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ON THE INTERFACE AND BOUNDARY CONDITIONS OF ELECTROMAGNETIC FIELDS

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Abstract

The field quantities **H**, **B**, **E**, **D**, satisfy some interface and boundary conditions on the boundary surface of two media. On the boundary surfaces there can be present electric or magnetic single or double charge or current layers. This article describes interface and boundary conditions for quantities **H**, **B**, **E**, **D** and for scalar and vector potentials. To the best knowledge of the author some of these conditions have not been published.

Keywords: electromagnetic field, scalar potential, vector potential, interface condition, boundary condition, Dirichlet boundary condition, Neumann boundary condition, electric charge layer, double charge layer, magnetic charge layer, double magnetic charge layer, electric current layer, double electric current layer, magnetic current layer, double magnetic current layer, charge layer, current layer.

The field quantities **H**, **B**, **E**, **D**, **J** in Maxwell's equations are frequently determined by potential functions. The application of the electric scalar potential φ and vector potential **A** or of the magnetic scalar potential ψ and vector potential **F** are usual. The use of other modified forms of these potentials may also occur.

The scalar potential φ and the vector potential **A** are usually applied if there are no magnetic charges and currents within the region examined. In this case

$$\mathbf{B} = \text{curl } \mathbf{A}, \tag{1}$$

$$\mathbf{E} = -\text{grad}\varphi - \frac{\partial \mathbf{A}}{\partial t} \tag{2}$$

The scalar potential ψ and the vector potential **F** are commonly used if there are no electric charges and currents within the region examined. Then

$$\mathbf{D} = \text{curl } \mathbf{F}, \tag{3}$$

$$\mathbf{H} = -\text{grad}\psi + \frac{\partial \mathbf{F}}{\partial t}. \tag{4}$$

It follows from (1) and (3) for an arbitrary surface S bounded by the curve c that

$$\int_S \mathbf{B} dS = \oint_c \mathbf{A} dl, \qquad \int_S \mathbf{D} dS = \oint_c \mathbf{F} dl. \tag{5}$$

The potentials satisfy differential equations (Laplace – Poisson, Helmholtz, wave equations) which are derived from the Maxwell equations. The solution of these equations is unique only if appropriate boundary conditions are satisfied. The continuity conditions for the potential functions describing the electromagnetic field on both sides of the boundary surface are called interface conditions. The boundary conditions give prescriptions on the boundary surface of the region examined under the assumption that no electromagnetic field exists outside this domain.

The two significant types of boundary conditions are those of Dirichlet and Neumann type. In case of Dirichlet boundary conditions the value of a scalar function or the tangential component of a vector function are prescribed on the boundary. In case of Neumann boundary conditions, the normal component of the gradient of a scalar function or the tangential component of the curl of a vector function are given. The boundary condition is of mixed type if Dirichlet condition is valid on the one part and Neumann condition on the other part of the bounding surface.

Single or double charge or current layers may be present on the boundary surfaces. The name of such layers, the characteristic quantities and their symbols are summarized in *Table 1*. On the bounding surfaces of the region examined such layers are always present, they form a closure of the electromagnetic field.

Table 1
Surface layers

Denomination	Characteristic parameter	Symbol
Electric charge layer	Surface charge density	ρ_S
Double electric charge layer	Moment	$\underline{\underline{\nu}}$
Magnetic charge layer	Surface charge density	η_S
Double magnetic charge layer	Moment	$\underline{\underline{\kappa}}$
Electric current layer	Surface current density	$\mathbf{J}_S$
Double electric current layer	Moment	$\underline{\underline{\lambda}}$
Magnetic current layer	Surface current density	$\mathbf{K}_S$
Double magnetic current layer	Moment	$\underline{\underline{\chi}}$

In the following, the quantities appearing on the one side of the boundary surface are denoted by the subscript 1, and those on the other side by the subscript 2.

Interface Conditions for Potentials on Boundary Surfaces without Surface Layers

The interface conditions known for **E**, **D**, **H**, **B** are in *Table 2* for the case if there are no surface layers on the boundary surface [1], [3]. (**n** is the unit normal.) Hence the interface conditions of the time independent scalar potentials are given in *Table 3* on the basis of (2) and (4).

Table 2
Interface conditions for field quantities on surfaces without layers

$\mathbf{n}\times(\mathbf{E}_1-\mathbf{E}_2)=\mathbf{0}$	$\mathbf{n}(\mathbf{D}_1-\mathbf{D}_2)=0$	$\mathbf{n}\times(\mathbf{H}_1-\mathbf{H}_2)=0$	$\mathbf{n}(\mathbf{B}_1-\mathbf{B}_2)=\mathbf{0}$

Table 3
Interface conditions for time independent scalar potentials on surfaces without layers

$\varphi_1-\varphi_2=0$	$\mathrm{n}\ (\varepsilon_1\,\mathrm{grad}\,\varphi_1-\varepsilon_2\,\mathrm{grad}\,\varphi_2)=0$
$\psi_1-\psi_2=0$	$\mathrm{n}\ (\mu_1\,\mathrm{grad}\,\psi_1-\mu_2\,\mathrm{grad}\,\psi_2)=0$

In (2) and (4) the gradients of scalar potentials are present, so the value of the potential can arbitrarily be chosen in one point of the region examined and this choice can be independent in the parts of the region separated by boundary surfaces. This choice is expedient if it simplifies the calculation. This means often that the zero potential points of these parts are common. In this case, the scalar potential is continuous on the interface free of layers.

The satisfaction of a Dirichlet boundary condition for scalar potential simultaneously determines the zero potential point, but in case of Neumann boundary condition this point must be specified.

Eqs. (1) and (3) yield the curl of the vector potentials. The divergence of vector potentials may be chosen arbitrarily. Usually, this choice corresponds to the Coulomb gauge:

$$\text{div } \mathbf{A} = 0, \qquad \text{div } \mathbf{F} = 0 \tag{6}$$

or to the Lorentz gauge:

$$\text{div } \mathbf{A} = -\sigma\mu\varphi - \mu\varepsilon\frac{\partial\varphi}{\partial t}, \qquad \text{div } \mathbf{F} = \mu\varepsilon\frac{\partial\psi}{\partial t}. \tag{7}$$

The Lorentz gauge is only practically useful in calculations for homogeneous media.

The scalar potentials can be eliminated from the Eqs. (2) and (4) of time dependent fields with the aid of the Lorentz gauge:

$$\frac{\partial \mathbf{E}}{\partial t} = -\frac{\partial^2 \mathbf{A}}{\partial t^2} + \frac{1}{\mu\varepsilon}\text{grad div } \mathbf{A}, \tag{8}$$

$$\frac{\partial \mathbf{H}}{\partial t} = \frac{\partial^2 \mathbf{F}}{\partial t^2} - \frac{1}{\mu\varepsilon}\text{grad div } \mathbf{F}. \tag{9}$$

On an interface without layers between two media the normal components of the magnetic flux density $\mathbf{B}$ and of the displacement vector $\mathbf{D}$ are continuous (*Table 2*), so for any surface S surrounded by a closed curve c on the boundary surface, we have

$$\int_S \mathbf{B}_1 d\mathbf{S} = \int_S \mathbf{B}_2 d\mathbf{S}, \qquad \int_S \mathbf{D}_1 d\mathbf{S} = \int_S \mathbf{D}_2 d\mathbf{S} \tag{10}$$

and so, taking (5) into consideration,

$$\oint_c \mathbf{A}_1 d\mathbf{l} = \oint_c \mathbf{A}_2 d\mathbf{l} \qquad \oint_c \mathbf{F}_1 d\mathbf{l} = \oint_c \mathbf{F}_2 d\mathbf{l}. \tag{11}$$

These are certainly satisfied if the tangential components of $\mathbf{A}$ and $\mathbf{F}$ are continuous. The interface conditions of the vector potentials are dependent from the choice of divergences of them. If the divergence is described with the same function in the two regions, then the continuity of tangential components of vector potentials on the boundary surface can be supposed. When according to Coulomb gauge div $\mathbf{A} = 0$, div $\mathbf{F} = 0$ in the two media, then can be supposed

$$\mathbf{n} \times (\mathbf{A}_1 - \mathbf{A}_2) = 0, \qquad (\text{div } \mathbf{A} = 0) \tag{12}$$

$$\mathbf{n} \times (\mathbf{F}_1 - \mathbf{F}_2) = 0, \qquad (\text{div } \mathbf{F} = 0) \tag{13}$$

However, the divergences of vector potentials are chosen according to Lorentz gauge, then the conditions for continuity of the tangential components of $\mathbf{E}$ and of $\mathbf{H}$ could be in contradiction with (12) and (13).

The interface conditions on the normal components of vector potentials are also dependent on the choice of divergences. The interface conditions on the normal components of the vector potentials shall be first examined for time-dependent fields with $\varphi \equiv 0$ and $\psi \equiv 0$ assumed. Then the Lorentz condition coincides with the Coulomb gauge (div $\mathbf{A} = 0$, div $\mathbf{F} = 0$). So from (8) and (9):

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{H} = \frac{\partial \mathbf{F}}{\partial t}. \tag{14}$$

It follows by time integration from the continuity condition of the normal components of $\mathbf{E}$, $\mathbf{H}$ for interfaces without layers that

$$\mathbf{n}(\varepsilon_1 \mathbf{A}_1 - \varepsilon_2 \mathbf{A}_2) = 0 \quad (\mathrm{div}\mathbf{A}_1 = \mathrm{div}\mathbf{A}_2, \quad \varphi \equiv 0) \tag{15}$$

$$\mathbf{n}(\mu_1 \mathbf{F}_1 - \mu_2 \mathbf{F}_2) = 0 \quad (\mathrm{div}\ \mathbf{F}_1 = \mathrm{div}\ \mathbf{F}_2, \quad \psi \equiv 0), \tag{16}$$

where the integration constants are assumed to be zero.

If the vector potentials satisfy the Lorentz gauge it shall be decomposed in normal and tangential components:

$$\mathbf{A} = \mathbf{A}_n + \mathbf{A}_\tau, \qquad \mathbf{F} = \mathbf{F}_n + \mathbf{F}_\tau. \tag{17}$$

It could be shown that the choice

$$\mathrm{div}\ \mathbf{A}_\tau = 0, \quad \mathrm{div}\ \mathbf{F}_\tau = 0 \tag{18}$$

constitutes no limitation [4]. Then we have from (8) and (9)

$$\frac{\partial E_n}{\partial t} = -\frac{\partial^2 A_n}{\partial t^2} + \frac{1}{\mu\varepsilon}\frac{\partial^2 A_n}{\partial n^2}, \qquad \frac{\partial H_n}{\partial t} = \frac{\partial^2 F_n}{\partial t^2} - \frac{1}{\mu\varepsilon}\frac{\partial^2 F_n}{\partial n^2}. \tag{19}$$

The normal components of vector potentials satisfy the homogeneous wave equations if $\mathbf{J} = \mathbf{0}$

$$\Delta A_n - \mu\varepsilon \frac{\partial^2 A_n}{\partial t^2} = 0, \qquad \Delta F_n - \mu\varepsilon \frac{\partial^2 F_n}{\partial t^2} = 0. \tag{20}$$

The operator Δ is written as the sum of a normal and a tangential operator:

$$\Delta = \Delta_\tau + \frac{\partial^2}{\partial n^2}. \tag{21}$$

Table 4
Interface conditions for vector potentials on surfaces without layers

		assumption		assumption
$\mathbf{n} \times (\mathbf{A}_1 - \mathbf{A}_2) = \mathbf{0},$	$\mathbf{n}\,(\varepsilon_1 \mathbf{A}_1 - \varepsilon_2 \mathbf{A}_2) = 0,$	$\operatorname{div}\mathbf{A}_1 = \operatorname{div}\mathbf{A}_2$ $\varphi \equiv 0,$	$\mathbf{n}\left(\frac{\mathbf{A}_1}{\mu_1} - \frac{\mathbf{A}_2}{\mu_2}\right) = 0,$	$\operatorname{div}\mathbf{A} = -\mu\sigma\varphi$ $-\mu\varepsilon\frac{\partial\varphi}{\partial t}$
$\mathbf{n} \times (\mathbf{F}_1 - \mathbf{F}_2) = \mathbf{0},$	$\mathbf{n}\,(\mu_1 \mathbf{F}_1 - \mu_2 \mathbf{F}_2) = 0,$	$\operatorname{div}\mathbf{F}_1 = \operatorname{div}\mathbf{F}_2$ $\psi \equiv 0,$	$\mathbf{n}\left(\frac{\mathbf{F}_1}{\mu_1} - \frac{\mathbf{F}_2}{\mu_2}\right) = 0,$	$\operatorname{div}\mathbf{F} = \mu\varepsilon\frac{\partial\psi}{\partial t}$

So

$$\varepsilon\frac{\partial E_n}{\partial t} = -\frac{1}{\mu}\Delta_\tau A_n, \qquad \mu\frac{\partial H_n}{\partial t} = \frac{1}{\varepsilon}\Delta_\tau F_n. \tag{22}$$

It follows from *Table 2* that the left sides of *Eqs.* (22) are continuous on boundary surfaces without layers, so the right sides are continuous, too. By integrating twice with respect to τ we have

$$\frac{1}{\mu_1}A_{1n} = \frac{1}{\mu_2}A_{2n}, \quad \mathbf{n}(\tfrac{1}{\mu_1}\mathbf{A}_1 - \tfrac{1}{\mu_2}\mathbf{A}_2) = 0 \quad (\text{div } \mathbf{A} = -\mu\sigma\varphi - \mu\varepsilon\tfrac{\partial\varphi}{\partial t}), \tag{23}$$

$$\frac{1}{\varepsilon_1}F_{1n} = \frac{1}{\varepsilon_2}F_{2n}, \quad \mathbf{n}(\tfrac{1}{\varepsilon_1}\mathbf{F}_1 - \tfrac{1}{\varepsilon_2}\mathbf{F}_2) = 0 \quad (\text{div } \mathbf{F} = \mu\varepsilon\tfrac{\partial\psi}{\partial t}), \tag{24}$$

where the integration constant is zero. (It can be remarked that the result is the same as well, when $\mathbf{J} \neq \mathbf{0}$.)

The interface conditions for vector potentials on surfaces without layers are summarized in *Table 4*.

Charge Layers and Double Charge Layers

The interface conditions for the electric and magnetic field on surfaces with charge layers and double charge layers are summarized in *Table 5*.

Table 5
Interface conditions for field quantities
on surfaces with charge layers and double charge layers

ρ_S	$\mathbf{n}\times(\mathbf{E}_1-\mathbf{E}_2)=0$	$\mathbf{n}(\mathbf{D}_2-\mathbf{D}_1) = \rho_S$
η_S	$\mathbf{n}\times(\mathbf{H}_1-\mathbf{H}_2)=0$	$\mathbf{n}(\mathbf{B}_2-\mathbf{B}_1) = \eta_S$
$\underline{\underline{\nu}}$	$\mathbf{n}\times(\mathbf{E}_1-\mathbf{E}_2) = -\frac{1}{\varepsilon_d}\text{ curl }\underline{\underline{\nu}}$	$\mathbf{n}(\mathbf{D}_1-\mathbf{D}_2) = 0$
$\underline{\underline{\kappa}}$	$\mathbf{n}\times(\mathbf{H}_1-\mathbf{H}_2) = -\frac{1}{\mu_d}\text{ curl }\underline{\underline{\kappa}}$	$\mathbf{n}(\mathbf{B}_1-\mathbf{B}_2) = 0$

The double charge layer consists of two charge layers at a distance $\Delta l \to 0$. On the surface element ΔS_1 of one layer the electric and magnetic charge densities are $-\rho_S$ and η_S, whereas on the surface element ΔS_2 of the other layer they are ρ_S and η_S, The distance between ΔS_1 and ΔS_2 is Δl and $\Delta S_1 = \Delta S_2$. The characteristic parameters of the double charge layers are the moments $\underline{\underline{\nu}}$ and $\underline{\underline{\kappa}}$ defined by

$$\underline{\underline{\nu}} = \mathbf{n}\nu = \rho_S\Delta\mathbf{l}, \quad \underline{\underline{\kappa}} = \mathbf{n}\kappa = \eta_S\Delta\mathbf{l}, \tag{25}$$

where $\Delta\mathbf{l} = \Delta l\mathbf{n}$ and $\mathbf{n}$ is directed from ΔS_1 to ΔS_2.

The first rows of *Tables 5* and *6* are valid in the case, if there is an electric charge density ρ_S on the boundary surface. The second rows are valid for surfaces with magnetic charge density η_S and the relations in this row are analogous to those in the first one. The third rows relate to the electric double charge layer with moment $\underline{\underline{\nu}}$ and the fourth rows to the magnetic one with moment $\underline{\underline{\kappa}}$. ε_d and μ_d are the permittivity and permeability of the homogeneous medium between the two layers. The second column of the third row in *Table 5*, which has not been published before, to the best knowledge of the author, follows for time independent fields from the relation in the second column of the third row in *Table 6*:

Table 6
Interface conditions for time independent scalar potentials on surfaces with charge layers and double charge layers

ρ_S	$\varphi_1 - \varphi_2 = 0$	$\mathbf{n}\ (\varepsilon_1 \text{ grad } \varphi_1 - \varepsilon_2 \text{ grad } \varphi_2) = \rho_S$
η_S	$\psi_1 - \psi_2 = 0$	$\mathbf{n}\ (\mu_1 \text{ grad } \psi_1 - \mu_2 \text{ grad } \psi_2) = \eta_S$
$\underline{\underline{\nu}}$	$\varphi_1 - \varphi_2 = -\frac{\nu}{\varepsilon_d}$	$\mathbf{n}\ (\varepsilon_1 \text{ grad } \varphi_1 - \varepsilon_2 \text{ grad } \varphi_2) = 0$
$\underline{\underline{\kappa}}$	$\psi_1 - \psi_2 = -\frac{\kappa}{\mu_d}$	$\mathbf{n}\ (\mu_1 \text{ grad } \psi_1 - \mu_2 \text{ grad } \psi_2) = 0$

$$\mathbf{n} \times (\text{grad}\varphi_1 - \text{grad}\varphi_2) = \mathbf{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = -\frac{1}{\varepsilon_d}\mathbf{n} \times \text{grad}\,\nu. \quad (26)$$

It can be proved that curl $\mathbf{n} = \mathbf{0}$ in any point of smooth surface, so

$$\text{curl}\,\underline{\underline{\nu}} = \text{curl}\,\mathbf{n}\nu = \text{grad}\,\nu \times \mathbf{n}, \quad (27)$$

i. e.

$$\mathbf{n} \times (\mathbf{E}_1 - \mathbf{E}_2) = -\frac{1}{\varepsilon_d}\text{curl}\,\underline{\underline{\nu}}. \quad (28)$$

The second column of *Table 6* is valid with the assumption that the zero potential points in the two parts of the region examined divided by the boundary surface are the same. In this case, the potential is continuous on single charge layers and jumps with $-\nu/\varepsilon_d$ and $-\kappa/\mu_d$ on double charge layers.

The relations of *Tables 2* and *3* follow from *Tables 5* and *6* in the case $\rho_S = 0$, $\eta_S = 0, \underline{\underline{\nu}} = 0, \underline{\underline{\kappa}} = 0$.

The boundary conditions are summarized in *Tables 7* and *8* on the basis of *Tables 5* and *6*. In this case no electromagnetic field is present on one side of the charge layer or double charge layer. For instance, the field quantities denoted by the subscript 1 are zero and the scalar potentials are constant ($\mathbf{n} \times \text{grad } \varphi = \mathbf{0}$, $\mathbf{n} \times \text{grad } \psi = \mathbf{0}$. The subscript 2 is omitted in *Tables 7* and *8*.

Table 7
Boundary conditions for field quantities
on surfaces with charge layers and double charge layers

ρ_S	$\mathbf{n}\times\mathbf{E}=0$	$\mathbf{nD}=\rho_S$
η_S	$\mathbf{n}\times\mathbf{H}=0$	$\mathbf{nB}=\eta_S$
$\underline{\underline{\nu}}$	$\mathbf{n}\times\mathbf{E}=\frac{1}{\varepsilon_d}$ curl $\underline{\underline{\nu}}$	$\mathbf{nD}=0$
$\underline{\underline{\kappa}}$	$\mathbf{n}\times\mathbf{H}=\frac{1}{\mu_d}$ curl $\underline{\underline{\kappa}}$	$\mathbf{nB}=0$

Table 8
Boundary conditions for time independent scalar potentials
on surfaces with charge layers and double charge layers

ρ_S	$\mathbf{n}\times$ grad $\varphi=0$ $\quad\varphi=$ const.	$\frac{\partial\varphi}{\partial n}=-\frac{\rho_S}{\varepsilon}$
η_S	$\mathbf{n}\times$ grad $\psi=0$ $\quad\psi=$ const.	$\frac{\partial\psi}{\partial n}=-\frac{\eta_S}{\mu}$
$\underline{\underline{\nu}}$	$\mathbf{n}\times$ grad $\varphi=-\frac{1}{\varepsilon_d}$ curl $\underline{\underline{\nu}}$	$\frac{\partial\varphi}{\partial n}=0$
$\underline{\underline{\kappa}}$	$\mathbf{n}\times$ grad $\psi=-\frac{1}{\mu_d}$ curl $\underline{\underline{\kappa}}$	$\frac{\partial\psi}{\partial n}=0$

Current Layers and Double Current Layers

It is well known that the tangential component of the magnetic field intensity changes abruptly on electric current layers, whereas the normal component of the magnetic flux density is continuous. In case of magnetic current layer, the tangential component of the electric field intensity jumps and the normal component of the electric displacement is continuous (*Table 9*).

Table 9
Interface conditions for field quantities
on surfaces with current layers and double current layers

$\mathbf{J}_S$	$\mathbf{n}\times(\mathbf{H}_2-\mathbf{H}_1)=\mathbf{J}_S$	$\mathbf{n}(\mathbf{B}_1-\mathbf{B}_2)=0$
$\mathbf{K}_S$	$\mathbf{n}\times(\mathbf{E}_1-\mathbf{E}_2)=\mathbf{K}_S$	$\mathbf{n}(\mathbf{D}_1-\mathbf{D}_2)=0$
$\underline{\underline{\lambda}}$	$\mathbf{n}\times(\mathbf{H}_1-\mathbf{H}_2)=0$	$\mathbf{n}(\mathbf{B}_1-\mathbf{B}_2)=\mu_d$ div $\underline{\underline{\lambda}}$
$\underline{\underline{\chi}}$	$\mathbf{n}\times(\mathbf{E}_1-\mathbf{E}_2)=0$	$\mathbf{n}(\mathbf{D}_1-\mathbf{D}_2)=-\varepsilon_d$ div $\underline{\underline{\chi}}$

The interface conditions of the normal and the tangential components of curl $\mathbf{A}$ and curl $\mathbf{F}$ are obtainable from the field components (*Table 10*).

If the divergences of $\mathbf{A}$ and of $\mathbf{F}$ resp. are described with the same function on both sides of the electric and of the magnetic current layer,

Table 10
Interface conditions for curl of vector potentials
on surfaces with current layers and double current layers

$\mathbf{J}_S$	$\mathbf{n}\times(\frac{1}{\mu_2}\text{ curl }\mathbf{A}_2-\frac{1}{\mu_1}\text{ curl }\mathbf{A}_1)=\mathbf{J}_S$	$\mathbf{n}(\text{curl }\mathbf{A}_2-\text{ curl }\mathbf{A}_1)=0$
$\mathbf{K}_S$	$\mathbf{n}\times(\frac{1}{\varepsilon_1}\text{ curl }\mathbf{F}_1-\frac{1}{\varepsilon_2}\text{ curl }\mathbf{F}_2)=\mathbf{K}_S$	$\mathbf{n}(\text{curl }\mathbf{F}_1-\text{ curl }\mathbf{F}_2)=0$
$\underline{\underline{\lambda}}$	$\mathbf{n}\times(\frac{1}{\mu_1}\text{ curl }\mathbf{A}_1-\frac{1}{\mu_2}\text{ curl }\mathbf{A}_2)=\mathbf{0}$	$\mathbf{n}(\text{curl }\mathbf{A}_1-\text{ curl }\mathbf{A}_2)=\mu_d\text{ div }\underline{\underline{\lambda}}$
$\underline{\underline{\chi}}$	$\mathbf{n}\times(\frac{1}{\varepsilon_1}\text{ curl }\mathbf{F}_1-\frac{1}{\varepsilon_2}\text{ curl }\mathbf{F}_2)=\mathbf{0}$	$\mathbf{n}(\text{curl }\mathbf{F}_1-\text{ curl }\mathbf{F}_2)=-\varepsilon_d\text{ div }\underline{\underline{\chi}}$

then the tangential components of **A** and of **F** are continuous independently from values of $\mathbf{J}_S$ and $\mathbf{K}_S$. This is always fulfilled at choice according to Coulomb gauge. At choice according to Lorentz gauge it is only fulfilled, when permeability and permittivity are the same on the two sides of current layer. Then

$$\mathbf{n}\times(\mathbf{A}_1-\mathbf{A}_2)=0, \qquad (\text{div }\mathbf{A}_1=\text{div }\mathbf{A}_2), \tag{29}$$

$$\mathbf{n}\times(\mathbf{F}_1-\mathbf{F}_2)=0, \qquad (\text{div }\mathbf{F}_1=\text{div }\mathbf{F}_2). \tag{30}$$

In double current layers, the current density on the element ΔS_1 of the surface S_1 is - $\mathbf{J}_S$ and - $\mathbf{K}_S$, on the element ΔS_2 of the surface S_2 at a distance Δl from ΔS_1 it is $\mathbf{J}_S$ and $\mathbf{K}_S$ ($\Delta l \to 0$, $\Delta S_1 = \Delta S_2$, *Fig. 1*).

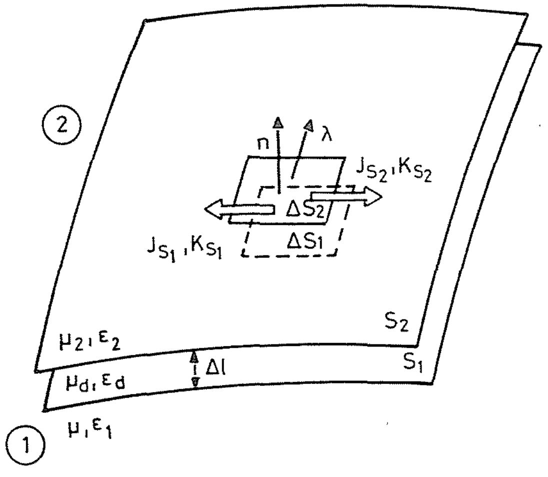

Fig. 1.

At an arbitrary point of an electric current layer we have with the notations of *Fig. 1*:

$$\mathbf{n} \times (\mathbf{H}_d - \mathbf{H}_1) = -\mathbf{J}_S \tag{31}$$

and

$$\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_d) = \mathbf{J}_S \tag{32}$$

and hence

$$\mathbf{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{0}, \tag{33}$$

i. e. the tangential component of the magnetic field intensity is continuous on double electric current layers. Thus, using (1)

$$\mathbf{n} \times (\frac{1}{\mu_1}\text{curl}\,\mathbf{A}_1 - \frac{1}{\mu_2}\text{curl}\,\mathbf{A}_2) = 0. \tag{34}$$

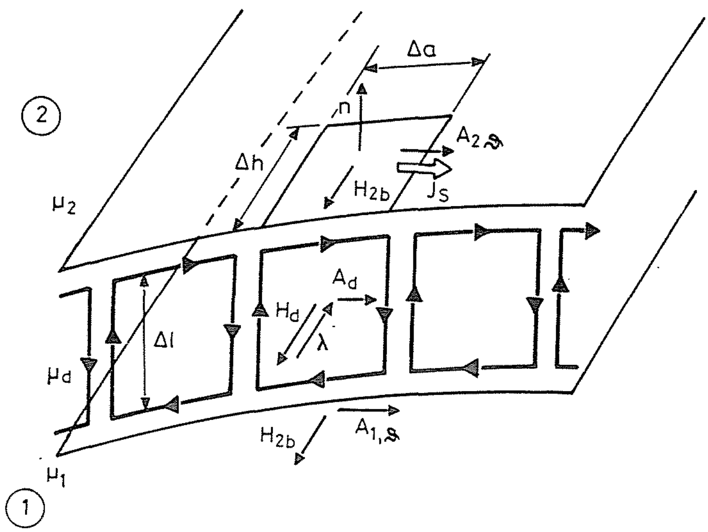

Fig. 2.

The double electric current layer can be regarded as consisting of current loops with the current

$$I = J_S \Delta h \tag{35}$$

as shown in *Fig. 2*. The electromagnetic moment of such a loop is

$$\mathbf{m} = \Delta\mathbf{l} \times \mathbf{J}_S \Delta h \Delta a, \tag{36}$$

where $\Delta\,\mathbf{l} = \Delta l\mathbf{n}$ and $\mathbf{n}$ is directed from ΔS_1 to ΔS_2. The moment of the double electric charge layer is defined by

$$\underline{\underline{\lambda}} = \frac{1}{\Delta a \Delta h}\mathbf{m} = \Delta\mathbf{l} \times \mathbf{J}_S, \qquad \lambda = J_S \Delta l. \tag{37}$$

Denoting the vector potential between the two layers by $\mathbf{A}^d$, and integrating it along a loop with the electromagnetic moment $\mathbf{m}$, we get

$$\oint_c \mathbf{A}^d d\mathbf{l} = (A^d_{2\vartheta} - A^d_{1\vartheta})\Delta a = -\mu_d H_{db} \Delta a \Delta l, \tag{38}$$

where μ_d is the permeability of the homogeneous medium between the two layers. The subscripts ϑ and b denote the two orthogonal tangential components. From (38), we have

$$H_{db}\Delta l = \frac{1}{\mu_d}(A^d_{1\vartheta} - A^d_{2\vartheta}). \tag{39}$$

It follows from (31), (39) and (37) that

$$H_{db}\Delta l - H_{1b}\Delta l = \frac{1}{\mu_d}(A^d_{1\vartheta} - A^d_{2\vartheta}) - H_{1b}\Delta l = -J_s \Delta l = -\lambda. \tag{40}$$

If $\Delta l \to 0$, then $H_{1b}\ \Delta l \to 0$ and so

$$\frac{1}{\mu_d}(A^d_{1\vartheta} - A^d_{2\vartheta}) = -\lambda. \tag{41}$$

Considering the directions:

$$\mathbf{n} \times (\mathbf{A}^d_2 - \mathbf{A}^d_1) = \mu_d \underline{\underline{\lambda}}. \tag{42}$$

The tangential component of the vector potential $\mathbf{A}$ is continuous on electric current layers, i. e. $A^d_{1\vartheta} = A_{1\vartheta}, A^d_{2\vartheta} = A_{2\vartheta}$. So

$$\mathbf{n} \times (\mathbf{A}_2 - \mathbf{A}_1) = \mu_d \underline{\underline{\lambda}} \qquad (\operatorname{div}\mathbf{A}_1 = \operatorname{div}\mathbf{A}_2), \tag{43}$$

when the divergence of $\mathbf{A}$ is described with the same function on both sides of the double current layer. This means that the tangential component of the vector potential has a jump proportional to the moment of the double electric current layer.

The divergence of the (43) is

$$\mathrm{div}[\mathbf{n}\times(\mathbf{A}_2-\mathbf{A}_1)] = (\mathbf{A}_2-\mathbf{A}_1)\mathrm{curl}\ \mathbf{n} - \mathbf{n}\mathrm{curl}\ (\mathbf{A}_2-\mathbf{A}_1) = \mu_d\ \mathrm{div}\ \underline{\underline{\lambda}}. \quad (44)$$

Since curl $\mathbf{n} = \mathbf{0}$ on smooth surfaces, we have

$$\mathbf{n}(\mathbf{B}_1-\mathbf{B}_2) = \mu_d\ \mathrm{div}\ \underline{\underline{\lambda}}. \quad (45)$$

The moment of a double magnetic current layer is

$$\underline{\underline{\chi}} = \Delta\mathbf{l}\times\mathbf{K}_S. \quad (46)$$

Its effect can be discussed similarly as above. The tangential component of the electric field intensity is continuous on double magnetic current layers (*Table 9*):

$$\mathbf{n}\times(\mathbf{E}_1-\mathbf{E}_2) = 0 \quad (47)$$

and so

$$\mathbf{n}\times\left(\frac{1}{\varepsilon_1}\mathrm{rot}\ \mathbf{F}_1 - \frac{1}{\varepsilon_2}\mathrm{rot}\ \mathbf{F}_2\right) = 0. \quad (48)$$

The normal component of $\mathbf{D}$ jumps here:

$$\mathbf{n}\times(\mathbf{D}_2-\mathbf{D}_1) = \varepsilon_d\ \mathrm{div}\ \underline{\underline{\chi}}, \quad (49)$$

ε_d is the permittivity of the homogeneous medium between the two layers. Similarly to (43), the tangential component of the vector potential $\mathbf{F}$ changes abruptly on double magnetic current layers, when the divergence of $\mathbf{F}$ is described with the same function on both sides of the double current layer.

$$\mathbf{n}\times(\mathbf{F}_1-\mathbf{F}_2) = \varepsilon_d\underline{\underline{\chi}}, \quad (\mathrm{div}\,\mathbf{F}_1 = \mathrm{div}\,\mathbf{F}_2). \quad (50)$$

The boundary conditions are summarized in *Table 11*. On the double electric current layer the boundary condition is according (34) $\mathbf{n}\times$ curl $\mathbf{A} = \mathbf{0}$. When $\mathbf{n}\ \mathbf{A} = 0$, it is equivalent with

$$\frac{\partial A}{\partial n} = 0, \quad (\mathbf{n}\ \mathbf{A} = 0). \quad (51)$$

Similarly, in the special case $\mathbf{n}\ \mathbf{F} = 0$ $\mathbf{n}\times$ curl $\mathbf{F} = \mathbf{0}$ may be described in the form

$$\frac{\partial F}{\partial n} = 0, \quad (\mathbf{n}\ \mathbf{F} = 0), \quad (52)$$

on double magnetic current layer the boundary condition.

Table 11

Boundary conditions
on surfaces with current layers and double current layers

$\mathbb{J}_S$	$\mathbf{n}\times\mathbb{H}=\mathbf{J}_S$	$\mathbf{nB}=0$	$\mathbf{n}\times\mathbf{A}=\mathbf{0}$	$\mathbf{n}\times$ curl $\mathbf{A}=\mu\mathbf{J}_S$	$\mathbf{n}$ curl $\mathbf{A}=\mathbf{0}$
$\mathbb{K}_S$	$\mathbf{n}\times\mathbb{E}=-\mathbb{K}_S$	$\mathbf{nD}=0$	$\mathbf{n}\times\mathbf{F}=\mathbf{0}$	$\mathbf{n}\times$ curl $\mathbf{F}=-\varepsilon\mathbb{K}_S$	$\mathbf{n}$ curl $\mathbf{F}=\mathbf{0}$
$\underline{\underline{\lambda}}$	$\mathbf{n}\times\mathbb{H}=\mathbf{0}$	$\mathbf{nB}=-\mu_d\mathrm{div}\underline{\underline{\lambda}}$	$\mathbf{n}\times\mathbf{A}=-\mu_d\underline{\underline{\lambda}}$	$\mathbf{n}\times$ curl $\mathbf{A}=\mathbf{0}$	$\mathbf{n}$ curl $\mathbf{A}=-\mu_d\mathrm{div}\ \underline{\underline{\lambda}}$
$\underline{\underline{\chi}}$	$\mathbf{n}\times\mathbb{E}=0$	$\mathbf{nD}=\varepsilon_d\mathrm{div}\underline{\underline{\chi}}$	$\mathbf{n}\times\mathbf{F}=-\varepsilon_d\underline{\underline{\chi}}$	$\mathbf{n}\times$ curl $\mathbf{F}=\mathbf{0}$	$\mathbf{n}$ curl $\mathbf{F}=\varepsilon_d\mathrm{div}\ \underline{\underline{\chi}}$

Table 12

Homogeneous boundary conditions on electric and on magnetic walls

Electric wall	$\mathbf{n}\times\mathbb{E}=\mathbf{0}$	$\mathbf{nB}=0$	$\mathbf{n}\times\mathbf{A}=\mathbf{0}$	$\mathbf{n}$ curl $\mathbf{A}=0$	$\mathbf{n}\times\mathrm{grad}\varphi=\mathbf{0}$	$\mathbf{n}\times$curl $\mathbf{F}=\mathbf{0}$
Magnetic wall	$\mathbf{n}\times\mathbb{H}=\mathbf{0}$	$\mathbf{nD}=0$	$\mathbf{n}\times\mathbf{F}=\mathbf{0}$	$\mathbf{n}$ curl $\mathbf{F}=0$	$\mathbf{n}\times\mathrm{grad}\psi=\mathbf{0}$	$\mathbf{n}\times$curl $\mathbf{A}=\mathbf{0}$

Equivalent Layers

Comparing *Table 5* and *Table 9* it can be established that some layers are equivalent with respect to the interface conditions. Thus, an electric charge layer is equivalent to a double magnetic current layer, provided

$$\rho_S = \varepsilon_d \operatorname{div} \underline{\underline{\chi}} \,. \tag{53}$$

Similarly, a magnetic charge layer and a double electric current layer are equivalent, if

$$\eta_S = -\mu_d \operatorname{div} \underline{\underline{\lambda}}. \tag{54}$$

A magnetic current layer and a double electric charge layer are equivalent, provided

$$-\mathbf{K}_S = \frac{1}{\varepsilon_d} \operatorname{curl} \underline{\underline{\nu}} \tag{55}$$

and an electric current layer and a double magnetic charge layer, if

$$\mathbf{J}_S = \frac{1}{\mu_d} \operatorname{curl} \underline{\underline{\kappa}} \tag{56}$$

In case of time dependent fields, further equivalences can be derived from the continuity equations

$$\operatorname{div} \mathbf{J}_S = -\frac{\partial \rho_S}{\partial t}, \qquad \operatorname{div} \mathbf{K}_S = -\frac{\partial \eta_S}{\partial t}. \tag{57}$$

Taking the time derivative of (53) and (54), we obtain

$$\frac{\partial \rho_S}{\partial t} = \varepsilon_d \operatorname{div} \frac{\partial \underline{\underline{\chi}}}{\partial t}, \qquad \frac{\partial \eta_S}{\partial t} = -\mu_d \operatorname{div} \frac{\partial \underline{\underline{\lambda}}}{\partial t}. \tag{58}$$

Comparing (57) with (58) it can be established that these are certainly satisfied, provided

$$\mathbf{J}_S = -\varepsilon_d \frac{\partial \underline{\underline{\chi}}}{\partial t}, \qquad \mathbf{K}_S = \mu_d \frac{\partial \underline{\underline{\lambda}}}{\partial t}. \tag{59}$$

These can also be deduced from the fact that a section of a magnetic current layer can be substituted by an electric loop current and a section of an electric current layer by a magnetic loop current.

The condition of the equivalence between a double magnetic charge layer and a double magnetic current layer is, according to (56) and (59):

$$\operatorname{curl} \underline{\underline{\kappa}} = -\mu_d \varepsilon_d \frac{\partial \underline{\underline{\chi}}}{\partial t} \tag{60}$$

Similarly, a double electric charge layer and a double electric current layer are equivalent, if

$$\operatorname{curl} \underline{\underline{\nu}} = -\mu_d \varepsilon_d \frac{\partial \underline{\underline{\lambda}}}{\partial t} \,. \tag{61}$$

It follows from the above discussion that it is sufficient to take two kinds of layers into consideration in case of time dependent fields. These may be, e.g. the electric and the magnetic current layers. In boundary value problems, the electric current layer occurs on ideal conductors (on electric walls), and the magnetic current layer on so-called magnetic walls. The electric wall is equivalent to the electric charge layer, to the double magnetic charge layer and to the double magnetic current layer and the magnetic wall is equivalent to the magnetic charge layer, to the double electric charge layer and to the double electric current layer, provided the appropriate relationships are satisfied.

The homogeneous boundary conditions on the two kinds of walls are summarized in *Table 12*. It can be seen that on electric walls the vector potential $\mathbf{A}$ satisfies the homogeneous Dirichlet boundary condition, the scalar potential φ is constant and the vector potential $\mathbf{F}$ satisfies the homogeneous Neumann boundary condition . On magnetic walls a Dirichlet boundary condition is valid for the vector potential $\mathbf{F}$, a Neumann boundary condition for the vector potential $\mathbf{A}$ and the scalar potential ψ is constant. These are only true, if the electromagnetic field is derived from the potential pairs $\mathbf{A}-\varphi$ or $\mathbf{F}-\psi$. When $\mathbf{A}$ and $\mathbf{F}$ are applied simultaneously, the boundary conditions must be satisfied by the resultant field quantities.

In the fourth column of *Table 12* the tangential components of $\mathbf{A}$ and $\mathbf{F}$ are written as zero. They could be any constant. However, this constant is arbitrary, so it is practical to choose it to be zero. The equations in the fifth column are valid under this assumption.

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