\partial t} = -\frac{\partial \mathbf{A}_\tau}{\partial t} - \frac{\varepsilon}{\sigma}\frac{\partial^2 \mathbf{A}_\tau}{\partial t^2} \tag{15}$$

Since relationships between quantities varying in time are investigated, the part constant in time is zero. $\mathbf{A}_\tau$ is of transversal direction, so its time-derivative is transversal, too, i.e. the field obtained is of TE mode.

According to (14)

$$\mathbf{A}_\tau = \text{curl } \mathbf{V} \tag{16}$$

can be written. This is ensured in view of (16), if $\mathbf{V}$ is longitudinal, i.e.

$$\mathbf{V} = V \mathbf{e}_l. \tag{17}$$

Indeed, in this case:

$$\mathbf{A}_\tau = \text{curl } (V\mathbf{e}_l) = \text{grad } V \times \mathbf{e}_l. \tag{18}$$

(17) imposes no limitation on generality, V can be obtained by integration from (18) if $\mathbf{A}_\tau$ is known and it satisfies (14).

According to (18), the TE mode field can be obtained from the scalar quantity V. The magnetic flux density is given by

$$\mathbf{B}_{\text{TE}} = \text{curl curl } (V\mathbf{e}_l) = \text{curl (grad } V \times \mathbf{e}_l) \tag{19}$$

The above discussion justifies the following statement. The vector potential can be decomposed into two parts without limiting the general case:

$$\mathbf{A} = \text{grad } V \times \mathbf{e}_l + A_l\mathbf{e}_l \,. \tag{20}$$

The field yielded by the first term on the right-hand side is of TE mode, the one derived from the second term is of TM mode. As a result, the entire field can be obtained from two scalar quantities: V and A_l.

Similarly, it will be shown that $\mathbf{F}$ may yield not only TE but also TM mode fields. To this end $\mathbf{F}$ is written as the sum of a transversal $\mathbf{F}_\tau$ and a longitudinal $\mathbf{F}_l$:

$$\mathbf{F} = \mathbf{F}_\tau + \mathbf{F}_l \,. \tag{21}$$

The TE mode solution can be obtained from $\mathbf{F}_l$. A TM mode field can be derived from $\mathbf{F}_\tau$ if

$$\text{div } \mathbf{F}_\tau = 0. \tag{22}$$

Then:

$$\mathbf{H}_{\text{TM}} = \frac{\sigma}{\varepsilon}\mathbf{F}_\tau + \frac{\partial \mathbf{F}_\tau}{\partial t}. \tag{23}$$

In view of (22)

$$\mathbf{F}_\tau = \text{curl } \mathbf{W} \tag{24}$$

can be written where

$$\mathbf{W} = W\mathbf{e}_l. \tag{25}$$

So

$$\mathbf{F} = \text{grad } W \times \mathbf{e}_l + F_l\mathbf{e}_l, \tag{26}$$

i.e. $\mathbf{F}$ can be defined with the aid of two scalar quantities: W and F_lA. TM mode solution can be obtained from W and a TE mode one from F_l.

Conditions of Two Solutions Coinciding

It will now be investigated what relationship should be maintained between **A** and **F** so that both yield the same electromagnetic field.

The TE mode solution obtained from the two potentials is identical if

$$\frac{1}{\varepsilon}F_l = -\frac{\partial V}{\partial t} \tag{27}$$

and the TM mode one if

$$\frac{1}{\mu}A_l = \frac{\partial W}{\partial t} \,. \tag{28}$$

The expression of the TE mode electric field intensity is, according to (15),(16),(17):

$$\mathbf{E}_{\mathrm{TE}} = -\frac{\partial}{\partial t}\mathrm{grad}\ V \times \mathbf{e}_l = -\mathrm{grad}\frac{\partial V}{\partial t} \times \mathbf{e}_l \tag{29}$$

and, according to (6) and (26):

$$\mathbf{E}_{\mathrm{TE}} = \frac{1}{\varepsilon}\mathrm{grad}F_l \times \mathbf{e}_l \,, \tag{30}$$

i.e. the satisfaction of (27) implies the equality of (29) and (30). Similarly, the expression of the magnetic field intensity is, according to (19):

$$\mathbf{H}_{\mathrm{TE}} = \frac{1}{\mu}\ \mathrm{curl}\ (\mathrm{grad}\ V \times \mathbf{e}_l) \tag{31}$$

and, according to (8) and (26):

$$\frac{\partial \mathbf{H}_{\mathrm{TE}}}{\partial t} = -\frac{1}{\mu\varepsilon}\mathrm{curl}\ (\mathrm{grad}\ F_l \times \mathbf{e}_l) \,. \tag{32}$$

From (31) follows that

$$\frac{\partial \mathbf{H}_{\mathrm{TE}}}{\partial t} = \frac{1}{\mu}\mathrm{curl}\ (\mathrm{grad}\ \frac{\partial V}{\partial t} \times \mathbf{e}_l) \,, \tag{33}$$

i.e. (32) and (33) are equal if (27) is satisfied.

It can be similarly verified that the satisfaction of (28) implies the equality of the TM mode fields derived from A_l and from W.

In view of (9) and (10), A_l and F_l satisfy the scalar wave equation, i.e.

$$\Delta A_l - \mu\sigma \frac{\partial A_l}{\partial t} - \mu\varepsilon \frac{\partial^2 A_l}{\partial t^2} = 0 \ , \tag{34}$$

$$\Delta F_l - \mu\sigma \frac{\partial F_l}{\partial t} - \mu\varepsilon \frac{\partial^2 F_l}{\partial t^2} = 0 \ . \tag{35}$$

So, it follows from (27) and (28) that V and W satisfy the wave equations

$$\Delta V - \mu\sigma \frac{\partial V}{\partial t} - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = 0 \tag{36}$$

and

$$\Delta W - \mu\sigma \frac{\partial W}{\partial t} - \mu\varepsilon \frac{\partial^2 W}{\partial t^2} = 0 \ . \tag{37}$$

So, if the electromagnetic field is derived from V and A or W and F_1, the calculations invariably involve functions satisfying the wave equation.

Functionals of the Potentials

In the time harmonic case functionals of A_l and F_l can be given:

$$I'_A = \frac{1}{2} \int_\Omega (\text{grad}^2 \ A_l - k^2 A_l^2) \text{d}V \tag{38}$$

and

$$I'_F = \frac{1}{2} \int_\Omega (\text{grad}^2 \ F_l - k^2 F_l^2) \text{d}V, \tag{39}$$

where

$$k^2 = (\sigma + j\omega\varepsilon) j\omega\mu. \tag{40}$$

In case of general time variation these two functionals can be given in current-free region:

$$I''_A = \frac{1}{2} \int_0^T \int_\Omega \left[\frac{1}{\mu} \text{grad}^2 \ A_l - \varepsilon (\frac{\partial A_l}{\partial t})^2 \right] \text{d}V \text{d}t \ , \tag{41}$$

$$I''_F = - \int_0^T \int_\Omega \left[\frac{1}{\varepsilon} \text{grad}^2 \ F_l - \mu (\frac{\partial F_l}{\partial t})^2 \right] \text{d}V \text{d}t. \tag{42}$$

References

1. MORSE, P.M. – FESHBACH, H. (1953): Methods of Theoretical Physics, Mc Graw-Hill Book Company, Inc. 1953.
2. FLÜGGE, S. (1958): Handbuch der Physik, Band XVI. Springer Verlag, 1958.
3. SIMONYI, K. (1963): Foundations of Electrical Engineering, Pergamon Press, London, 1963.