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EQUIVALENCE OF EXTRAORDINARY WAVES IN A UNIAXIAL MEDIUM AND SCALAR WAVES IN VACUUM

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Abstract

We use the equation of the wave-vector surface in uniaxially anisotropic media to construct an equation for the amplitude of extraordinary waves. This equation is identical to the one derived directly from Maxwell's equations for the component of the electric field vector parallel to the optic axis. In principal axis coordinates, the extraordinary wave equation is a scaled version of the scalar Helmholtz equation. Consequently, there is a one-to-one correspondence between extraordinary waves and scalar waves in vacuum. This equivalence can be used to find the solution of problems on diffraction and beam propagation in uniaxial media from known solutions of the corresponding isotropic problems. As a simple example we determine the size of the focal region of converging beams in uniaxial crystals.

Keywords: propagation, crystal optics.

Introduction

The widespread use of birefringent materials in integrated optical devices [1–3] has raised interest also in computational problems relating to the propagation of waves in anisotropic media. Since these problems [4–9] are often more difficult to solve than their counterparts in isotropic optics, it is desirable to find methods for the reduction of anisotropic problems to isotropic ones.

In this paper we present a method by which some problems concerning the propagation, focusing and diffraction of monochromatic extraordinary waves in homogeneous but uniaxially anisotropic media may be transformed into equivalent problems relating to an isotropic medium.

At first we assume that the wave field to be described is characterized by a scalar amplitude and that the latter is expressible as a superposition of plane waves whose wave vectors are determined by the ellipsoid of wave vectors known from elementary crystal optics. From this assumption we construct a homogeneous scalar wave equation for the extraordinary waves and prove that this equation is the same as the one derived from Maxwell's equations for the component of the electric vector that is parallel to the optic axis.

Next we note that the anisotropic wave equation, if written in principal axis coordinates, is a transformed version of the Helmholtz equation. The transformation is trivial: each coordinate is scaled by one of the refractive indices. Consequently, there is a very simple one-to-one relation between extraordinary waves and waves in vacuum.

Finally, we use this similarity argument to determine the spot size in the focal line of a converging extraordinary beam.

Derivation of the Extraordinary Wave Equation from the Scalar Angular Spectrum

We consider a monochromatic extraordinary wave in a homogeneous uniaxial crystal. We assume that 1. the extraordinary wave is characterized by a position-dependent scalar amplitude Ψ, 2. the amplitude is the superposition of plane waves, 3. the end points of the wave vectors of the plane wave components lie on the ellipsoid of wave vectors.

By assumptions 1) and 2), the scalar amplitude is given by

$$\Psi(\mathbf{r}) = \int\int A(k_x, k_y) \exp[-i(k_x x + k_y y + k_z z)] dk_x dk_y \ , \tag{1}$$

where $\mathbf{k} = (k_x, k_y, k_z)$ denotes the wave vector of a plane-wave component and $A(k_x, k_y)$ is the angular (or plane-wave) spectrum of Ψ. (Note that k_z is a two-valued function of k_x and k_y. Therefore, the above integral is in fact the sum of two integrals containing the two branches of the function $k_z(k_x, k_y)$.) If the x direction is parallel with the optic axis of the uniaxial medium then assumption 3) states that

$$\frac{k_x^2}{n_o^2} + \frac{k_y^2}{n_\varepsilon^2} + \frac{k_z^2}{n_\varepsilon^2} = k_0^2 \ , \tag{2}$$

where n_o and n_ε are the ordinary and extraordinary refractive indices of the medium and k_0 is the vacuum wave number [10]. Taking the second partial derivatives of (1) with respect to x, y and z, respectively we find that

$$\frac{\partial^2 \Psi}{\partial x^2} = -\int\int k_x^2 A(k_x, k_y) \exp[-i(k_x x + k_y y + k_z z)] dk_x dk_y \ , \tag{3a}$$

$$\frac{\partial^2 \Psi}{\partial y^2} = -\int\int k_y^2 A(k_x, k_y) \exp[-i(k_x x + k_y y + k_z z)] dk_x dk_y \ , \tag{3b}$$

$$\frac{\partial^2 \Psi}{\partial z^2} = -\int\int k_z^2 A(k_x, k_y) \exp[-i(k_x x + k_y y + k_z z)] dk_x dk_y \ , \tag{3c}$$

We first divide (3a) by n_o^2 and (3b) and (3c) by n_e^2 and then we add the three resulting equations to obtain

$$\frac{1}{n_o^2}\frac{\partial^2\Psi}{\partial x^2}+\frac{1}{n_e^2}\frac{\partial^2\Psi}{\partial y^2}+\frac{1}{n_e^2}\frac{\partial^2\Psi}{\partial z^2}=$$

$$=-\int\int\left(\frac{k_x^2}{n_o^2}+\frac{k_y^2}{n_e^2}+\frac{k_z^2}{n_e^2}\right)A(k_x,k_y)\exp[-i(k_xx+k_yy+k_zz)]dk_xdk_y\ . \quad (4)$$

According to (2) the factor under the integral before A is equal to k_0^2 and by (1) the right hand side of (4) is just $-k_0^2\Psi$. Therefore, Ψ is a solution to the differential equation

$$\frac{1}{n_o^2}\frac{\partial^2\Psi}{\partial x^2}+\frac{1}{n_e^2}\frac{\partial^2\Psi}{\partial y^2}+\frac{1}{n_e^2}\frac{\partial^2\Psi}{\partial z^2}+k_0^2\Psi=0\ . \quad (5)$$

Up to this point we have not given yet any physical interpretation for the scalar amplitude Ψ. As the above 'derivation' is based on a scalar description, it is inherently incapable of producing a physical justification of itself. Before accepting *Eq.* (5) we must therefore compare it to the wave equation(s) obtained from the exact electromagnetic theory. To this end, we repeat the steps described by FLECK and FEIT [7] to derive the vectorial wave equation for the electric field vector $\mathbb{E}$ of a monochromatic extraordinary wave in a homogeneous, uniaxial, nonmagnetic crystal.

In a rectangular coordinate system whose x axis is parallel with the optic axis, the (relative) dielectric tensor ε is diagonal with elements

$$\varepsilon_x=n_e^2,\quad \varepsilon_y=\varepsilon_z=n_o^2\ . \quad (6)$$

Assuming a time dependence of $e^{i\omega t}$, the elimination of $\mathbb{H}$ from Maxwell's two curl equations

$$\text{rot }\mathbb{H}=i\omega\varepsilon_0\varepsilon\mathbb{E}\ ,\quad \text{rot}\mathbb{E}=-i\omega\mu_0\mathbb{H} \quad (7)$$

yield

$$\Delta\mathbb{E}=-\text{grad div}\mathbb{E}+k_0^2\varepsilon\mathbb{E}=0\ . \quad (8)$$

Using the divergence relation

$$\text{div }\varepsilon\mathbb{E}=\varepsilon_x\frac{\partial\mathbb{E}_x}{\partial x}+\varepsilon_z\left(\frac{\partial\mathbb{E}_y}{\partial y}+\frac{\partial\mathbb{E}_z}{\partial z}\right)=0 \quad (9)$$

to replace div $\mathbf{E}$ in (8) by

$$\operatorname{div} \mathbf{E} = \left(1 - \frac{\varepsilon_x}{\varepsilon_z}\right) \frac{\partial \mathbf{E}_x}{\partial x} \tag{10}$$

we find that the x component of (8) is

$$\frac{\partial^2 \mathbf{E}_x}{\partial x^2} + \frac{\partial^2 \mathbf{E}_y}{\partial y^2} + \frac{\partial^2 \mathbf{E}_z}{\partial z^2} - \left(1 - \frac{\varepsilon_x}{\varepsilon_z}\right) \frac{\partial^2 \mathbf{E}_x}{\partial x^2} + k_0^2 \varepsilon_x \mathbf{E}_x = 0 \tag{11}$$

or, after division by ε_x,

$$\frac{1}{\varepsilon_z} \frac{\partial^2 \mathbf{E}_x}{\partial x^2} + \frac{1}{\varepsilon_x} \frac{\partial^2 \mathbf{E}_y}{\partial y^2} + \frac{1}{\varepsilon_x} \frac{\partial^2 \mathbf{E}_z}{\partial z^2} + k_0^2 \mathbf{E}_x = 0\ . \tag{12}$$

According to definition (6) of the ordinary and extraordinary refractive indices, (12) is seen to be equivalent to (5). Thus we conclude that the originally unspecified scalar amplitude Ψ must be identified with E_x, i.e., as the component of $\mathbf{E}$ parallel with the optic axis.

The exact equations in Ref. 7 for the y and z components of $\mathbf{E}$ are different from (5) in that they contain also a term with the mixed second partial derivatives of E_x:

$$(\Delta + k_0^2 \varepsilon_y) \mathbf{E}_y = \left(1 - \frac{\varepsilon_x}{\varepsilon_z}\right) \frac{\partial^2 \mathbf{E}_x}{\partial x \partial y}\ , \tag{12b}$$

$$(\Delta + k_0^2 \varepsilon_z) \mathbf{E}_z = \left(1 - \frac{\varepsilon_x}{\varepsilon_z}\right) \frac{\partial^2 \mathbf{E}_x}{\partial x \partial z}\ . \tag{12c}$$

If E_x is known, these terms may be treated as source terms and E_y, E_z may be found as the solutions of the inhomogeneous scalar wave equations (12b), (12c). If the beam is ordinary then $E_x = 0$ (because ordinary waves are TE with respect to the optic axis) and the equations for E_y, E_z reduce to the homogeneous wave equation for an isotropic medium with refractive index n_o.

Correspondence between Extraordinary Waves in a Uniaxial Medium and Scalar Waves in Vacuum

It is noteworthy that *Eq.* (5) is just the scalar Helmholtz equation written in a scaled coordinate system. We show how this fact leads to an obvious equivalence between scalar waves in vacuum and extraordinary waves in a uniaxial medium.

Let $\Psi^{iso}(\xi,\eta,\zeta)$ denote a solution of the scalar Helmholtz equation:

$$\frac{\partial^2\Psi^{iso}}{\partial\xi^2}+\frac{\partial^2\Psi^{iso}}{\partial\eta^2}+\frac{\partial^2\Psi^{iso}}{\partial\zeta^2}+k_0^2\Psi^{iso}=0\;. \tag{13}$$

The second partial derivatives of the function

$$\Psi^{ani}(x,y,z)=\Psi^{iso}(n_o x,n_e y,n_e z) \tag{14}$$

are

$$\frac{\partial^2\Psi^{ani}}{\partial x^2}+n_o^2\frac{\partial^2\Psi^{iso}}{\partial\xi^2}\bigg|_{\xi=n_o x}\,,\quad \frac{\partial^2\Psi^{ani}}{\partial y^2}+n_e^2\frac{\partial^2\Psi^{iso}}{\partial\eta^2}\bigg|_{\eta=n_e y}\,,$$
$$\frac{\partial^2\Psi^{ani}}{\partial z^2}+n_e^2\frac{\partial^2\Psi^{iso}}{\partial\zeta^2}\bigg|_{\zeta=n_e z}\,. \tag{15}$$

Consequently, $\Psi^{ani}(x,y,z)$ given by (14) is a solution of the extraordinary wave equation (5). This means that from each known solution $\Psi^{iso}(\xi,\eta,\zeta)$ of the Helmholtz equation one can construct a function $\Psi^{ani}(x,y,z)$ satisfying (5) and vice versa. In other words, there is a trivial one-to-one correspondence between waves in vacuum and extraordinary waves in a uniaxial medium.

The Size of the Focal Region of a Converging Extraordinary Wave

In this section we use the above correspondence between vacuum waves and extraordinary waves to determine the spot size of a focused extraordinary beam that propagates in a direction perpendicular to the optic axis.

According to diffraction theory [10] relating to isotropic media, the spot diameter (or, generally speaking, any characteristic transverse linear dimension measured in the focal plane) of a converging paraxial beam in vacuum is given by

$$D^{iso}=C_1\frac{f^{iso}}{k_0A^{iso}}\;, \tag{16}$$

where f^{iso} is the focal length (i.e., the distance of the focal point from a fixed 'aperture plane'), A^{iso} is the diameter of the aperture (or, more generally, a characteristic transverse linear dimension of the beam measured far from the focal point), k_0 is the vacuum wave number, and C_1 is a numerical factor that depends on the shape but not on the size of the amplitude distribution along the 'aperture plane'.

The generalization of (16) to extraordinary waves is readily found from the similarity rule (14). The transverse dimensions A, D and the longitudinal dimension f characterize a solution Ψ^{iso} of the Helmholtz equation. Therefore, the corresponding dimensions of the equivalent anisotropic amplitude function Ψ^{ani} are

$$A^{ani} = A^{iso}/n_o \ , \quad D^{ani} = D^{iso}/n_o, f^{ani} = f^{iso}/n_e \ . \tag{17}$$

(Here it is assumed that the transverse dimensions A and D are measured in the principal section i.e., in the (x, z) plane which is parallel with the optic axis.) Expressing the isotropic quantities from these relations and inserting them in (16) , we find for the spot diameter (measured in the principal section)

$$D^{ani} = \left(\frac{n_e}{n_o}\right)^2 C_1 \frac{f^{ani}}{n_e k_0 A^{ani}} \ . \tag{18}$$

This means that the spot size (in the principal section) of a focused extraordinary beam is $(n_e/n_o)^2$ times the value calculated for a beam that propagates in an isotropic medium whose refractive index is n_e. Previously we have established this relation by more involved Fourier optical [8, 9] and numerical [9] investigations.

Conclusion

We have shown that

1. the scalar angular spectrum representation combined with the equation of the wave-vector surface of extraordinary waves in a uniaxial medium is sufficient for the derivation of the exact wave equation of the component of $\mathbb{E}$ parallel with the optic axis, and that

2. this wave equation is a scaled version of the scalar Helmholtz equation.

As a consequence, we have found that

3. there is one-to-one correspondence between scalar vacuum waves and extraordinary waves in a uniaxial crystal, and that

4. if measured in the principal section, the spot size of a focused beam in a uniaxial medium is $(n_e/n_o)^2$ times the value obtained from the formula for the spot size of a beam (with the same far-field beam diameter) that propagates in a homogeneous medium with refractive index n_e.

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